Inequalities defining polyhedral realizations and monomial realizations of crystal bases

Combinatorics and Arithmetic for Physics, IHES

Yuki Kanakubo (Ibaraki University)

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Goal

We give a conjecture on explicit forms of polyhedral realizations for crystal bases of quantum groups in terms of monomial realizations.

Crystal bases $B(\infty)$, $B(\lambda)$: a powerful tool to study representations of quantum groups

Polyhedral realization : a combinatorial description of $B(\infty)$
Monomial realization : a combinatorial description of $B(\lambda)$

Plan

1. Quantum groups
2. Crystal bases and polyhedral realizations
3. Monomial realizations
4. Main results

1. Quantum groups

Lie algebras

Lie algebra $\mathfrak{g}$: Vector space/$\mathbb{C}$ with Lie bracket product $[\ ,\ ]$.

For $x, y, z \in \mathfrak{g}$,

- $[x, x] = 0$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

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(I) Finite dimensional simple Lie algebras

(II) Kac-Moody algebras

(III) Quantum groups

(IV) Quantum groups for Kac-Moody algebras

(I) Finite dimensional **simple** Lie algebra $\mathfrak{g}$

simple means $\mathfrak{g}$ has no ideal other than $\{0\}$ and $\mathfrak{g}$.

Example) $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{C}, a + d = 0 \right\}$.

$$[x, y] = xy - yx, \qquad x, y \in \mathfrak{g} \Rightarrow [x, y] \in \mathfrak{g}.$$

Putting

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

these are generators of $\mathfrak{g}$. It holds

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Theorem (Serre(1966))

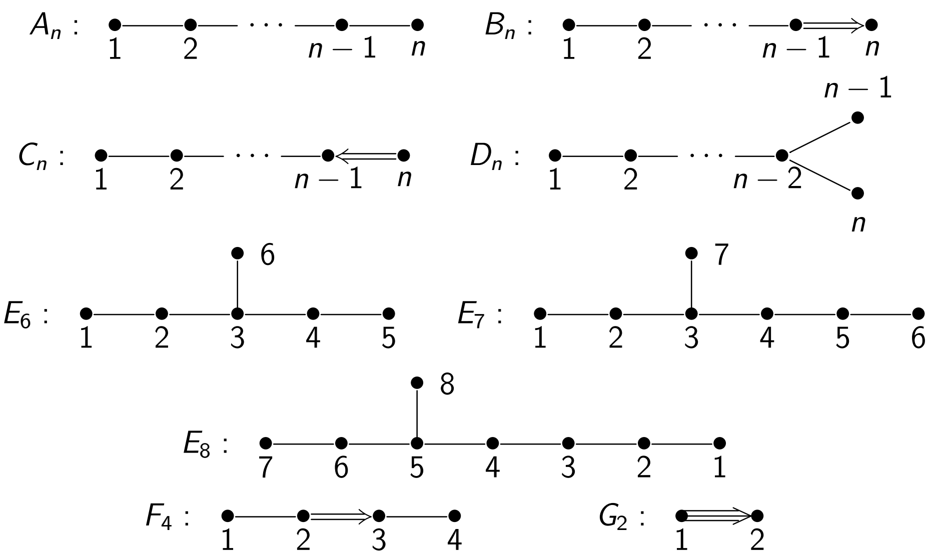
$\mathfrak{g}$: finite dimensional simple Lie algebra. Then $\mathfrak{g}$ has generators e_i, f_i and h_i ($i \in I = \{1, 2, \cdots, n\}$) s.t.

1. $[h_i, h_j] = 0$,
2. $[e_i, f_j] = \delta_{i,j} h_i$,
3. $[h_i, e_j] = a_{ij} e_j$,
4. $[h_i, f_j] = -a_{ij} f_j$,
5. $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0 \quad$ for $i \neq j$,
6. $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0 \quad$ for $i \neq j$.

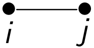
Here, $(\mathrm{ad}\ x)y := [x, y]$. $A = (a_{ij})_{i,j \in I}$ is called a **Cartan matrix** and

- $a_{ii} = 2$ ($i \in I$), $a_{ij} \leq 0$ if $i \neq j$,
- if $a_{ij} = 0$ then $a_{ji} = 0$,
- (a_{ij}) is symmetrizable (i.e., $\exists D$: diagonal matrix s.t. DA is symmetric) and DA is positive definite.

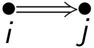
(a_{ij}) is classified by Dynkin diagrams:



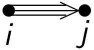
Here,



implies $a_{ij} = a_{ji} = -1$,



implies $a_{ij} = -1$, $a_{ji} = -2$,



implies $a_{ij} = -1$, $a_{ji} = -3$. If i and j are not connected then $a_{ij} = a_{ji} = 0$.

The **structure** and **classification** of cpx. finite dimensional simple Lie algebras are well known.

$\rightsquigarrow$The theory of fin. dim. simple Lie algebras are **quite successful**.

Considering natural **variations** of fin.dim.simple Lie algebras, it can be expected to obtain some new interesting theories.

Variations

- Removing the positive definite condition for $(a_{i,j}) \to$ **Kac-Moody algebra**.

  Recall : When $\mathfrak{g}$ is fin. dim. simple,

  - $a_{ii} = 2$ $(i \in I)$, $a_{ij} \leq 0$ if $i \neq j$,
  - if $a_{ij} = 0$ then $a_{ji} = 0$,
  - (a_{ij}) is symmetrizable (i.e., $\exists D$: diagonal matrix s.t. DA is symmetric) and DA is positive definite.
- Quantization $\to$ **quantum group**.
- Both $\to$ **quantum group** for KM algebra.

(II) Kac-Moody algebras

Let $A = (a_{ij})_{i,j \in I}$ be a matrix s.t.

- $a_{ii} = 2$ $(i \in I)$, $a_{ij} \leq 0$ if $i \neq j$,
- if $a_{ij} = 0$ then $a_{ji} = 0$,
- (a_{ij}) is symmetrizable.

We need the following:

- Let $\mathfrak{h}$ be a $2|I| - \mathrm{rank}(A)$-dimensional vector space/$\mathbb{C}$ and we assume that $\{h_i\}_{i \in I} \cup \{d_j\}_{1 \leq j \leq |I| - \mathrm{rank}(A)}$ is a base of $\mathfrak{h}$.
- We take $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ s.t. $\alpha_j(h_i) = a_{ij}$, $\alpha_j(d_k) \in \{0, 1\}$.

Definition

Let $\mathfrak{g}$ be a Lie algebra with generators e_i, f_i ($i \in I$) and $\mathfrak{h}$ s.t. for $h, h' \in \mathfrak{h} \subset \mathfrak{g}$,

1. $[h, h'] = 0$,
2. $[e_i, f_j] = \delta_{i,j} h_i$,
3. $[h, e_j] = \alpha_j(h) e_j$,
4. $[h, f_j] = -\alpha_j(h) f_j$,
5. $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0 \quad$ for $i \neq j$,
6. $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0 \quad$ for $i \neq j$.

$\mathfrak{g}$ is called a symmetrizable **Kac-Moody algebra**.

Unlike fin.dim.simple Lie alg,

- Kac-Moody algebras are infinite dimensional.
- All non-trivial representations are infinite dimensional.

In particular, when DA is nonnegative-definite and not positive definite, $\mathfrak{g}$ is called an **affine Lie algebras**.

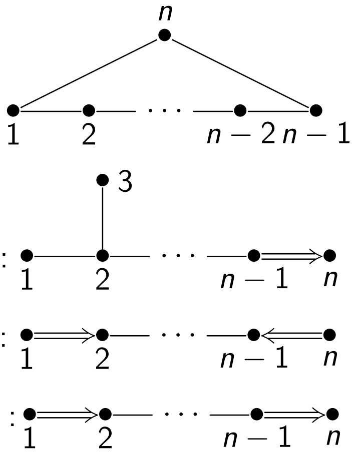
Nonnegative-definite (a_{ij}) is classified by affine Dynkin diagrams:

$A^{(1)}_{n-1}$ $(n \geq 3)$:

$B^{(1)}_{n-1}$ $(n \geq 3)$:

$C^{(1)}_{n-1}$ $(n \geq 3)$:

$A^{(2)}_{2n-2}$ $(n \geq 3)$:



and $D^{(2)}_n$, $D^{(1)}_{n-1}$, $A^{(2)}_{2n-3}$, $D^{(3)}_4$, $E^{(1)}_6$, $E^{(1)}_7$, $E^{(1)}_8$, $F^{(1)}_4$, $G^{(1)}_2$ and $E^{(2)}_6$.

(III)Quantum groups (Drinfeld,Jimbo(1985)) (='similar' algebra to $\mathfrak{g}$)

- q : indeterminant,

$[r]_q := \frac{q^r - q^{-r}}{q - q^{-1}}$ for $r \in \mathbb{Z}_{\geq 0}$, $[r]_q! := [r]_q[r-1]_q \cdots [1]_q$,

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}.$$

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra.

Let $U_q(\mathfrak{g})$ be $\mathbb{C}(q)$-algebra with unit 1 with generators e_i, f_i and q^h ($h \in \oplus_{i\in I}\mathbb{Z}h_i$) satisfying

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \ (i \neq j)$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \ (i \neq j)$.

Here, $D = \mathrm{diag}(d_1, \cdots, d_n)$. $U_q(\mathfrak{g})$ is called a **quantum group** associated with $\mathfrak{g}$.

When '$q \to 1$', we get relations of (universal env. alg. of) $\mathfrak{g}$.

(IV) quantum groups for Kac-Moody algebras

We take $A = (a_{ij})$, vector space $\mathfrak{h}$ and $\alpha_i \in \mathfrak{h}^*$ just as in the definition of Kac-Moody algebras :

Let $(a_{ij})_{i,j \in I}$ be a matrix s.t.

- $a_{ii} = 2$ $(i \in I)$, $a_{ij} \leq 0$ if $i \neq j$,
- if $a_{ij} = 0$ then $a_{ji} = 0$,
- A is symmetrizable.

Let $\mathfrak{h}$ be a $2|I| - \mathrm{rank}(A)$-dimensional vector space$/\mathbb{C}$ with base $\{h_i\}_{i \in I} \cup \{d_j\}_{1 \leq j \leq |I| - \mathrm{rank}(A)}$.

We take $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ s.t. $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d_k) \in \{0, 1\}$.

We can define $U_q(\mathfrak{g})$ by the same way as in the case $\mathfrak{g}$ is fin. dim. simple:

Let $U_q(\mathfrak{g})$ be $\mathbb{C}(q)$-algebra with unit 1 and with generators e_i, f_i and q^h ($h \in \oplus_{i\in I}\mathbb{Z}h_i \oplus \bigoplus_j \mathbb{Z}d_j$) satisfying

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ $(i \neq j)$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ $(i \neq j)$.

This $U_q(\mathfrak{g})$ is called **quantum group** associated with KM-algebra $\mathfrak{g}$.

2. Crystal bases and polyhedral realizations

Representations of $U_q(\mathfrak{g}) = \langle e_i, f_i, q^h \rangle$ for a KM-algebra $\mathfrak{g}$

$$P := \{\lambda \in \mathfrak{h}^* | \lambda(h_i), \lambda(d_k) \in \mathbb{Z}\} : \text{weight lattice}$$
$$P^+ := \{\lambda \in \mathfrak{h}^* | \lambda(h_i), \lambda(d_k) \in \mathbb{Z}_{\geq 0}\}.$$

For $\lambda \in P^+$, there exists a representation $V(\lambda)$ s.t. $\exists v_\lambda \in V(\lambda)$ and

$$e_i v_\lambda = 0 \ (\forall i \in I), \quad q^h v_\lambda = q^{\lambda(h)} v_\lambda \ (\forall h \in \oplus_{i \in I} \mathbb{Z} h_i \oplus \bigoplus_k \mathbb{Z} d_k),$$

$$V(\lambda) = \langle f_{j_1} \cdots f_{j_r} v_\lambda | j_1, \cdots, j_r \in I \rangle_{\mathbb{C}(q)-\text{vect.sp.}}.$$

- $V(\lambda)$ is an analog of finite dimensional irreducible representation $L(\lambda)$ of fin.dim.simple Lie algebra.
  c.f.) $\exists \ell_\lambda \in L(\lambda)$ and

$$e_i \ell_\lambda = 0 \ (\forall i \in I), \quad h \ell_\lambda = \lambda(h) \ell_\lambda \ (\forall h \in \oplus_{i \in I} \mathbb{Z} h_i),$$

$$L(\lambda) = \langle f_{j_1} \cdots f_{j_r} \ell_\lambda | j_1, \cdots, j_r \in I \rangle_{\mathbb{C}-\text{vect.sp.}}.$$

Crystal base $B(\lambda)$ (Properties)

- $B(\lambda)$ is a set with maps

$$\tilde{e}_i, \tilde{f}_i : B(\lambda) \to B(\lambda) \sqcup \{0\},$$

which are 'combinatorial analogs' of e_i, $f_i : V(\lambda) \to V(\lambda)$.

- $\exists\ \overline{v}_\lambda \in B(\lambda)$ s.t.

$$B(\lambda) = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_k} \overline{v}_\lambda | k \in \mathbb{Z}_{\geq 0},\ j_1, \cdots, j_k \in I\} \setminus \{0\}.$$

It holds $\sharp B(\lambda) = \dim V(\lambda)$.

- There is a map $\mathrm{wt} : B(\lambda) \to P$ s.t.

$$\mathrm{wt}(\tilde{f}_{j_1} \cdots \tilde{f}_{j_k} \cdot \overline{v}_\lambda)(h_i) = \text{Eigen value of } f_{j_1} \cdots f_{j_k} \cdot v_\lambda \in V(\lambda) \text{ for } q^{h_i}.$$

Let $U_q^-(\mathfrak{g}) := \langle f_i | i \in I \rangle \subset U_q(\mathfrak{g})$.

- $q^h \curvearrowright U_q^-(\mathfrak{g})$ by $q^h x q^{-h}$ for $x \in U_q^-(\mathfrak{g})$.
- $f_i \curvearrowright U_q^-(\mathfrak{g})$ by the multiplication from the left.
- $\exists\ e_i' \curvearrowright U_q^-(\mathfrak{g})$, which is a modification of the multiplication of e_i

Crystal base $B(\infty)$ for $U_q^-(\mathfrak{g})$ (Properties)

- $B(\infty)$ is a set with maps

$$\tilde{e}_i, \tilde{f}_i : B(\infty) \to B(\infty) \sqcup \{0\},$$

which are 'combinatorial analogs' of e_i', $f_i : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$.

- $\exists\ \overline{v}_\infty \in B(\infty)$ s.t.

$$B(\infty) = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_k} \overline{v}_\infty | k \in \mathbb{Z}_{\geq 0},\ j_1, \cdots, j_k \in I\}.$$

- There is a map $\mathrm{wt} : B(\infty) \to P$ s.t.

$\mathrm{wt}(\tilde{f}_{j_1} \cdots \tilde{f}_{j_k} \cdot \overline{v}_\infty)(h_i) =$ Eigen value of $f_{j_1} \cdots f_{j_k} \cdot v_\infty \in U_q^-(\mathfrak{g})$ for q^{h_i}.

- For any $\lambda \in P^+$, there is a map $B(\infty) \twoheadrightarrow B(\lambda)$, which is an analog of $U_q^-(\mathfrak{g}) \twoheadrightarrow V(\lambda)$, $f_{j_1} \cdots f_{j_r} \mapsto f_{j_1} \cdots f_{j_r} v_\lambda$.

By studying $B(\lambda)$, we know eigenvalues of $V(\lambda)$, $\dim V(\lambda)$ and structures of tensor products and so on. $B(\infty)$ tells us a structure of $U_q^-(\mathfrak{g})$.

To study $B(\infty)$, $B(\lambda)$, **combinatorial descriptions** are useful. They have a bunch of combinatorial descriptions.

Today, we consider the **polyhedral realization** that is a description of $B(\infty)$ and **monomial realization** that is a description of $B(\lambda)$.

Polyhedral realizations

Let $\iota = (\cdots, i_2, i_1)$ be a sequence from $I = \{1, 2, \cdots, n\}$.

$$\mathbb{Z}_\iota^\infty := \{(\cdots, a_2, a_1) | a_j \in \mathbb{Z},\ a_k = 0\ (k \gg 0)\}.$$

One can define maps denoted by $\tilde{f}_i$, $\tilde{e}_i : \mathbb{Z}_\iota^\infty \to \mathbb{Z}_\iota^\infty \sqcup \{0\}$ and $\mathrm{wt} : \mathbb{Z}_\iota^\infty \to P$ as follows:

For $r \in \mathbb{Z}_{\geq 1}$ and $\mathbf{a} = (\cdots, a_2, a_1) \in \mathbb{Z}_\iota^\infty$,

$$\sigma_r(\mathbf{a}) := a_r + \sum_{j>r} a_{i_r, i_j} a_j \quad (r \in \mathbb{Z}_{\geq 1}),$$

$$\sigma^{(k)} := \max_{r; i_r = k}\{\sigma_r(\mathbf{a})\},$$

$$M^{(k)}(\mathbf{a}) := \{r \in \mathbb{Z}_{\geq 1} | i_r = k,\ \sigma_r(\mathbf{a}) = \sigma^{(k)}(\mathbf{a})\}.$$

$\mathrm{wt} : \mathbb{Z}^\infty \to P$ is defined by

$$\mathrm{wt}(\mathbf{a}) := -\sum_{j=1}^{\infty} a_j \alpha_{i_j}.$$

One can define

$$\tilde{e}_i, \tilde{f}_i : \mathbb{Z}_\iota^\infty \to \mathbb{Z}_\iota^\infty \sqcup \{0\}$$

as

$$(\tilde{f}_k(\mathbf{a}))_r := a_r + \delta_{r,\min M^{(k)}(\mathbf{a})},$$

$$(\tilde{e}_k(\mathbf{a}))_r := a_r - \delta_{r,\max M^{(k)}(\mathbf{a})} \text{ if } \sigma^{(k)}(\mathbf{a}) > 0 \text{ ; o.w. } \tilde{e}_k(\mathbf{a}) = 0$$

for $\mathbf{a} = (\cdots, a_j, \cdots, a_2, a_1) \in \mathbb{Z}_\iota^\infty$.

Essential point : Above $\tilde{e}_i$, $\tilde{f}_i$ and wt are defined by only sum of integers (in particular, **without** representation theory). Everybody can compute them by following the above rule.

Theorem (Nakashima-Zelevinsky)

There is an injective map

$$\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}_\iota^\infty$$

s.t. $\Psi_\iota(\overline{v}_\infty) = (\cdots, 0, 0, 0)$ and Ψ_ι commutes with $\tilde{e}_i$, $\tilde{f}_i$ and preserves wt.

By this theorem, we can reduce calculations of $\underbrace{\tilde{e}_i, \tilde{f}_i, \mathrm{wt}}_{\text{important}}$ on $B(\infty)$ to

$\underbrace{\tilde{e}_i, \tilde{f}_i, \mathrm{wt}}_{\text{computable}}$ on $\mathbb{Z}_\iota^\infty$.

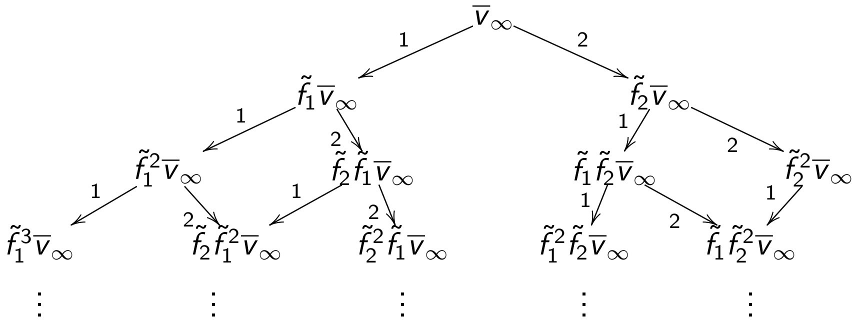
Definition

$\mathrm{Im}(\Psi_\iota)(\cong B(\infty))$ is called a **polyhedral realization** for $B(\infty)$.

Example) $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle| a + e + i = 0 \right\}$: type A_2,

$\iota = (\cdots, 2, 1, 2, 1, 2, 1)$.

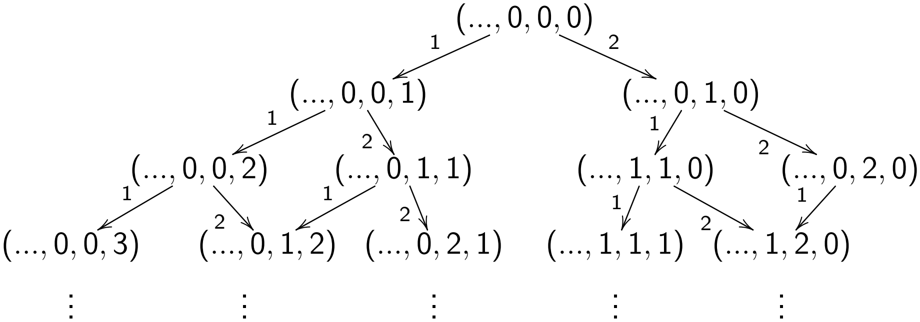
The crystal graph of $B(\infty)$ is as follows:



Example) $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle| a+e+i=0 \right\}$: type A_2,

$\iota = (\cdots, 2, 1, 2, 1, 2, 1)$.

The crystal graph of $(B(\infty) \cong)\mathrm{Im}(\Psi_\iota) \subset \mathbb{Z}^\infty$ is as follows:



$$\begin{aligned} &\mathrm{Im}(\Psi_\iota) \\ =\ &\{(\cdots, a_3, a_2, a_1) \in \mathbb{Z}^\infty \mid a_1 \geq 0,\ a_2 \geq a_3 \geq 0,\ a_k = 0\ (k > 3)\} \end{aligned}$$

Problem

We'd like to know inequalities defining polyhedral realization $\mathrm{Im}(\Psi_\iota)$ such as $a_1 \geq 0,\ a_2 \geq a_3 \geq 0,\ a_k = 0\ (k > 3)$.

Nakashima-Zelevinsky, Hoshino, Kim-Shin

- $\mathfrak{g}$: fin.dim.simple Lie alg., $\iota = (\cdots, n, \cdots, 2, 1, n, \cdots, 2, 1) \Rightarrow$ an explicit form of the inequalities of $\mathrm{Im}(\Psi_\iota)$.

Hoshino (2013)

- $\mathfrak{g}$: classical affine type, $\iota = (\cdots, n, \cdots, 1, n, \cdots, 1) \Rightarrow$ an explicit form of the inequalities of $\mathrm{Im}(\Psi_\iota)$.

Littlemann(1998)

- $\mathfrak{g}$: fin. dim. simple, ι : 'nice decomposition' $\Rightarrow$ an explicit form of **string cone** $\mathcal{S}_\iota$.
  ($\mathrm{Im}(\Psi_\iota)$ coincides with integer pts of a string cone.)

K-Nakashima (2020), K(2023)

- If $\mathfrak{g}$ is a classical Lie algebra or classical affine Lie algebra and ι is 'adapted' then the inequalities of $\mathrm{Im}(\Psi_\iota)$ are combinatorially described by column tableaux or Young walls.

K-Koshevoy-Nakashima(2024)

- If $\mathfrak{g}$ is a classical Lie algebra and arbitrary ι then we give an algorithm to compute the inequalities of $\mathrm{Im}(\Psi_\iota)$.

Goal

In this talk, we consider the case $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra and ι is **adapted** and give a conjecture that claims inequalities of $\mathrm{Im}(\Psi_\iota)$ are expressed in terms of monomial realizations.

Definition

Let $A = (a_{i,j})_{i,j \in I}$ be the symmetrizable generalized Cartan matrix of $\mathfrak{g}$ and $\iota = (\cdots, i_3, i_2, i_1)$. If the following condition holds then ι is said to be *adapted* to A: For $i, j \in I$ s.t. $a_{i,j} < 0$, the subsequence of ι consisting of all i, j is either

$$(\cdots, i, j, i, j, i, j, i, j) \quad \text{or} \quad (\cdots, j, i, j, i, j, i, j, i).$$

Example) $\mathfrak{g}$: type $A_3^{(1)}$, $\iota = (\cdots, 2, 1, 3, 2, 1, 3, 2, 1, 3)$

- subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 1, 2, 1)$
- subsequence consisting of 2, 3 : $(\cdots, 2, 3, 2, 3, 2, 3)$
- subsequence consisting of 1, 3 : $(\cdots, 1, 3, 1, 3, 1, 3)$

Thus, ι is adapted to A.

3. Monomial realizations

We consider the set of Laurent monomials as follows:

$$\mathcal{Y} := \left\{ \prod_{s\in\mathbb{Z},\ i\in I} X_{s,i}^{\zeta_{s,i}} \,\middle|\, \zeta_{s,i}\in\mathbb{Z},\ \zeta_{s,i}=0 \text{ except for finitely many } (s,i) \right\}.$$

We define maps wt, $\tilde{f}_i$, $\tilde{e}_i$ on $\mathcal{Y}$ associated with an adapted sequence ι as follows: Let $(p_{i,j})_{i,j\in I;a_{i,j}<0}$ be integers s.t.

$$p_{i,j} = \begin{cases} 1 & \text{if the subseq. of } \iota \text{ consisting of } i,j \text{ is } (\cdots,j,i,j,i,j,i), \\ 0 & \text{if the subseq. of } \iota \text{ consisting of } i,j \text{ is } (\cdots,i,j,i,j,i,j). \end{cases}$$

For $X = \prod_{s\in\mathbb{Z},\, i\in I} X_{s,i}^{\zeta_{s,i}} \in \mathcal{Y}$, one sets $\mathrm{wt}(X) := \sum_{s,i} \zeta_{s,i}\Lambda_i$

$$\varphi_i(X) := \max\left\{ \sum_{k\le s} \zeta_{k,i} \,\middle|\, s\in\mathbb{Z} \right\}, \quad \varepsilon_i(X) := \varphi_i(X) - \mathrm{wt}(X)(h_i)$$

$$A_{s,k} := X_{s,k} X_{s+1,k} \prod_{j \in I; a_{j,k} < 0} X_{s+p_{j,k},j}^{a_{j,k}} \quad (s \in \mathbb{Z},\ k \in I).$$

For $i \in I$, let us define actions of Kashiwara operators as follows:

$$\tilde{f}_i X := \begin{cases} A_{n_{f_i},i}^{-1} X & \text{if} \quad \varphi_i(X) > 0, \\ 0 & \text{if} \quad \varphi_i(X) = 0, \end{cases} \qquad \tilde{e}_i X := \begin{cases} A_{n_{e_i},i} X & \text{if} \quad \varepsilon_i(X) > 0, \\ 0 & \text{if} \quad \varepsilon_i(X) = 0, \end{cases}$$

where we set

$$n_{f_i} := \min \left\{ r \in \mathbb{Z} \,\middle|\, \varphi_i(X) = \sum_{k \le r} \zeta_{k,i} \right\},$$

$$n_{e_i} := \max \left\{ r \in \mathbb{Z} \,\middle|\, \varphi_i(X) = \sum_{k \le r} \zeta_{k,i} \right\}.$$

Theorem (Kashiwara, Nakajima)

We take $X \in \mathcal{Y}$ as $\tilde{e}_i(X) = 0$ for all $i \in I$ and put $\lambda = \mathrm{wt}(X) \in P^+$. There exists a bijection from $B(\lambda)$ to the set of monomials

$$\{\tilde{f}_{j_m} \cdots \tilde{f}_{j_1} X | m \in \mathbb{Z}_{\geq 0},\ j_1, \cdots, j_m \in I\} \setminus \{0\} \subset \mathcal{Y},$$

which is compatible with wt, $\tilde{f}_i$, $\tilde{e}_i$.

$\mathcal{M}_{s,k,\iota} := \{\tilde{f}_{j_m} \cdots \tilde{f}_{j_1} X_{s,k} | m \in \mathbb{Z}_{\geq 0},\ j_1, \cdots, j_m \in I\} \setminus \{0\} (\cong B(\Lambda_k))$ for $s \in \mathbb{Z}$, $k \in I$.

Here, $\Lambda_k \in P^+$ is defined by $\Lambda_k(h_j) = \delta_{k,j}$, $\Lambda_k(d_s) = 0$.

e.g.) $\mathfrak{g}$: A_2, $\iota = (\cdots, 2, 1, 2, 1)$. $\mathcal{M}_{s,1,\iota}(\cong B(\Lambda_1))$ is expressed as

$$X_{s,1} \xrightarrow{1} \frac{X_{s,2}}{X_{s+1,1}} \xrightarrow{2} \frac{1}{X_{s+1,2}}$$

and $\mathcal{M}_{s,2,\iota}(\cong B(\Lambda_2))$ is expressed as

$$X_{s,2} \xrightarrow{2} \frac{X_{s+1,1}}{X_{s+1,2}} \xrightarrow{1} \frac{1}{X_{s+2,1}}.$$

Tropicalizing them and considering inequalities $Trop(M) \geq 0$ for $M \in \mathcal{M}_{s,1,\iota} \cup \mathcal{M}_{s,2,\iota}$, for $(\cdots, a_{2,2}, a_{2,1}, a_{1,2}, a_{1,1}) \in \mathbb{Z}^{\infty}$ we have

$$a_{s,1} \geq 0, \ a_{s,2} - a_{s+1,1} \geq 0, \ -a_{s+1,2} \geq 0,$$

$$a_{s,2} \geq 0, \ a_{s+1,1} - a_{s+1,2} \geq 0, \ -a_{s+2,1} \geq 0 \quad (s \in \mathbb{Z}_{\geq 1}).$$

By $a_{s,1} \geq 0$, $-a_{s+2,1} \geq 0$ and $a_{s,2} \geq 0$, $-a_{s+1,2} \geq 0$, we see that $a_{m+1,1} = a_{m,2} = 0$ for all $m \geq 2$.

Simplifying other inequalities, we get

$$a_{1,1} \geq 0,\ a_{1,2} \geq a_{2,1} \geq 0,\ a_{m+1,1} = a_{m,2} = 0\ (m \in \mathbb{Z}_{\geq 2}).$$

Now we identify

$$(\cdots, a_3, a_2, a_1) = (\cdots, a_{2,2}, a_{2,1}, a_{1,2}, a_{1,1}) \in \mathbb{Z}^{\infty}$$

Recall)

$$\begin{aligned} & \mathrm{Im}(\Psi_\iota) \\ = \ & \{(\cdots, a_3, a_2, a_1) \in \mathbb{Z}^{\infty} | a_1 \geq 0,\ a_2 \geq a_3 \geq 0,\ a_k = 0\ (k > 3)\} \end{aligned}$$

4. Polyhedral realizations and Monomial realizations

Let $\iota = (\cdots, i_3, i_2, i_1)$. For $(\cdots, a_3, a_2, a_1) \in \mathbb{Z}^{\infty}_{\iota}$, we rewrite the variable a_k as

$$a_k = a_{s,j}$$

if $i_k = j$ and j is appearing s times in $i_1, i_2, \cdots, i_k$.

Conjecture

Let $\mathfrak{g}$ be a symmetrizable KM alg. and ι be an adapted sequence and $\mathcal{M}_{s,k,\iota}$ be the set of monomials for $\mathfrak{g}^L$. Here $\mathfrak{g}^L$ is a KM alg. whose generalized Cartan matrix is transposed matrix of that of $\mathfrak{g}$. Then

$$\mathrm{Im}(\Psi_\iota) = \left\{ \mathbf{a} \in \mathbb{Z}^{\infty}_{\iota} \middle| \varphi(\mathbf{a}) \geq 0 \text{ for all } \varphi \in \bigcup_{s \in \mathbb{Z}_{\geq 1},\ k \in I} Trop(\mathcal{M}_{s,k,\iota}) \right\}$$

If it is true, inequalities characterising the **polyhedral realization** are expressed in terms of **monomial realizations**.

Theorem

(1) When $\mathfrak{g}$ is a finite dimensional simple Lie algebra of type A_n, B_n, C_n or D_n, the conjecture is true.

(2) When $\mathfrak{g}$ is a Kac-Moody algebra of rank 2, the conjecture is true.

(3) When $\mathfrak{g}$ is a classical affine Lie algebra of type $A^{(1)}_{n-1}$, $B^{(1)}_{n-1}$, $C^{(1)}_{n-1}$, $D^{(1)}_{n-1}$, $A^{(2)}_{2n-2}$, $A^{(2)}_{2n-3}$ or $D^{(2)}_{n}$, the conjecture is true.

Example) $\mathfrak{g}$: type C_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$. Then $\mathcal{M}_{s,1,\iota}$ is

$$X_{s,1} \xrightarrow{1} \frac{X_{s,2}}{X_{s+1,1}} \xrightarrow{2} \frac{X_{s+1,1}}{X_{s+1,2}} \xrightarrow{1} \frac{1}{X_{s+2,1}}$$

and $\mathcal{M}_{s,2,\iota}$ is

$$X_{s,2} \xrightarrow{2} \frac{X_{s+1,1}^2}{X_{s+1,2}} \xrightarrow{1} \frac{X_{s+1,1}}{X_{s+2,1}} \xrightarrow{1} \frac{X_{s+1,2}}{X_{s+2,1}^2} \xrightarrow{2} \frac{1}{X_{s+2,2}}.$$

Tropicalizing them, we get

$$a_{s,1} \geq 0,\ a_{s,2} - a_{s+1,1} \geq 0,\ a_{s+1,1} - a_{s+1,2} \geq 0,\ -a_{s+2,1} \geq 0,$$

$$a_{s,2} \geq 0,\ 2a_{s+1,1} - a_{s+1,2} \geq 0,\ a_{s+1,1} - a_{s+2,1} \geq 0,$$

$$a_{s+1,2} - 2a_{s+2,1} \geq 0,\ -a_{s+2,2} \geq 0.$$

Simplifying the inequalities, $\mathrm{Im}(\Psi_\iota)$ for type B_2 is as follows:

$\mathrm{Im}(\Psi_\iota) =$

$\{(a_{m,j}) \in \mathbb{Z}^\infty | a_{1,2} \geq a_{2,1} \geq a_{2,2} \geq 0,\ a_{1,1} \geq 0,\ a_{m,1} = a_{m,2} = 0\ (m \geq 3)\}$