Crystal Structure of Localized Quantum Unipotent Coordinate Category

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References

[KKOP] Kashiwara M., Kim M., Oh S-J. and Park E., Localization for quiver Hecke algebras, Pure Appl.Math. Q. 17(2021), no.4, 1465–1548.

[KKOP2] ——, Localization for quiver Hecke algebras II, Proc. Lond. Math. Soc. (3) **127** (2023), no. 4, 1134–1184. (arXiv:2208.01255.)

[KKOP3] ——, Localization for quiver Hecke algebras III, arXiv:2308.09243.

[KKOP4] ——, Affinizations , R-matices and Reflection fuctors, arXiv:2304.00238.

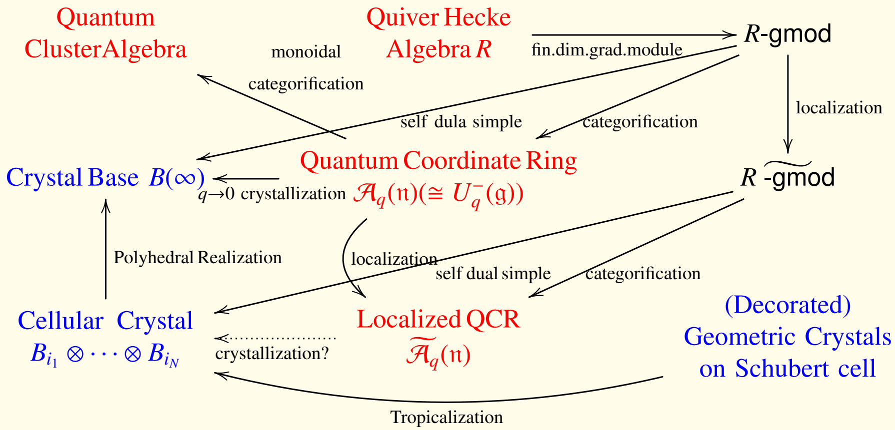
[Kanakubo-N] Kanakubo Y. and Nakashima T., Half potential on geometric crystals and connectedness of cellular crystals, Transformation Groups **28** (2023), no.1, 327–373.

[Lauda-Vazirani] Lauda A.D. and Vazirani M., Crystals from categorified quantum groups, Adv.Math., **228**, (2011), 803–861.

[N] Nakashima T.,Categorified Crystal Structure on Localized Quantum Coordinate Ring, arXiv:2208.08396

[Kashiwara-N] in preparation

Introduction



Preliminaries

- $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_- = \langle e_i, h_i, f_i \rangle$: Simple Lie algebra ($\to$ Kac-Moody),
$A = (a_{ij})_{i,j \in I:=\{1,2,\cdots,n\}}$: Cartan matrix for $\mathfrak{g}$
- $\{\alpha_i : i \in I\}$: set of simple roots, $\{h_i : i \in I\}$: set of simple coroots such that $a_{ij} = \alpha_j(h_i)$. Define the root lattice $Q := \bigoplus_i \mathbb{Z}\alpha_i \supset Q_+ := \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k} \in Q_+$, define the height of β by $|\beta| = k$.
- (,) : symm.bilinear form on $\mathfrak{t}^*$ s.t. $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\lambda(h_i) = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{t}^*$. (We shall use the notation $\langle h_i, \lambda \rangle$ for $\lambda(h_i)$.)
- $P := \{\lambda \in \mathfrak{t}^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z} (\forall i \in I)\}$: weight lattice$\supset P_+$: dominant weights
- $P^* := \{h \in \mathfrak{t} \mid \langle h, P \rangle \subset \mathbb{Z}\}$: dual weight lattice
- $W = \langle s_i \mid i \in I \rangle$: Weyl group ass. P.
- $U_q(\mathfrak{g}) := \langle e_i,\ f_i,\ q^h \rangle_{i \in I, h \in P^*}$: quantum algebra/$\mathbb{Q}(q)$
- $U_q^-(\mathfrak{g}) := \langle f_i \rangle_{i \in I}$, $U_q^+(\mathfrak{g}) := \langle e_i \rangle_{i \in I}$: nilpotent subalgebras

Quantum coordinate ring

(Unipotent) quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ is defined as a restricted dual of $U_q^+(\mathfrak{g})$:

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta \in Q_-} \mathcal{A}_q(\mathfrak{n})_\beta \qquad \mathcal{A}_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q))$$

Note that we get the isomorphism of $\mathbb{Q}(q)$-algebras

$$U_q^-(\mathfrak{g}) \quad \xrightarrow{\sim} \quad \mathcal{A}_q(\mathfrak{n}) \qquad (f_i \longmapsto F_i^*).$$

The $\mathbb{Z}$-form $\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ is defined by:

$$\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]} := \{a \in \mathcal{A}(\mathfrak{n}) \mid \langle a, U^+_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})\rangle \subset \mathbb{Z}[q,q^{-1}]\}.$$

Crystal Base I

Definition

Let $A \subset \mathbb{Q}(q)$ be the subring regular at $q = 0$. A pair (L, B) is a crystal base of $M \in \mathcal{O}_{\mathrm{int}}(\mathfrak{g})$(resp. $U_q^-(\mathfrak{g})$), if it satisfies:

1. L *is a free* A*-submodule of* M *(resp.* $U_q^-(\mathfrak{g})$*) such that*

$$M \cong \mathbb{Q}(q) \otimes_A L \quad \text{(resp. } U_q^-(\mathfrak{g}) \cong \mathbb{Q}(q) \otimes_A L\text{)}$$
$$L = \oplus_\lambda L_\lambda \ (L_\lambda := L \cap M_\lambda).$$

2. B *is a basis of the* $\mathbb{Q}$*-vector space* L/qL *and*

$$B = \sqcup_\lambda B_\lambda \ (B_\lambda := B \cap L_\lambda / qL_\lambda).$$

3. $\tilde{e}_i L \subset L$ *and* $\tilde{f}_i L \subset L$. *(*$\tilde{e}_i, \tilde{f}_i \in \mathrm{End}_{\mathbb{Q}(q)}(M)$ *Kashiwara operator)*
4. $\tilde{e}_i B \subset B \sqcup \{0\}$ *and* $\tilde{f}_i B \subset B \sqcup \{0\}$.
5. *For* $u, v \in B$, $\tilde{f}_i u = v \Longleftrightarrow \tilde{e}_i v = u$.

Crystal Base II

By ④ of the definition, B holds a colored oriented graph structure, called crystal graph:

Definition

The crystal graph of a crystal B is a colored oriented graph given by the rule:

$$b_1 \xrightarrow{i} b_2 \quad \Longleftrightarrow \quad b_2 = \tilde{f}_i b_1 \qquad (b_1, b_2 \in B).$$

Let $V(\lambda)$ (resp. $U_q^-(\mathfrak{g})$) be the integrable simple h.w.module (resp. nilp. negative subalg. of $U_q(\mathfrak{g})$) with the h.w.v u_λ $(\lambda \in P_+)$ (resp. $1 := u_\infty$). Define

$$L(\lambda) := \sum_{i_j \in I, l \geq 0} A\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\lambda, \quad B(\lambda) := \{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\lambda \bmod qL(\lambda) \,|\, i_j \in I, l \geq 0\} \setminus \{0\},$$

$$L(\infty) := \sum_{i_j \in I, l \geq 0} A\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty, \quad B(\infty) := \{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty \bmod qL(\infty) \,|\, i_j \in I, l \geq 0\} \setminus \{0\}.$$

Theorem (Kashiwara)

The pair $(L(\lambda), B(\lambda))$ (resp. $(L(\infty), B(\infty))$) is a crystal base of $V(\lambda)$. (resp. $U_q^-(\mathfrak{g})$).

Tensor product of Crystal Bases I

Tensor product of crystal bases is one of the most beautiful and useful results.

Theorem

Let (L_j, B_j) *be a crystal base of finite dimensional* $U_q(\mathfrak{g})$*-module* M_j $(j = 1, 2)$*. Set* $\underline{L = L_1 \otimes_A L_2}$ *and* $\underline{B = \{b_1 \otimes b_2;\ b_j \in B_j\ (j = 1, 2)\} \subset L/qL}$*. Then we have*

1. (L, B) *is a crystal base of* $M_1 \otimes M_2$*.*
2.

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \le \varepsilon_i(b_2), \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\ \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \ge \varepsilon_i(b_2). \end{cases}$$

where $\boxed{\varepsilon_i(b) = \max\{k \ge 0;\ \tilde{e}_i^k b \ne 0\}}$ $\boxed{\varphi_i(b) = \max\{k \ge 0;\ \tilde{f}_i^k b \ne 0\}}$

$$\underbrace{\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow}_{\varepsilon_i(u)} \underset{u}{\bullet} \underbrace{\longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet}_{\varphi_i(u)}$$

Crystals

"Crystal" is a combinatorial object abstracting the properties of crystal bases.

Definition (Crystal)

A 6-tuple $(B, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i\in I}$ *is a crystal if* B *is a set and* $\exists 0 \notin B$ *and maps:*

$$\mathrm{wt} : B \to P, \quad \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \quad (i \in I) \tag{1}$$

$$\tilde{e}_i : B \sqcup \{0\} \to B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \ (i \in I), \tag{2}$$

satisfying :

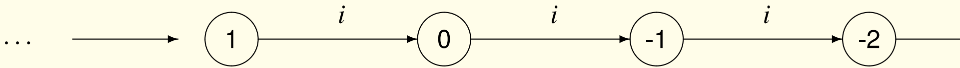
1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$.
2. *If* $b, \tilde{e}_i b \in B$, *then* $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
3. *If* $b, \tilde{f}_i b \in B$, *then* $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.
4. *For* $b, b' \in B$ *and* $i \in I$, *one has* $\tilde{f}_i b = b' \iff b = \tilde{e}_i b'$.
5. *If* $\varphi_i(b) = -\infty$ *for* $b \in B$, *then* $\tilde{e}_i b = \tilde{f}_i b = 0$ *and* $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Example

For $i \in I$, set $B_i := \{(n)_i \mid n \in \mathbb{Z}\}$ and its crystal structure is given by

$$\mathrm{wt}((n)_i) = n\alpha_i,\ \ \varepsilon_i((n)_i) = -n,\ \ \varphi_i((n)_i) = n,$$
$$\varepsilon_j((n)_i) = \varphi_j((n)_i) = -\infty\ \ (i \neq j),$$
$$\tilde{e}_i((n)_i) = (n+1)_i,\quad \tilde{f}_i((n)_i) = (n-1)_i,$$
$$\tilde{e}_j((n)_i) = \tilde{f}_j((n)_i) = 0\ \ (i \neq j)$$

Crystal graph of B_i:



Fix a sequence of indices $\mathbf{i} = (i_1, \cdots, i_m) \in I^m$ and write

$$(x_1, \cdots, x_m) := \tilde{f}_{i_1}^{x_1}(0)_{i_1} \otimes \cdots \otimes \tilde{f}_{i_m}^{x_m}(0)_{i_m} = (-x_1)_{i_1} \otimes \cdots \otimes (-x_m)_{i_m},$$

where if $n < 0$, then $\tilde{f}_i^n(0)_i$ means $\tilde{e}_i^{-n}(0)_i$.
The crystal structure on $B_{i_1} \otimes \cdots \otimes B_{i_m}$ is given by: Identifying $x = (x_1, \cdots, x_m)$ with an element in $\mathbb{Z}^m$, define

$$\sigma_k(x) := x_k + \sum_{j<k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j \quad (k \in [1, m]),$$

$$\widetilde{\sigma}^{(i)}(x) := \max\{\sigma_k(x) \mid 1 \leq k \leq m \text{ and } i_k = i\}, \quad (i \in I),$$

$$\widetilde{M}^{(i)} = \widetilde{M}^{(i)}(x) := \{k \mid 1 \leq k \leq m,\ i_k = i,\ \sigma_k(x) = \widetilde{\sigma}^{(i)}(x)\} \quad (i \in I),$$

$$\widetilde{m}_f^{(i)} = \widetilde{m}_f^{(i)}(x) := \max \widetilde{M}^{(i)}(x), \quad \widetilde{m}_e^{(i)} = \widetilde{m}_e^{(i)}(x) := \min \widetilde{M}^{(i)}(x) \quad (i \in I).$$

the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ and the functions wt and ε_i, φ_i as

$$\tilde{f}_i(x)_k := x_k + \delta_{k, \widetilde{m}_f^{(i)}}, \qquad \tilde{e}_i(x)_k := x_k - \delta_{k, \widetilde{m}_e^{(i)}},$$

$$\mathrm{wt}(x) := -\sum_{k=1}^{m} x_k \alpha_{i_k}, \quad \varepsilon_i(x) := \widetilde{\sigma}^{(i)}(x), \quad \varphi_i(x) := \langle h_i, \mathrm{wt}(x) \rangle + \varepsilon_i(x).$$

Cellular Crystal $\mathbb{B}_{\mathbf{i}} = \mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes \cdots \otimes B_{i_k}$

For a reduced word $\mathbf{i} = i_1 i_2 \cdots i_k$ of $w \in W$, we call the crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_k}$ a **cellular crystal** associated with $\mathbf{i}$. Indeed, it is obtained by the tropicalization from the positive geometric crystal on the Langlands dual Schubert cell $X_w^\vee$ $(w = s_{i_1} \cdots s_{i_k})$.

Theorem ([Kanakubo-N])

For any simple Lie algebra $\mathfrak{g}$ and any reduced word $i_1 i_2 \cdots i_k$, the cellular crystal $\mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ is connected (as a crystal graph).

$N = l(w_0)$: the length of the longest element. For $\forall k \leq N$,

$$\mathbb{B}_{i_1 i_2 \cdots i_N} \text{ is connected} \Longrightarrow \mathbb{B}_{i_1 i_2 \cdots i_k} \text{ is connected}$$

since $B_1 \otimes B_2$ is connected $\Longrightarrow$ both B_1 and B_2 are connected.

Cellular Crystal $\mathbb{B}_{\mathbf{i}}$ – Subspace $\mathcal{H}_{\mathbf{i}}$

Fix a longest reduced word $\mathbf{i} = i_1 \cdots i_N$, define the function β_k by

$$\beta_k(x) := \sigma_{k^+}(x) - \sigma_k(x) = x_k + \sum_{k<j<k^+} \langle h_{i_k} \alpha_{i_j} \rangle x_j + x_{k^+}$$

$$(k^+ := \min\{\{m \mid k < m \leq N,\ i_k = i_m\} \sqcup \{N+1\}\},\ 1 \leq k, k^+ \leq N)$$

Define $\mathcal{H}_{\mathbf{i}} \subset \mathbb{Z}^N$:

$$\mathcal{H}_{\mathbf{i}} := \{x \in \mathbb{Z}^N (= \mathbb{B}_{\mathbf{i}}) \,|\, \beta_k(x) = 0 (\forall k \text{ s.t. } k^+ \leq N)\} \subset \mathbb{B}_{\mathbf{i}}$$

Proposition (Kanakubo-N)

For $\mathbf{i} = i_1 i_2 \cdots i_N$, $k = 1, 2, \cdots, N$ *and a fundamental weight* Λ_i, *we define*

$$h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle, \qquad \mathbf{h}_i := (h_i^{(1)}, h_i^{(2)}, \ldots, h_i^{(N)}) \in \mathbb{B}_{\mathbf{i}}$$

$$\Longrightarrow \mathcal{H}_{\mathbf{i}} = \mathbb{Z}\mathbf{h}_1 \oplus \mathbb{Z}\mathbf{h}_2 \oplus \cdots \oplus \mathbb{Z}\mathbf{h}_n$$

$B(\infty)$: crystal of $U_q^-(\mathfrak{g})$. We can realize $B(\infty) \subset \mathbb{B}_{\mathbf{i}} = \mathbb{Z}^N$ by e.g., polyhedral realization.

Example

$\mathfrak{g} = A_2$*-case:* $B(\infty) = \{(x, y, z) \in \mathbb{Z}^3 \mid 0 \le x \le y,\ z \ge 0\} \subset \mathbb{Z}^3 = \mathbb{B}_{121}$

Lemma (Kanakubo-N, N)

For $h \in \mathcal{H}_{\mathbf{i}}$, define

$$B^h(\infty) := \{x + h \in \mathbb{Z}^N = \mathbb{B}_{\mathbf{i}} \mid x \in B(\infty)\}.$$

1. *For any $x + h \in B^h(\infty)$ and $i \in I$, we obtain*

$$\tilde{e}_i(x + h) = \tilde{e}_i(x) + h, \qquad \tilde{f}_i(x + h) = \tilde{f}_i(x) + h,$$

   and then $B^h(\infty)$ is connected.

2. *For any $h \in \mathcal{H}_{\mathbf{i}}$, we have $B(\infty) \cap B^h(\infty) \neq \emptyset$.*
3. $$\mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty)$$

By the fact that $B(\infty)$ is connected and this lemma, we can show that $\mathbb{B}_{\mathbf{i}}$ is connected.

Quiver Hecke Algebra

For a finite index set I and a field $\mathbf{k}$, let $(Q_{i,j}(u,v))_{i,j\in I} \subset \mathbf{k}[u,v]$ be polynomials satisfying: $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, $Q_{i,i}(u,v) = 0$ for any $i, j \in I$ and some other conditions. For $\beta = \sum_i m_i\alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$.

Definition

For $\beta \in Q_+$, the quiver Hecke algebra $R(\beta)$ associated with a Cartan matrix $A = (a_{ij})_{i,j=1,2,\ldots,n}$ and polynomials $(Q_{ij}(u,v))_{i,j\in I}$ is the algebra generated by

$$\{e(\nu) | \nu \in I^\beta := \{((\nu_1,\cdots,\nu_m) \mid \sum_{k=1}^{m} \alpha_{\nu_k} = \beta\}\}, \quad \{x_k\}_{1\le k\le m}, \quad \{\tau_i\}_{1\le i\le m-1} \qquad \text{set } R := \bigoplus_{\beta\in Q_+} R(\beta)$$

Grading

The defining relations are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a $\mathbb{Z}$-graded algebra. Here we define the *weight* of $R(\beta)$-module M as $\mathrm{wt}(M) = -\beta$.

R-modules I

R-modules

1. Define the *graded shift functor* q on a graded $R(\beta)$-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$ by:

$$qM := \bigoplus_{k\in\mathbb{Z}} (qM)_k, \quad \text{where } (qM)_k = M_{k-1}.$$

2. For $f \in \mathrm{Hom}_R(q^k M, N)$, define $\deg(f) = k$.
3. For $M \in R(\beta)$-Mod and $N \in R(\beta')$-Mod, define the convolution product by

$$M \circ N := R(\beta+\beta')e(\beta,\beta') \otimes_{R(\beta)\otimes R(\beta')} (M \otimes N) \quad (e(\beta,\beta') := \sum_{\nu\in I^{\beta},\, \nu'\in I^{\beta'}} e(\nu,\nu'))$$

4. $M \nabla N := \mathrm{hd}(M \circ N)$ (head), $M \Delta N := \mathrm{soc}(M \circ N)$ (socle), where the head of a module is the quotient by its radical and the socle of a module is the sum of all simple submodules.
5. A simple R-module M is *real* $\Longleftrightarrow$ $M \circ M$ is simple.
6. If $M \cong M^*$, we say M is *self-dual*.

Categorification of $U_q^-(\mathfrak{g})$ and $\mathcal{A}_q(\mathfrak{n})$

$R(\beta)$-gmod: Category of finite-dimensional graded $R(\beta)$-modules
$R(\beta)$-gproj: Category of finitely generated graded projective $R(\beta)$-modules
Define the functors

$$E_i \;:\; R(\beta)\text{-gmod} \to R(\beta-\alpha_i)\text{-gmod by} \quad E_i(M) := e(\alpha_i, \beta-\alpha_i)M$$
$$F_i \;:\; R(\beta)\text{-gmod} \to R(\beta+\alpha_i)\text{-gmod by} \quad F_i(M) = L(i) \circ M,$$

where $e(\alpha_i, \beta-\alpha_i) := \sum_{\nu \in I^\beta, \nu_1 = i} e(\nu)$ and $L(i)$ is a 1-dim. simple $R(\alpha_i)$-module. They satisfy e.g., $E_i F_i = q_i^{-2} F_i E_i + \mathrm{id}$ (q-boson relation) and q-Serre relations.

Theorem ([Khovanov-Lauda, Rouquier])

Let $\mathcal{K}(R\text{-gmod})$ (resp. $\mathcal{K}(R\text{-gproj})$) be the Grothendieck ring of the monoidal category R-gmod (resp. R-gproj). Then we obtain

$$\mathcal{K}(R\text{-gproj}) \cong U_q^-(\mathfrak{g})_{\mathbb{Z}[q,q^{-1}]}, \qquad \mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$$

Categorfication of $B(\infty)$ by Lauda and Vazirani

For a simple module $M \in R(\beta)$-gmod, define

$$\begin{aligned}
&\mathrm{wt}(M) = -\beta, \\
&\varepsilon_i(M) = \max\{n \in \mathbb{Z} \mid E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle, \\
&\widetilde{E}_i M := q_i^{1-\varepsilon_i(M)} \mathrm{soc}(E_i M) \quad (q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}), \\
&\widetilde{F}_i M := q_i^{\varepsilon_i(M)} \mathrm{hd}(F_i M).
\end{aligned}$$

Set $\mathbb{B}(R\text{-gmod}) := \{S \mid S$ is a self-dual simple module in R-gmod$\}$

Theorem (Lauda-Vazirani)

The 6-tuple, $(\mathbb{B}(R\text{-}gmod), \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\})$ holds a crystal structure and there exists the following isomorphism of crystals:

$$\Psi : \mathbb{B}(R\text{-}gmod) \xrightarrow{\sim} B(\infty)$$

Braiders and Real Commuting Family I

Let Λ be $\mathbb{Z}$-lattice and $\mathcal{T} = \oplus_{\lambda \in \Lambda} \mathcal{T}_\lambda$ be a $\mathbf{k}$-linear Λ-graded monoidal category with $1 \in \mathcal{T}_0$ and the bifunctor $\circ : \mathcal{T}_\lambda \times \mathcal{T}_\mu \to \mathcal{T}_{\lambda+\mu}$. (Later Λ will be the root lattice Q)

Definition ([KKOP])

q: central obj.in $\mathcal{T}_0$. A ***graded braider*** *is a triple (C, R_C, ϕ), where $C \in \mathcal{T}$, $\mathbb{Z}$-linear map $\phi : \Lambda \to \mathbb{Z}$ and a morphism:*

$$R_C : C \circ X \to q^{\phi(\lambda)} X \circ C \quad (X \in \mathcal{T}_\lambda),$$

which is functorial in X and satisfying the commutative diagram

$$\begin{array}{ccc} C \circ X \circ Y & \xrightarrow{R_C(X)\circ Y} & q^{\phi(\lambda)} X \circ C \circ Y \\ & \searrow^{R_C(X \circ Y)} & \downarrow {\scriptstyle X \circ R_C(Y)} \\ & & q^{\phi(\lambda+\mu)} (X \circ Y) \circ C \end{array} \qquad (X \in \mathcal{T}_\lambda,\ Y \in \mathcal{T}_\mu)$$

Braiders and Real Commuting Family II

Let I be an index set and $\Gamma := \oplus_{i\in I}\mathbb{Z}e_i$ and $\Gamma_+ := \oplus_{i\in I}\mathbb{Z}_{\geq 0}e_i$. (Later Γ will be the weight lattice P and Γ_+ be the set of dominant weights P_+.)

Definition ([KKOP])

We say $(C_i, R_{C_i}, \phi_i)_{i\in I}$ *a* **real commuting family(RCF)of graded braiders** *in* $\mathcal{T}$ *if*

1. $C_i \in \mathcal{T}_{\lambda_i}$ *for some* $\lambda_i \in \Lambda$, *and* $\phi_i(\lambda_i) = 0$, $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$ $(i, j \in I)$.
2. $R_{C_i}(C_i) \in \mathbf{k}^\times \mathrm{id}_{C_i \circ C_i}$ $(i \in I)$, $\quad R_{C_i}(C_j) \circ R_{C_j}(C_i) \in \mathbf{k}^\times \mathrm{id}_{C_i \circ C_j}$ $(i, j \in I)$.
   (Note: $R_{C_i}(C_j)$*'s satisfy the "Yang-Baxter equation" on* $C_i \circ C_j \circ C_k$*.)*

Lemma ([KKOP])

For a RCF $(C_i, R_{C_i}, \phi_i)_{i\in I}$, $\exists$*bilin.map* $H : \Gamma \times \Gamma \to \mathbb{Z}$ *such that* $\phi_i(\lambda_j) = H(e_i, e_j) - H(e_j, e_i)$ *and there exist* $\{C^\alpha \in \mathcal{T} \mid \alpha \in \Gamma_+\}$ *and isom.* $\xi_{\alpha,\beta} : C^\alpha \circ C^\beta \xrightarrow{\sim} q^{H(\alpha,\beta)} C^{\alpha+\beta} (\forall \alpha, \beta \in \Gamma_+)$.

Define for $\alpha, \beta \in \Gamma$, $\mathcal{D}_{\alpha,\beta} := \{\delta \in \Gamma \mid \alpha + \delta, \beta + \delta \in \Gamma_+\}$
$^{\exists}$Inductive system $\{H_\delta((X,\alpha),(Y,\beta))\}_{\delta \in \mathcal{D}_{\alpha,\beta}} \subset \mathrm{Hom}(C^{\alpha+\delta} \circ X, Y \circ C^{\beta+\delta})$

Definition (Localization [KKOP])

We define the **localization** *of monoid.cat.* $\mathcal{T}$ *denoted by* $\widetilde{\mathcal{T}}$ *or* $\mathcal{T}[C_i^{\circ -1} \mid i \in I]$*:*

$$\mathrm{Ob}(\widetilde{\mathcal{T}}) := \mathrm{Ob}(\mathcal{T}) \times \Gamma,$$

$$\mathrm{Hom}_{\widetilde{\mathcal{T}}}((X,\alpha),(Y,\beta)) := \varinjlim_{\delta \in \mathcal{D}(\alpha,\beta),\ \lambda + L(\alpha) = \mu + L(\beta)} H_\delta((X,\alpha),(Y,\beta)),$$

$$(X,\alpha) \circ (Y,\beta) := (q^{-\phi(\beta,\lambda) + H(\alpha,\beta)}(X \circ Y), \alpha + \beta),$$

where $X \in \mathcal{T}_\lambda$, $Y \in \mathcal{T}_\mu$ *and* $L : \Gamma \to \Lambda$ *(*$e_i \mapsto \lambda_i$*)*

Determinantial Modules

To localize R-gmod let us find "**real commuting family of graded braiders**". Take a simple $R(n\alpha_i)$-module $L(i^n) := q_i^{\frac{n(n-1)}{2}} L(i)^{\circ n}$ satisfying $\mathrm{qdim}(L(i^n)) = [n]_i!$.

Definition

For $M \in R$-gmod, define

$$\widetilde{F}_i^n(M) := L(i^n)\nabla M.$$

For a Weyl group element w, let $s_{i_1} \cdots s_{i_l}$ be its reduced expression. For $\Lambda \in P_+$ and w, set $m_k = m_k(\Lambda) := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l}\Lambda\rangle \qquad (k = 1, \cdots, l)$. We define the **determinantial module** $\mathsf{M}(w\Lambda, \Lambda)$ *associated with w and Λ by*

$$\mathsf{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l}\mathbf{1},$$

which does not depend on the choice of reduced word $i_1 \ldots i_l$.

$C_i := \mathsf{M}(w\Lambda_i, \Lambda_i)$ affords an affinization $\Rightarrow$ $^{\exists}$ R-matrix $R_{C_i} : C_i \circ X \to q^{\phi} X \circ C_i$.

Set $C_\Lambda := \mathrm{M}(w_0\Lambda, \Lambda)$. In particular, for $i \in I$ set $C_i = C_{\Lambda_i}$. Then we obtain

Theorem ([KKOP])

Define the function $\phi_{C_i} : Q \to \mathbb{Z}$ by $\phi_{C_i}(\beta) := -(\beta, w_0\Lambda_i + \Lambda_i)$

$\Longrightarrow \{(C_i, R_{C_i}, \phi_{C_i})\}_{i\in I}$ *a real comm. family of graded braiders in R-gmod.*

Take $\Gamma = P$ and $\Gamma_+ = P_+$. Then, we obtain the localization of R-gmod

$$\widetilde{R\text{-gmod}} := R\text{-gmod}[C_i^{\circ -1} \mid i \in I]$$

by $\{(C_i, R_{C_i}, \phi_{C_i})\}_{i\in I}$.

Its Grothendieck ring $\mathcal{K}(\widetilde{R\text{-gmod}})$ defines the **localized quantum coordinate ring** $\widetilde{\mathcal{A}_q(\mathfrak{n})} := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{K}(\widetilde{R\text{-gmod}})$.

Localization $\widetilde{R\text{-gmod}}$ II

Definition

For a ring R(not necessarily commutative) and a multiplicative set $S \subset R$, a ring R' is said to be a left ring of quotients of R w.r.t. S if $\exists$*hom.* $\varphi : R \to R'$ *s.t.*

1. $\forall s' \in \varphi(S)$ *is invertible in* R'.
2. $\forall m \in R'$ *is in the form* $m = \varphi(s)^{-1}\varphi(a)$ *for some* $s \in S,\ a \in R$.
3. $\operatorname{Ker}\varphi = \{r \in R \mid sr = 0$ *for some* $s \in S\}$.

R' *is denoted by* $S^{-1}R$.

Proposition (KKOP)

We get $\mathcal{K}(\widetilde{R\text{-gmod}}) \cong \mathcal{S}^{-1}\mathcal{K}(R\text{-gmod})$= *the left ring of quotients of the ring* $\mathcal{K}(R\text{-gmod})(\cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]})$ *with respect to the multiplicative set*

$$\mathcal{S} := \{q^k \prod_{i\in I}[C_i]^{a_i} \mid k \in \mathbb{Z},\ (a_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}\}$$

Localization $\widetilde{R\text{-gmod}}$ III

Proposition (KKOP)

Let $\Phi : R\text{-gmod} \to \widetilde{R\text{-gmod}}$ be the canonical functor. Then,

1. $\widetilde{R\text{-gmod}}$ is an abelian category and the functor Φ is exact.
2. $\widetilde{C}_i := \Phi(C_i)$ $(i \in I)$ is invertible central graded braider in $\widetilde{R\text{-gmod}}$.
3. $S \in R\text{-gmod}$ is simple $\Longrightarrow \Phi(S)$ is simple in $\widetilde{R\text{-gmod}}$.

   For $\nu \in P$, define $\widetilde{C}_\nu$ by $\widetilde{C}_{\lambda+\mu} = \widetilde{C}_\lambda \circ \widetilde{C}_\mu$ (up to grading) and $\widetilde{C}_{-\Lambda_i} = C_i^{\circ -1}$
4. For $\forall$simple $M \in \widetilde{R\text{-gmod}}$, simple $\exists S \in R\text{-gmod}$ and $\exists \Lambda \in P$ s.t. $M \cong \widetilde{C}_\Lambda \circ \Phi(S)$ (Λ and S are not necessarily unique) .

Crystal Structure on $R\text{-}\widetilde{\mathrm{gmod}}$ I

For a simple object $\widetilde{C}_\Lambda \circ \Phi(S) \in R\text{-}\widetilde{\mathrm{gmod}}$ we write simply $C_\Lambda \circ S$.
Set $\mathbb{B}(R\text{-}\widetilde{\mathrm{gmod}}) := \{S \mid S$ is a self-dual simple object in $R\text{-}\widetilde{\mathrm{gmod}}\}$

Lemma (KKOP)

$^\forall$simple module $M \in R\text{-}\widetilde{\mathrm{gmod}}$, $^{\exists!}n \in \mathbb{Z}$ such that $q^n M$ is self-dual, denoted by $\delta(M)$.

The actions of the Kashiwara operators [N]

Define the Kashiwara operators $\widetilde{F}_i$ and $\widetilde{E}_i$ $(i \in I)$ on $\mathbb{B}(R\text{-}\widetilde{\mathrm{gmod}})$:

$$\widetilde{F}_i(C_\Lambda \circ S) = q^{\delta(C_\Lambda \circ \widetilde{F}_i S)} C_\Lambda \circ \widetilde{F}_i S,$$

$$\widetilde{E}_i(C_\Lambda \circ S) = \begin{cases} q^{\delta(C_\Lambda \circ \widetilde{E}_i S)} C_\Lambda \circ \widetilde{E}_i S & \text{if } E_i S \neq 0, \\ q^{\delta(C_{\Lambda - \Lambda_{i^*}} \circ (\widetilde{E}_i C_{\Lambda_{i^*}} \circ S))} C_{\Lambda - \Lambda_{i^*}} \circ (\widetilde{E}_i C_{\Lambda_{i^*}} \circ S) & \text{if } E_i S = 0, \end{cases}$$

where δ is given in the above lemma and $i^* \in I$ is the index satisfying $\Lambda_{i^*} = -w_0 \Lambda_i$.

Crystal Structure on $\widetilde{R\text{-gmod}}$ II

Crystal structure: ε_i and wt [N]

Let $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ (Lauda-Vazirani). For $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R\text{-gmod}})$, define

$$\mathrm{wt}(C_\Lambda \circ S) = \mathrm{wt}(\Psi(S)) + w_0\Lambda - \Lambda,$$
$$\varepsilon_i(C_\Lambda \circ S) = \varepsilon_i(\Psi(S)) - \langle h_i, w_0\Lambda\rangle,$$
$$\varphi_i(C_\Lambda \circ S) = \varepsilon_i(\Psi(C_\Lambda \circ S)) + \langle h_i, \mathrm{wt}(C_\Lambda \circ S)\rangle.$$

Theorem ([N])

The 6-tuple $(\mathbb{B}(\widetilde{R\text{-gmod}}), \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\widetilde{E}_i\}, \{\widetilde{F}_i\})_{i\in I}$ *is a crystal.*

Indeed, we should show that well-definedness, i.e.,all data do not depend on the presentation $C_\Lambda \circ S \cong C_{\Lambda'} \circ S'$ and for $b = C_\Lambda \circ S$,

$$\widetilde{E}_i\widetilde{F}_i b = \widetilde{F}_i\widetilde{E}_i b = b,$$
$$\varepsilon_i(\widetilde{F}_i(b)) = \varepsilon_i(b) + 1, \qquad \varepsilon_i(\widetilde{E}_i(b)) = \varepsilon_i(b) - 1,$$
$$\mathrm{wt}(\widetilde{E}_i b) = \mathrm{wt}(b) + \alpha_i, \qquad \mathrm{wt}(\widetilde{F}_i b) = \mathrm{wt}(b) - \alpha_i.$$

As we have seen above that the set $\mathcal{H}_{\mathbf{i}} \subset \mathbb{B}_{\mathbf{i}}$ is presented by

$$\mathcal{H}_{\mathbf{i}} = \bigoplus_{i \in I} \mathbb{Z}\mathbf{h}_i, \quad \mathbf{h}_i = ((h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle)_{k=1,\cdots,N}$$

Lemma ([N])

For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$ *and* $\Lambda \in P_+$, *set* $m_k = m_k(\Lambda) := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda \rangle)_{k=1,\cdots,N}$. *Then, we obtain*

$$\begin{aligned} \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) &= \tilde{f}_{i_1}^{m_1}(0)_{i_1} \otimes \tilde{f}_{i_2}^{m_2}(0)_{i_2} \otimes \cdots \otimes \tilde{f}_{i_N}^{m_N}(0)_{i_N} \\ &= (m_1, m_2, \cdots, m_N) =: \mathbf{h}_\Lambda \in \mathcal{H}_{\mathbf{i}}, \end{aligned}$$

where note that for $\Lambda = \Lambda_i$, *one has* $m_k(\Lambda_i) = h_i^{(k)}$. *Then in this case we obtain*

$$\tilde{f}_{i_1}^{m_1(\Lambda_i)} \tilde{f}_{i_2}^{m_2(\Lambda_i)} \cdots \tilde{f}_{i_N}^{m_N(\Lambda_i)}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) = \mathbf{h}_i$$

Cellular Crystal $\mathbb{B}_{\mathbf{i}}$ and $\mathbb{B}(\widetilde{R\text{-gmod}})$ II

Observation: Determinantial modules $\{C_\Lambda = \mathrm{M}(w_0\Lambda, \Lambda)\} \longleftrightarrow \mathcal{H}_{\mathbf{i}}$

$$\{C_\Lambda \mid \Lambda \in P_+\} \subset R\text{-gmod} \quad \longleftrightarrow \quad \mathcal{H}_{\mathbf{i}}$$

$$C_\Lambda = \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_N}^{m_N} \mathbf{1} \quad \longleftrightarrow \quad \mathbf{h}_\Lambda = \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N})$$

Theorem ([N])

For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$, ∃isomorphism of crystals:

$$\widetilde{\Psi} : \mathbb{B}(\widetilde{R\text{-gmod}}) \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty)$$

$$C_\Lambda \circ S \quad \longmapsto \quad \mathbf{h}_\Lambda + \Psi(S) \in B^{\mathbf{h}_\Lambda}(\infty),$$

where $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ (Lauda-Vazirani), S is simple in $\mathbb{B}(R\text{-gmod})$ and for $\Lambda = \sum_i a_i \Lambda_i$ we have $\mathbf{h}_\Lambda = \sum_i a_i \mathbf{h}_i$.

Localized Quantum Unipotent Coordinate Category I

In an arbitrary "symmetrizable Kac-Moody" setting, for any Weyl group element $w \in W$, there exists the full subcategory $\mathscr{C}_w$ of R-gmod defined as follows: For $M \in R(\beta)$-gmod, set $\mathsf{W}(M) := \{\gamma \in Q_+ \cap (\beta - Q_+) \mid e(\gamma, \beta-\gamma)M \neq 0\} \subset Q_+$.

Define the category $\quad \mathscr{C}_w := \{M \in R\text{-gmod} \mid \mathsf{W}(M) \subset Q_+ \cap wQ_-\}$

Note: $\mathscr{C}_{w_0} = R$-gmod for a semi-simple $\mathfrak{g}$.
Indeed, $\mathscr{C}_w$ categorifies $\mathcal{A}_q(\mathfrak{n}(w))$ = quantum unipotent coordinate ring ass w, that is, $\mathcal{K}(\mathscr{C}_w) \cong \mathcal{A}_q(\mathfrak{n}(w))$. It admits a localization

$$\widetilde{\mathscr{C}_w} = \mathscr{C}_w[C_i^{\circ -1} \mid i \in I], \quad (C_i = M(w\Lambda_i, \Lambda_i)),$$

called localized quantum unipotent coordinate category ass.$w \in W$.
Let $Q_w : R\text{-gmod} \to \widetilde{\mathscr{C}_w}$ be the localization functor.

Localized Quantum Unipotent Coordinate Category II

Proposition (KKOP2, KKOP3)

1. $Q_{w|\mathscr{C}_w} : \mathscr{C}_w \to \widetilde{\mathscr{C}_w}$ *is fully faithful.*
2. *There exists the category equivalence* R-*gmod*$[C_i^{\circ -1} \mid i \in I] \xrightarrow{\sim} \widetilde{\mathscr{C}_w}$
3. *We obtain* $\mathrm{Ker}(Q_w) = R$-*gmod* $\setminus \mathscr{B}_w$ *where* $\mathscr{B}_w = \{M \in R\text{-gmod} \mid {}^{\forall}\text{simple subquotient } S \text{ of } M, \Psi(S) \in B_w(\infty)\}$ *and* $B_w(\infty)$ *is a Demazure crystal in* $B(\infty)$.

Problem

Q: Does the category $\widetilde{\mathscr{C}_w}$ hold a crystal $\mathbb{B}(\widetilde{\mathscr{C}_w}) := \{\text{self-dual simple} \in \widetilde{\mathscr{C}_w}\}$? If so,

$$\mathbb{B}(\widetilde{\mathscr{C}_w}) \xrightarrow{\sim} B_{i_1} \otimes \cdots \otimes B_{i_m} \ ?$$

where $i_1 \cdots i_m$ is a reduced word of w.

Localized Quantum Unipotent Coordinate Category III

Answer

The answer is “YES” (joint work with M.Kashiwara)

Key tools

For the answer, we need the following three objects:

Rigidity Affinization R-matrix

Rigid category

Definition

$X, Y \in \mathcal{T}$ monoidal category, and $\varepsilon : X \circ Y \to 1$ and $\eta : 1 \to Y \circ X$ morphisms in $\mathcal{T}$. A pair (X, Y) is a dual pair or X is a left dual to Y, denoted $\mathcal{D}^{-1}(Y)$ and Y is a right dual to X, denoted $\mathcal{D}(X)$ if the following compositions are identities:

$$X \simeq X \circ 1 \xrightarrow{\mathrm{id} \circ \eta} X \circ Y \circ X \xrightarrow{\varepsilon \circ \mathrm{id}} 1 \circ X \simeq X, \quad Y \simeq 1 \circ Y \xrightarrow{\eta \circ \mathrm{id}} Y \circ X \circ Y \xrightarrow{\mathrm{id} \circ \varepsilon} Y \circ 1 \simeq Y$$

Definition

A monoidal cat. $\mathcal{T}$ is left rigid (resp. right rigid) if ${}^{\forall}X \in \mathcal{T}$, ${}^{\exists}\mathcal{D}^{-1}(X)$ (resp. $\mathcal{D}(X)$). We say $\mathcal{T}$ is rigid if $\mathcal{T}$ is left and right rigid, .

Theorem ([KKOP,KKOP2])

For a quiver Hecke algebra R associated with an arbitrary symmetrizable Kac-Moody Lie algebra and any $w \in W$, the category $\widetilde{\mathscr{C}_w}$ is rigid, i.e., ${}^{\forall}X \in \widetilde{\mathscr{C}_w}$, ${}^{\exists}\mathcal{D}(X), \mathcal{D}^{-1}(X)$. ($\widetilde{\mathscr{C}_{w,v}}$ is right rigid, but not yet known to be left rigid.)

Affinzation in $\widetilde{\mathscr{C}}_w$ and R-matrix I

In [KKOP4], for a monoidal category $\mathcal{C}$ with several "good" conditions, the following categories are defined

$$\begin{aligned} \mathcal{C} \subset \quad & \mathrm{Pro}(\mathcal{C}) = \{\text{pro-object=some projective limit}\}, \\ & \cup \\ & \mathrm{Modg}(k[z], \mathrm{Pro}(\mathcal{C})) \supset \mathrm{Pro}_{coh}(k[z], \mathcal{C}) \supset \mathrm{Aff}_z(\mathcal{C}). \end{aligned}$$

An object $(\mathsf{M}, z) \in \mathrm{Aff}_z(\mathcal{C})$ is called an affine object of $M = \mathsf{M}/z\mathsf{M} \in \mathcal{C}$ and an affine object (M, z) of $M = \mathsf{M}/z\mathsf{M}$ with a "rational center R_{M}" is called an affinization of M.

Proposition (KKOP4, Prop 5.6)

If a category $\mathcal{C}$ is rigid, then the monoidal category $\mathrm{Aff}_z(\mathcal{C})$ is also rigid.

Example

If R is symmetric, $^\forall$simple real object M in $\mathscr{C}_w$ affords an affinization $\mathsf{M} = M \otimes_k k[z]$.

Affinization in $\widetilde{\mathscr{C}}_w$ and R-matrix II

Theorem (KKOP4, Prop 6.2, Thm 6.10, Prop 6.18)

Let $\mathcal{C}$ be an abelian rigid monoidal category with bi-exact tensor product.

1. *For an affreal(=real and admits affinization) $M \in \mathcal{C}$ and a simple $N \in \mathcal{C}$, $M \circ N$ and $N \circ M$ have simple heads and simple socles, moreover,*

$$\mathrm{Hom}_{\mathcal{C}}(M \circ N, N \circ M) = \mathrm{Hom}_{\mathcal{C}}(N \circ M, M \circ N) = k^{\exists} R_{M,N} \quad \textit{(R-matrix).}$$

2. *Let $(\mathsf{M}, z_{\mathsf{M}})$ be the affinization of M as in ① and $N \in \mathcal{C}$ a simple. Then in $\mathrm{Pro}_{coh}(k[z], \mathcal{C})$ $^{\exists!}$renormailzed R-matrix $R^{\mathrm{ren}}_{\mathsf{M},N}$ s.t.*

$$k[z] \cdot R^{\mathrm{ren}}_{\mathsf{M},N} = \mathrm{Hom}_{k[z]}(\mathsf{M} \circ N, N \circ \mathsf{M})$$

satisfying YB-eq. and $R^{\mathrm{ren}}_{\mathsf{M},N}|_{z=0} = R_{M,N}$, etc.

Affinzation in $\widetilde{\mathscr{C}_w}$ and R-matrix III

Applying the above results to $C = \widetilde{\mathscr{C}_w}$, we obtain R-matrix and rigidity of the category $\mathrm{Aff}_z(\widetilde{\mathscr{C}_w})$.
Then by the theorem above, for any affreal object $M \in \widetilde{\mathscr{C}_w}$, its affinization $(\mathsf{M}, z, R_{\mathsf{M}})$ and a simple $N \in \widetilde{\mathscr{C}_w}$, we obtain renormalized R-matrix and R-matrix

$$R^{\mathrm{ren}}_{\mathsf{M},N} \in \mathrm{H\scriptstyle{OM}}_{k[z]}(\mathsf{M} \circ N, N \circ \mathsf{M}) \qquad R_{M,N} = R^{\mathrm{ren}}_{\mathsf{M},N}|_{z=0} \in \mathrm{H\scriptstyle{OM}}_{\widetilde{\mathscr{C}_w}}(M \circ N, N \circ M)$$

$R : M \circ N \to N \circ M$ be an R-matrix in $\mathscr{C}_w$. Then, we define $\Lambda(M, N) := \deg(R)$.
Similarly, for an R-matrix $\widetilde{R} : X \circ Y \to Y \circ X$ in $\widetilde{\mathscr{C}_w}$, define $\Lambda(X, Y) := \deg(\widetilde{R})$, and

$$\widetilde{\Lambda}(X, Y) := \frac{1}{2}(\Lambda(X, Y) + (\mathrm{wt}(X), \mathrm{wt}(Y))).$$

Crystal Structure on $\widetilde{\mathscr{C}}_w$

Using the rigidity and R-matrix of $\widetilde{\mathscr{C}}_w$, we can define the crystal structure on

$$\mathbb{B}(\widetilde{\mathscr{C}}_w) := \{M \in \widetilde{\mathscr{C}}_w \mid M \cong M^* \text{ is self-dual simple object}\}$$

by defining for $X \in \mathbb{B}(\widetilde{\mathscr{C}}_w)$, e.g.,

$$\widetilde{\mathrm{E}}_i\, X := q_i^{\varphi_i(X)} X \nabla \mathscr{D}\mathrm{Q}_w(L(i)),$$
$$\varepsilon_i(X),\ \varphi_i(X),\ \widetilde{\mathrm{F}}_i\, X,\ \widetilde{\mathrm{E}}_i^*\, X,\ \widetilde{\mathrm{F}}_i^*\, X, \quad etc....$$

Note that $\tilde{E}_i = \tilde{F}_i^{-1}$ and $\tilde{E}_i^* = \tilde{F}_i^{*\,-1}$.

Theorem

$\mathbb{B}(\widetilde{\mathscr{C}}_w)$ becomes a crystal and we obtain an isomorphism of crystals :

$$\mathbb{B}(\widetilde{\mathscr{C}}_w) \xrightarrow{\sim} B_{i_1} \otimes \cdots \otimes B_{i_m}$$

where $i_1 \cdots i_m$ is a reduced word of $w \in W$.

Corollary

For any reduced word $i_1 \cdots i_m$ of any $w \in W$, the cellular crystal $B_{i_1} \otimes \cdots \otimes B_{i_m}$ is connected as a crystal graph.

Merci beaucoup pour votre attention