FUNCTIONAL EQUATIONS FOR MOTIVIC GENERATING SERIES OF KRONECKER MODULI

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(joint with. A. Astruc, F. Chapoton, K. Martinez, arXiv:2410.07913)

1. A linear algebra problem

Let V and W be complex vector spaces of dimension d and e, respectively, and let m be a positive integer.

Problem: Classify m-tuples $f_1, \ldots, f_m : V \to W$ of linear maps up to equivalence/base change $(f_k)_k \rightsquigarrow (hf_kg^{-1})_k$ for $g \in \mathrm{GL}(V)$, $h \in \mathrm{GL}(W)$.

$m = 1$: solved by elementary linear algebra, equivalence classes depend only $d, e, \mathrm{rk}(f_1) \leq d, e$.
$m = 2$: solved by Kronecker 1890, of similar complexity as the Jordan canonical form.
$m \geq 3$: unsolved; depends on many continuous parameters, for example $d = e$, f_1, f_2, f_3 given by square matrices

$$E_d,\ diag(\lambda_1, \ldots, \lambda_d), \begin{bmatrix} * & * & \ldots & * & * \\ 1 & * & \ldots & * & * \\ * & 1 & \ldots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & * & * \\ * & * & \ldots & 1 & * \end{bmatrix}:$$

then for pairwise different λ_i, only finitely many such matrices are equivalent to a given one, thus $d^2 + 1$ continuous parameters.

We assume $m \geq 3$ from now on.

2. Kronecker moduli

Construct moduli spaces; first impose a genericity assumption:

$(f_k)_k$ is called stable if for all proper non-zero subspaces $U \subset V$, we have

$$\dim \sum_{k=1}^{m} f_k(U) > \frac{e}{d} \cdot \dim U.$$

Theorem: (Mumford's GIT) There exists a complex connected algebraic manifold $K_{d,e}^{(m)}$ whose points correspond bijectively to equivalence classes of stable tuples $(f_k)_k$ as above. $K_{d,e}^{(m)}$ is of dimension $mde - d^2 - e^2 + 1$, and compact if $\gcd(d, e) = 1$.

We need a so-called framed version of these moduli spaces which are always compact manifolds (and solve a closely related linear algebra problem): we consider tuples $(f_k)_k$ as before, together with a vector $w \in W$, again up to the natural base change action of $\mathrm{GL}(V) \times \mathrm{GL}(W)$. We call this data stable if, for all subspaces $U \subset V$, we have

$$\dim \sum_{k=1}^{m} f_k(U) \leq \frac{e}{d} \cdot \dim U,$$

with strict inequality if $w \in \sum_k f_k(U)$.

Theorem: There exists a complex compact connected algebraic manifold $K_{d,e}^{(m),\mathrm{fr}}$ parametrizing equivalence classes of stable tuples $((f_k)_k, w)$. If $\gcd(d,e) = 1$, it is a rank $e-1$ projective space bundle over $K_{d,e}^{(m)}$.

3. Generating series

Fix a slople $\mu \in \mathbb{Q}$ such that $\mu^2 - m\mu + 1 < 0$, and write $\mu = \frac{e}{d}$ for $\gcd(d,e) = 1$. Define the generating series of Euler characteristic of framend Kronecker moduli of slope μ by

$$F_\mu(t) = 1 + \sum_{n \geq 1} \chi(K_{nd,ne}^{(m),\mathrm{fr}}) t^n \in \mathbb{Z}[[t]].$$

Theorem: (M. Kontsevich, Y. Soibelman) $F_\mu \in \mathbb{Q}((t))$ is an algebraic series.

Theorem: (R. 2011) We have

$$F_1(t) \cdot (1 - tF_1(t)^{m-2})^m = 1.$$

Refine these series to Betti numbers in singular cohomology:

$$F_\mu(v,t) = 1 + \sum_{n \geq 1} \sum_i \dim H^i(K_{nd,ne}^{(m),\mathrm{fr}}, \mathbb{Q}) v^{i - \dim K_{nd,ne}^{(m),\mathrm{fr}}} t^n.$$

The main new result, generalizing the above equation, is:

Theorem: (ACMR 24) The series $F_1(v,t)$ is determined by $F_1(v,0) = 1$ and

$$F_1(v,t) = \prod_{i=1}^{m} (1 - v^{2i-m-1} t \cdot \prod_{j=1}^{m-2} F_1(v, v^{2i-2j-2} t))^{-1}.$$

4. Back to Kronecker moduli

Similarly, we define

$$G_1(v,t) = 1 + \sum_{n \geq 1} \sum_i \dim H^i(K_{n,n-1}^{(m)}, \mathbb{Q}) v^{1 - \dim K_{n,n-1}^{(m)}} t^n.$$

Define the v-derivative of a series by $D_v H(v,t) = \frac{H(v,vt) - H(v,v^{-1}t)}{(v-v^{-1})t}$.

Theorem: (ACMR 24) The series $G_1(v,t)$ is related to $F_1(v,t)$ by

$$D_v G_1(v,t) = t \cdot \prod_{i=1}^{m-1} F_1(v, v^{m-2i} t).$$

Consequently (!), the Euler characteristic $\chi(K^{(m)}_{n,n-1})$ equals the number of length n intervals in the $(m-2)$-Tamari lattice.