New identities for diff.-polynomial structures built of Jacobians.

Arthemy Kiselev (Groningen, NL), joint work: M. Jagoe Brown, F. Schipper; software by R. Buring (INRIA, France).

Def. Nambu-determinant Poisson bracket on $\mathbb{R}^{d\geq 2} \ni \underline{x} = (x^1, \ldots, x^d)$

$$\{f,g\}_d(\underline{x}) = \varrho(\underline{x}) \cdot \det\left(\frac{\partial(f,g,a_1,\ldots,a_{d-2})}{\partial(x^1,\ldots,x^d)}\right),$$

where $f, g, a_1, \ldots, a_{d-2} \in C^\infty(\mathbb{R}^d)$, $\varrho(\underline{x})\partial_{x^1}\wedge\ldots\wedge\partial_{x^d} \in \mathfrak{X}^d(\mathbb{R}^d)$.

Ex. $\{f,g\}_2(x,y) = \varrho(x,y)\cdot \begin{vmatrix} f_x & g_x \\ f_y & g_y \end{vmatrix}$; $\{f,g\}_3 = \varrho \cdot \begin{vmatrix} f_x & g_x & a_x \\ f_y & g_y & a_y \\ f_z & g_z & a_z \end{vmatrix}$

Ex. $a = \frac{1}{2}(x^2+y^2+z^2)$: $\{x,y\}_3 = z$, etc.

Refs: Kontsevich graphs act on Nambu–Poisson brackets,

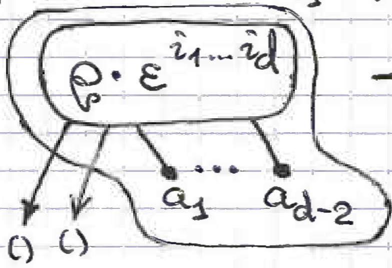
I. New identities for Jacobian determinants;
II. The tetrahedral flow is a coboundary in 4D;
III. Uniqueness aspects. (& refs therein)

← [AVK, MJB, FS]

§1 Mystery: embedding of identities, $d \hookrightarrow d+1$.

$\{f,g\}_d = \varrho \cdot \varepsilon^{i_1 \ldots i_d} \cdot \partial_{i_1}(f)\cdot\partial_{i_2}(g)\cdot\partial_{i_3}(a_1)\cdot\ldots\cdot\partial_{i_d}(a_{d-2})$;

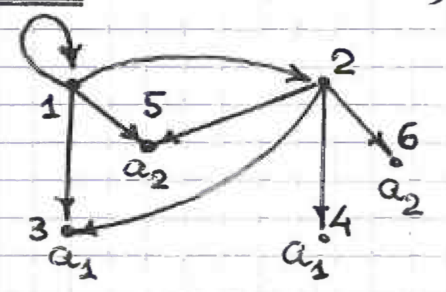
$P_d(\varrho, [\underline{a}]) =$



← **Def.** Nambu micro-graph: built from whole P_d.

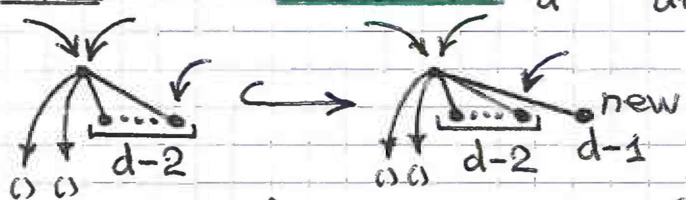
Def. $\xrightarrow{\ i\ } = \sum_{i=1}^d \partial/\partial x^i$

Ex. $\mathcal{H}am = [1,2,3,5;\ 3,4,5,6]$



$= \emptyset$ as f-la.

Def. Graph embedding $\Gamma_d \hookrightarrow \hat{\Gamma}_{d+1}$



Rem. $\text{F-la}(\hat{\Gamma}_{d+1}) = \text{F-la}(\Gamma_d)\cdot\left(\partial(a_{new})/\partial(x^{new})\right)^{power} + \langle\text{cross-terms}\rangle$.

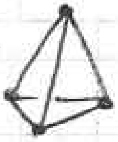
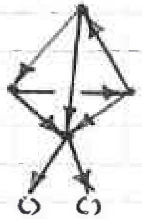
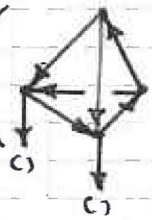
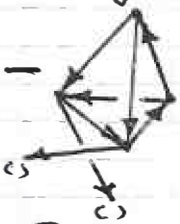
?! $\left(\text{Valid lin. relation}\big|_{\dim=d}\right) \hookrightarrow \left(\text{Still valid lin. relation}\big|_{\dim=d+1}\right)$

§2 MYSTERY: GRAPH COCYCLES ACT ON ϱ AND a IN NAMBU.

?: DEFORM POISSON $\mid P \mapsto P + \varepsilon Q(P) + \bar{o}(\varepsilon)$ STAYS POISSON:

$\mathrm{JAC}(P + \varepsilon Q + \bar{o}(\varepsilon)) = \mathrm{JAC}(P)\cdot\varepsilon^0 + [\text{we require } 0]\cdot\varepsilon^1 + \bar{o}(\varepsilon) = \bar{o}(\varepsilon)$.

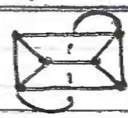
TH. (M.K., 1996). $\exists$ CLASS $\{Q = Q^{\gamma}_{\forall d}([\![P]\!]) \mid$ GOOD GRAPHS $\gamma\}$.

Ex. $\gamma_3 =$  $\xrightarrow{\mathrm{Or}}$ $Q^{\gamma}(P) =$  $+ 3\cdot\Big($  $-$  $\Big)$

- $Q^{\gamma} = \vec{\mathrm{Or}}(\gamma)(P\otimes\cdots\otimes P)$.

DEF. $\frac{d}{d\varepsilon}(P) = Q(P)$ RESTRICTS TO NAMBU $\{P(\varrho, [\underline{a}])\}$ (if)

$\exists \frac{d}{d\varepsilon}(\varrho)$, $\exists \frac{d}{d\varepsilon}(a_i) \mid Q(P(\varrho,[\underline{a}])) \overset{\checkmark}{=} \frac{d}{d\varepsilon}(P(\varrho,[\underline{a}])) \leftarrow$ LEIBNIZ.

Ex. γ_3 RESTRICTS OVER $d=3,4$; $\gamma_5 =$  $+ \frac{5}{2}$  OVER $d=3$.

?! F-las $\dot{\varrho}, \dot{a}_i$ EXPLICITLY FROM γ. $\} \longleftarrow$ [M.K., 2019] AVK

$\frac{d}{dt}(a_i) := \vec{\mathrm{Or}}(\gamma)(a_i\otimes P\otimes\cdots\otimes P) + \cdots + \vec{\mathrm{Or}}(\gamma)(P\otimes\cdots\otimes P\otimes a_i)$;

$$\frac{d}{dt}(\varrho) := \frac{\Big(Q^{\gamma}_d(P) - \Sigma_i P\big(\varrho, [a_1], \ldots, [\tfrac{d}{dt}(a_i)], \ldots, [a_{d-2}]\big)\Big)(f,g)}{\det\big(\partial(f,g,a_1,\ldots,a_{d-2})/\partial(x^1,\ldots,x^d)\big)};$$

is DIFF.-POLYNOMIAL IN ϱ, a_i; $] \longleftarrow$ "CROSS-TERMS CANCEL OUT."

& $\frac{d}{dt}(P(\varrho,[\underline{a}])) \overset{\checkmark}{=} Q^{\gamma}_d(P) \overset{\mathrm{def}}{=} \frac{d}{d\varepsilon}(P(\varrho,[\underline{a}]))$.

§3 MYSTERY: EACH NAMBU-POISSON $P(\varrho,[\underline{a}])$ STABLE W.R.T. γ.

?! $P_d + \varepsilon Q^{\gamma}_d + \bar{o}(\varepsilon)$ DOES NOT LEAVE THE CLASS $\{P \bmod [\![P, \vec{X}]\!]\}$

$\Longleftrightarrow Q^{\gamma}_d \overset{\checkmark}{=} [\![P, \vec{X}^{\gamma}_d]\!]$. $\} \leftarrow$ change $\underline{x}$ along $\vec{X}$.

& $\dot{a}_i = (-\vec{X}^{\gamma}_d)(a_i) \overset{\checkmark}{=} \frac{d}{dt}(a_i)$; $\dot{\varrho}\,\partial_{\underline{x}} = [\![\varrho\,\partial_{\underline{x}}, \vec{X}]\!] \overset{\checkmark}{=} \frac{d}{dt}(\varrho)\cdot\partial_{\underline{x}}$,

$\vec{X} = \vec{X}^{\gamma}_d + [\![P, \text{Hamiltonian}]\!]$.

REM. $P(\varrho, \ldots(-a_i)\ldots) = \ominus P(\varrho, \ldots a_i \ldots)$;

$P(\varrho, \sigma(\underline{a})) = (-)^{\sigma} P(\varrho, \underline{a})$ but $Q^{\gamma_j}_d(-a_i) = \oplus Q^{\gamma_j}_d(a_i)$

$\overset{?}{\longrightarrow} \dot{P} = Q^{\gamma}(P)$ EITHER LEAVES NAMBU CLASS OR FIXES EACH $P(\varrho,[\underline{a}])$.