Łukasiewicz logic and Tsallis entropy related to (conditionally) free probability

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IHES - Bures-sur-Yvette. PARIS.

Historical remarks on J. Łukasiewicz

multi-valued logic

1918, 18 III – was born 3-valued logic of Jan Łukasiewicz (born LWÓW 1878)

1920 – Emil Leon POST (born AUGUSTÓW 1897) give model for multi-valued

1918 – L.E.J. Brouwer – constructive mathematics.

1925 – Heisenberg – quantum logic

1930 – Łukasiewicz – Tarski – model of ∞-valued logic $[0,1] \cap \mathbb{Q}$

$\aleph_1$ – " – $[0,1]$

E.T. BELL: „After KOPERNIK NEWTON DARWIN EINSTEIN ŁUKASIEWICZ
– (1935)

in the last 6000 years multi-valued logic of J. Łukasiewicz is 5-th – great result".

Everything start from ARYSTOTELES – IV before Chr. BC.

$\boxed{\text{„} p \vee (\neg p) \text{" is not always TRUE !}}$

„Tomorrow will be sea battle"

<u>Abstract</u>: 1. Free product of groups with states (pos def. functions) as models of the <u>Free</u> and <u>Boolean</u> probability.

2. New version of J. Łukasiewicz and A. Tarski theorem on free projections.

3. Theorem: Let $\left.\begin{array}{l} P = P^2 = P^* \\ Q = Q^2 = Q^* \end{array}\right\}$ free projections

in some v. Neumann algebra $(\mathcal{A}, tr)$, with faithful, normal trace: $tr : \mathcal{A} \to \mathbb{C}$, $tr(\mathbb{1}) = 1$

then either

$$P \vee Q = I \quad \text{or} \quad P \wedge Q = 0$$

Moreover:

$$tr(P \wedge Q) = \max(tr(P) + tr(Q) - 1, 0)$$

$$tr(P \vee Q) = \min(tr(P) + tr(Q), 1)$$

Th (Verges, Voiculescu, 2017)

If P, Q - Boolean independent projections in some probability system $(\mathcal{B}, \Phi)$, then

[redacted] $\Phi(P \wedge Q) = (x^{-1} + y^{-1} - 1)^{-1}$, where

$x = \Phi(P)$, $y = \Phi(Q)$.

Remark: If $t_q(x,y) = \max\left[(x^{1-q} + y^{1-q} - 1), 0\right]^{\frac{1}{1-q}}$

$x, y \in (0,1)$, $q \in \mathbb{R}$ is so called

Tsallis entropy, then (multiplication ?) : $x \otimes_q y$

for $q = 0$ we have free case.

$q = 2$, we have Boolean case

who HUGO STEINHAUS

$q = 1$, $t_1(x,y) = x \cdot y$ (Z.Z. JUREK informed)

$q \to +\infty$, $t_{\infty}(x,y) = \min(x,y)$

$q \to -\infty$ $t_{-\infty}(x,y) = \max(x,y)$.

Def 1. Free product group $G = G_1 * G_2 * G_3 * \cdots$

$= \{ e, g_1 g_2 \cdots g_k, \; g_j \in G_{i_j} \setminus \{e\}, \quad i_1 \neq i_2 \neq i_3 \neq \cdots \}$

Def 2. If $\Phi_j : G_j \to B(\mathcal{H})$ is ~~completely~~ positive, $\Phi_j(e) = I$, and we define regular free product (Boolean product)

$\Phi : *G_j \to B(\mathcal{H})$ as follows:

(α) $\Phi|_{G_j} = \Phi_j$

(β) If $x = g_1 g_2 \cdots g_k$ as before, then

$$\Phi(x) = \Phi(g_1)\Phi(g_2)\cdots\Phi(g_k).$$

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Ex 1. Let $G_j = \mathbb{Z}$, $j = 1, 2 ..$

$G_1 * G_2 * .. * G_N = \mathbb{F}_N$ - the free grp on N-free gen.

Let $\varphi_j(n) = r^{|n|}$, $n \in \mathbb{Z}$, $r \in [-1, 1]$

then the regular free product

$P_r(x) = (*_r \varphi_j)(x) = r^{|x|}$, where $|x| = k$ the length funct.

if $r = 0$, $P_0(x) = \begin{cases} 1, & x = e \\ 0 & x \neq e \end{cases}$

Ex 2 (α) The $(VN(\mathbb{F}_N), P_r)_{0 < r < 1}$ is the model of the free Probability

(β) $(VN(G_1 * G_2 * .. * G_N), r^{|x|})$ more good model of free

(γ) If $\underline{r \neq 0}$, we get model of Boolean prob.

Example of J. Łukasiewicz model of 3-valued logic.

Take $G_i = \mathbb{Z}_2 = \{e, R_i\}$, $R_i = R_i^{-1}$

and take on $VN(\mathbb{Z}_2 \times \mathbb{Z}_2)$ the von Neumann trace

$$tr(f) = f(e), \text{ then}$$

the projections $P_i = \frac{I + R_i}{2}$ are

free independent, $tr(P_i) = \frac{1}{2}$, and

$$P_i \wedge P_j = 0, \text{ if } i \neq j, \text{ so}$$

the projections $\{0, P_i, I\}$ are model of J. Łukasiewicz 3-valued logic $\{0, \frac{1}{2}, 1\}$.

and we have the following ~~table~~ of J. Łukasiewicz

~~pq~~ $Cpq \equiv p \Rightarrow q$

$q \backslash p$	0	$\frac{1}{2}$	1
0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	1	1	1

by definition $Cpq \equiv (p \Rightarrow q) \equiv (1-p) \vee q$

The table for $P \wedge Q$ is following:

$x \wedge y = (x+y-1)_+$

$q \backslash p$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

why if $tr(P)=tr(Q)=\frac{1}{2}$, free

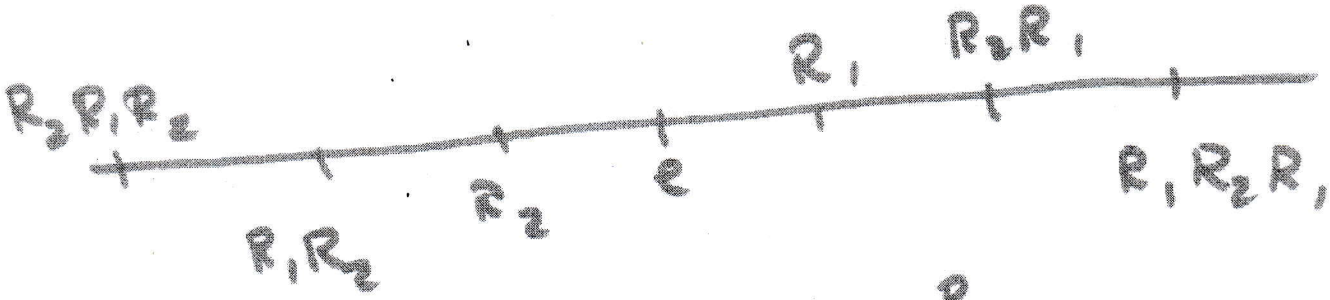
then $tr(P\wedge Q)=0$

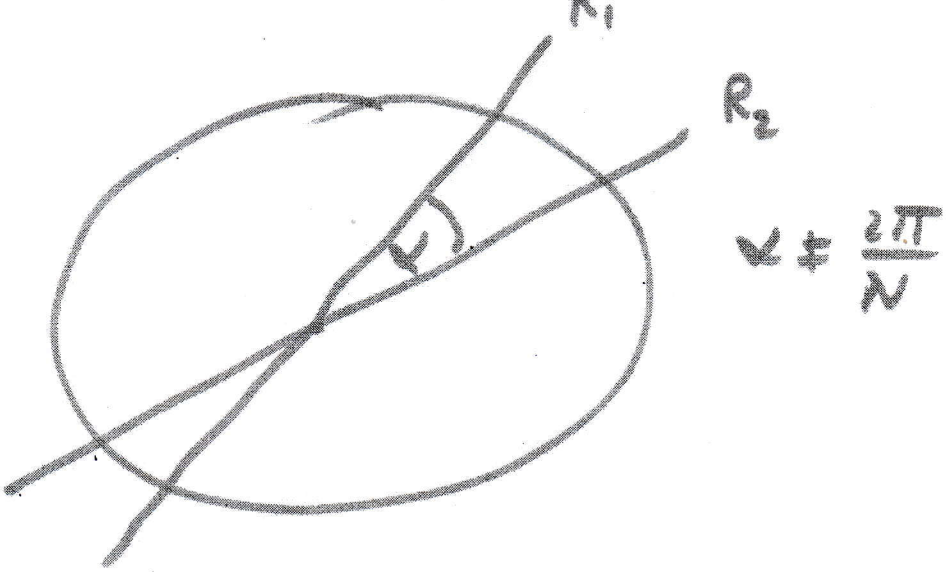
Take $P_j=\frac{1+R_j}{2}$, ,

$R_i^2=1$, $\{R_i,e\}=\mathbb{Z}_2$

and gp $\{R_1,R_2\}=\mathbb{Z}_2*\mathbb{Z}_2=\mathbb{D}_\infty$ - dihedral group

i.e. Cayley graph of $\mathbb{D}_\infty$.





We have to show that in $\mathcal{H}\ell^2(\mathbb{D}_\infty)$ the subspaces

$$\mathcal{H}_j = P_j(\mathcal{H})$$

$$\underline{\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}.}$$

But $\mathcal{H}_j = \{ f \in \mathcal{H} : \mathbb{R}_j * f = f \}$

$$= \{ f \in \mathcal{H} : P_j * f = f \}$$

$$f = P_j * f = \frac{1+R_j}{2} * f = \frac{1}{2}(f + R_j * f)$$

$$\Rightarrow \quad f = R_j * f \quad , j = 1, 2 .$$

Therefore $\quad f(e) = f(R_1) = f(R_2) =$

$$= f(R_1 R_2) = f(R_2 R_1) = \ldots \quad c = f(x)$$

but $\quad f \in \ell^2(\mathbb{D}_\infty) \Rightarrow \underline{c \equiv 0}.$

9(a)

Remarks on non-commutative independence.

1. Let $(\mathcal{A},\Phi)$, $*$-algebra with $\mathbb{1}$, Φ-state

$\mathcal{A}$ - v. Neumann alg.

Quantum sets $\longleftrightarrow$ projections in $\mathcal{A}$

P, Q - projections are „Quant. independent" if

(QI) $$\Phi(P\wedge Q)=\Phi(P)\,\Phi(Q)$$

$P\wedge Q$ = projection onto $P(\mathcal{H})\wedge P(\mathcal{H})$ =

$$= s\text{-}\lim_{n\to\infty}(PQ)^n.$$

If $PQ=QP$, then $P\wedge Q=PQ$

and we have „classical" independence.

Problem: ? Find more examples using Tsallis entropy!

A.2] Free and C-free probability

Let $\mathcal{A}$ - $*$-algebra (observables) with unit $\mathbb{1}$, and

$\varphi : \mathcal{A} \to B(\mathcal{H})$ - operator valued - state
(completely pos. map)

$\psi : \mathcal{A} \to \mathbb{C}$, state ($\psi(x^* x) \geq 0$, $\psi(\mathbb{1}) = 1$.

Definition :

Let $\mathcal{A}_i$ - subalgebras in $\mathcal{A}$, are called C-free independent, if for each $n = 1, 2, 3, ..$

$$\varphi(x_1 x_2 \dots x_n) = \varphi(x_1)\varphi(x_2) \dots \varphi(x_n), \text{ for}$$

$x_i \in \mathcal{A}_{i_j}$, $i_1 \neq i_2 \neq i_3 \neq \dots$, and

$$\psi(x_i) = 0, \quad i = 1, 2, .., n,$$

$\cancel{\varphi \mathcal{A}_i}$

Subdefinition : If $\varphi = \psi$, then the $\mathcal{A}_i$ - are called FREE sub-algebras independent

Sub-algebras $\mathcal{A}_i \subset (\mathcal{A}, \varphi)$ are Boolean - independent, if

$$\varphi(x_1 x_2 \cdots x_n) = \varphi(x_1) \cdots \varphi(x_n), \text{ for } x_j \in \mathcal{A}_{i_j}$$

Definition: Random variables $X_i \in (\mathcal{A}, \varphi, \psi)$

are FREE (C-FREE) (BOOLEAN) independent, if

the sub-algebra $\mathcal{A}_i$ = generat by X_i, are

FREE (C-FREE) (Boolean) independent.

Theorem: Let $q \geq 0$, $x, y \in (0,1)$, the Tsallis entropy

product $T_q(x,y) \stackrel{def}{=} \left[x^{(1-q)} + y^{(1-q)} - 1 \right]_+^{1/1-q}$

Then (1) If $P, Q \in (\mathcal{A}, \varphi)$ FREE projections, φ - trace state

then

$$\underline{\varphi(P \wedge Q) = T_0(x,y)}, \text{ where}$$

$\varphi(P) = x$, $\varphi(Q) = y$.

(D. VOICULESCU + VARGAS)

② If P, Q - are Boolean independent projections ($P^2 = P$, $Q^2 = Q$, $P = P^*$, $Q = Q^*$), then

$$\varphi(P \wedge Q) = T_2(x,y) = \left[x^{-1} + y^{-1} - 1\right]_+^{(-1)}$$

where $x = \varphi(P)$, $y = \varphi(Q)$.

Here $P \wedge Q$ is the projection onto $P(\mathcal{H}) \cap Q(\mathcal{H})$

③ If $P, Q \in (\mathcal{A}, \varphi)$ are classical independent ($\varphi(P \wedge Q) = \varphi(P)\varphi(Q)$, $PQ = QP$), then

$$\varphi(P \wedge Q) = T_1(x,y) = x \cdot y = \lim_{s \to 1} T_s(x,y)$$

Note: If $e_q^x = \left[1 + (1-q)x\right]_+^{\frac{1}{1-q}}$, $q < 1$, $x > 0$

(TSALLIS exponential function), then inverse (logarithm)

$$\log_q(x) = \frac{x^{1-q} - 1}{1-q}$$

Also we have (see TSALLIS books)

if $T_q(x,y) = x \otimes_q y$, for $x \geq 0$, $y \geq 0$ (product of Tsallis)

then one verify that:

$$(\alpha) \quad e_q^{(x+y)} = e_q^x \otimes_q e_q^y$$

$$(\beta) \quad e_q^x \cdot e_q^y = e_q^{[x+y+(1-q)xy]}$$

$$(\gamma)! \quad e_q^x \cdot e_{(2-q)}^{-x} = 1.$$

Def: q-entropy of TSALLIS (2002) (HAVRDA-CHARVAT (1967) Cech math

Entropy	SHannon		HC
Formula =	$-\sum_{i=1}^{N} p_i \ln p_i$	$-\sum p_i \ln_q(p_i)$	$= S(P)$

Here $\sum_{i=1}^{N} p_i = 1$, $p_i \geq 0$, $P = (p_1, p_2, \ldots, p_N)$

MAIN THEOREM (Arizmendi + Vargas + MB)

Let P, Q are projections in some prob. space $(\mathcal{A}, \varphi, \psi)$ which are $\underline{C}$-free,

If $(\varphi(P), \psi(P)) = (x_1, x_2)$

$(\varphi(Q), \psi(Q)) = (y_1, y_2)$, and

$\varphi(P \vee Q), \psi(P \vee Q) = (a, b)$, then

$$\frac{a}{b} = \left(\frac{x_1}{x_2} + \frac{y_1}{y_2} - 1\right)_+$$

Corollary: If $x_1 = x$, $x_2 = x^q$, $y_1 = y$, $y_2 = y^q$, $b = a^q$, then

$$a = T_q(x, y) = x \otimes_q y .$$

Ad 4)

Completely positive maps and quantum channels

Let $\mathcal{H}$ - Hilbert space ($\mathcal{H} = l^2(\mathbb{N}) = \{ x = (\alpha_1, \alpha_2, \dots) : \sum_j |\alpha_j|^2 = \|x\|^2 \}$)
separable

and $B(\mathcal{H}) = \{ T : \mathcal{H} \to \mathcal{H}$, linear bounded map $\}$

<u>Definition</u>: Let $\Phi : \mathcal{A} \to B(\mathcal{H}_0)$ - unital, c. positive map

$\mathcal{A}$ - $*$-sub-algebra of $B(\mathcal{H})$.

We call Φ <u>comp. positive</u>, if for $\xi_j \in \mathcal{H}_0$

$$\sum_{i,j=1}^{N} \langle \Phi(x) \xi_i | \xi_j \rangle \geq 0, \text{ for all } N = 1, 2, 3, \dots$$

<u>Th</u>: Φ is c. pos. if and only if $\forall n = 1, 2, 3, \dots$

$$\Phi_n : \mathcal{A}(M_n) \to B(\mathcal{H}_0 \otimes \mathbb{C}^n), \text{ defined}$$

$$\Phi_n (x_{ij}) \stackrel{\text{def}}{=} \left(Q(x_{ij}) \right)_{i,j=1}^{N} \text{ is}$$

<u>positive</u>.

Def: Quantum CHANNEL : $\mathcal{A}$ – $*$-sub-algebra of $B(\mathcal{H})$

Let $\Phi : \mathcal{A} \to B(\mathcal{H}_0)$, unital c.p. map,

$\psi : \mathcal{A} \to \mathbb{C}$, state, which is TRACE : $\psi(xy) = \psi(yx)$ $\forall x, y \in \mathcal{A}$.

Φ is called Quantum CHANNEL, if

$$\boxed{\psi(\Phi(x)) = \psi(x), \text{ for } x \in \mathcal{A}}$$

See book of HOLEVO for classical case

Typical EXAMPLES : q-second quantization channel MB (Studia Mat. '95)

Let $\mathcal{A} = \Gamma_q(\mathcal{H})$ – von Neumann algebra

generated by q-Gaussian r.v. $G_i = e_i + e_i^*$:

$$e_i e_j^* - q\, e_j^* e_i = \delta_{ij} I, \quad e_i(\Omega) = 0.$$

$\|\Omega\| = 1$ vacuum vect

Take a real contraction

$$T : \mathcal{H} \to \mathcal{H}, \quad \|T\| \leq 1, \text{ and define}$$

$$\Gamma_q(T) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}) \quad \text{as follows:}$$

$$\Gamma_q(T)\, W(f_1 \otimes \dots f_n) = W(T(f_1) \otimes \dots \otimes T f_n)$$

where $W(f_1 \otimes \dots \otimes f_n)\Omega = f_1 \otimes f_2 \otimes \dots \otimes f_n$

(WICK PRODUCT)!

If we take $\varphi(x) = \langle x\Omega | \Omega \rangle$, $x \in \Gamma_q(\mathcal{H})$, then

$(\Gamma_q(T), \varphi)$ is the quantum channel.

THEOREM (M. Bożejko, W. Bożejko) If $(\mathcal{A}_i, \Phi_i, \varphi_i)$ are

q. channels, $i \in I$, then the conditionally [on] FREE product $(* \mathcal{A}_i, \Phi, \varphi)$ is

again a q. channel,

Here $\Phi|_{\mathcal{A}_i} = \Phi_i$, $\varphi|_{\mathcal{A}_i} = \varphi_i$.

The proof uses the results of W. Młotkowski and B+L+S (Bożejko–Leinert–Speicher) Theory.

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Remarks on H-C (Tsallis) entropy)
(see I. Bengtsson + K. Życzkowski, Geometry of Q. States).

On page 58, [B+Ż] H-CH entropy is def. for a probability vector $p=(p_1, p_2, .., p_N)$:

$$S_q^{HC}(p) = \frac{1}{1-q}\left[\sum_{i=1}^{N} p_i^q - 1\right] =$$

$$= \frac{1}{1-q}\left[\sum_{i=1}^{N} p_i^q - \sum_{i=1}^{N} p_i\right] =$$

$$= \frac{1}{1-q}\left[\sum p_i \left(p_i^{q-1} - 1\right)\right] = \underline{-\sum_{i=1}^{N} p_i \ln_q(p_i)}$$

Fact: H-C (Tsallis) entropy is concave for $\underline{q>0}$.

Fact: $S_q(P) = \frac{1}{1-q}\left(\sum p_i^q - 1\right) = -\left[D_q\left(\sum p_i^x\right)\right]_{x=1}$

where „multiplicative" Jackson q-derivation:

$$D_q(f(x)) = \frac{f(qx) - f(x)}{qx - x} = \frac{f(x) - f(qx)}{x(1-q)}$$