

Macdonald Functions and Quantum Dilogarithm

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- Quantum dilogarithm : def and props
- Macdonald functions
- The case of SL_2 : dualities and reps.
- General case (SL_N)
- Q-operators and conjectures

QUANTUM DILOGARITHM

THE QUANTUM DILOGARITHM

Def:

$$E(z) = \prod_{n=0}^{\infty} (1 - q^n z)$$

$q \in \mathbb{C}^*$ fixed
 $|q| < 1$

Why the name? $q \rightarrow 1$ classical limit

$$q = e^{-\varepsilon}, \quad \varepsilon \rightarrow 0 :$$

$$\begin{aligned} E(z) &\sim e^{\sum_{n=0}^{\infty} \log(1 - q^n z)} = e^{-\sum_{p \geq 1} \sum_{n=0}^{\infty} \frac{(q^n z)^p}{p}} \\ &= e^{\sum_{p \geq 1} \frac{z^p}{p(1 - q^p)}} \sim e^{\frac{1}{\varepsilon} \sum_{p \geq 1} \frac{x^p}{p^2}} = e^{\frac{1}{\varepsilon} \text{Li}_2(x)} \end{aligned}$$

A.K.A. QUANTUM EXPONENTIAL

$$\frac{1}{E(z)} = \sum_{n=0}^{\infty} \frac{z^n}{(q)_n}$$

$$\begin{aligned} (q)_n &= (1-q)(1-q^2)\cdots(1-q^n) \\ &= [n]_q! \\ &= E(q)/E(q^{n+1}) \end{aligned}$$

• also = q -analogue of Γ -function:

$$\frac{1}{E(qz)} = (1-z) \frac{1}{E(z)}$$

• gives a proof of above: $\sum_{n=0}^{\infty} \frac{z^n - z^{n+1}}{(q)_n} = \sum_{n=0}^{\infty} z^n \left(\frac{1}{(q)_n} - \frac{1}{(q)_{n+1}} \right) = \sum_{n=0}^{\infty} \frac{(zq)^n}{(q)_n}$
by parts (Abel)

A q-BINOMIAL FORMULA

$$\frac{E(t^x)}{E(t)} = \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n} t^n \quad ; \quad (x)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \\ = E(x)/E(q^n x)$$

Proof: let $T \equiv T_{q,x}$ = "shift operator": $Tf(x) = f(qx)$

then: $((1-x)T)^n = (x)_n T^n$; $T \cdot 1 = 1$

$$\sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n} t^n = \frac{1}{E(t(1-x)T)} \cdot 1$$

$$\Rightarrow \frac{E(t)}{E(tx)} \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n} t^n = \underbrace{\left[E(t(1-x)T) E(tx) \right]^{-1}}_q \cdot E(t)$$

A QUANTUM PROPERTY

$$u, v \in \mathcal{A} ; uv = quv \quad (\text{Heisenberg}).$$

then

$$E(u)E(v) = E(u+v)$$

Proof:

$$\frac{1}{E(u)E(v)} = \sum_{n,m=0}^{\infty} \frac{v^n u^m}{(q)_n (q)_m}$$

$$\binom{n+m}{n}_q = \frac{(q)_{n+m}}{(q)_n (q)_m}$$

$$\sum_{n,m=0}^{\infty} \frac{1}{(q)_{n+m}} \underbrace{\binom{n+m}{n}_q v^n u^m}_{(u+v)^{n+m}} = \frac{1}{E(u+v)}$$

(q-binomial formula)

Apply to previous case:

$$E(\underbrace{t(1-x)}_u T) E(\underbrace{tx}_v) = E(\underbrace{tT}_u + \underbrace{tx(1-T)}_v) = E(tT) E(tx(1-T))$$

Finally:

$$\begin{aligned} \frac{E(t)}{E(tx)} \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n} t^n &= [E(t(1-x)T)E(tx)]^{-1} \cdot E(t) \\ &= \underbrace{E(tx(1-T))^{-1}}_1 \underbrace{E(tT)^{-1}}_1 \cdot E(t) \end{aligned}$$

□

Lesson 1: use non-commuting arguments!

PENTAGON IDENTITY

$u, v \in \mathcal{A}$, $uv = qvu$, then:

$$E(u) E(-q^{-1}uv) E(v) = E(v) E(u)$$

Proof: by quantum property: $E(u) E(v(1-u)) \frac{1}{E(u)} = E(v)$

$$\frac{1}{E(v(1-u))} \frac{1}{E(u)} = \sum_{n=0}^{\infty} \frac{(v(1-u))^n}{(q)_n} \frac{1}{E(u)} = \sum_{n=0}^{\infty} \frac{v^n}{(q)_n} \frac{(u)_n}{E(u)} = \sum_{n=0}^{\infty} \frac{v^n}{(q)_n} \frac{1}{E(q^n u)}$$

$$\Rightarrow E(u) \frac{1}{E(v(1-u))} \frac{1}{E(u)} = \sum_{n=0}^{\infty} \frac{1}{(q)_n} \underbrace{E(u) v^n}_{v^n} \frac{1}{E(q^n u)} = \frac{1}{E(v)} \quad \square$$

MACDONALD FUNCTIONS

MACDONALD OPERATORS (A_{N-1})

$\{D_a, a=1, 2, \dots, N\}$ acting on functions $f(x_1, x_2, \dots, x_N)$

$$D_a = \sum_{\substack{I \subset \{1, 2, \dots, N\} \\ |I| = a}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t^{x_i - x_j}}{x_i - x_j} \cdot \underbrace{\prod_{\ell \in I} T_{q, x_\ell}}_{\text{order } a \text{ shift operator}}$$

NB: $D_N = T_{q, x_1} T_{q, x_2} \dots T_{q, x_N}$; $D_0 = 1$ (convention)

$$\textcircled{\text{Thm}} [D_a, D_b] = 0 \quad \forall a, b \in \{1, 2, \dots, N\}$$

Nice formulation: DAHA [Cherednik]

MACDONALD POLYNOMIALS

$P_\lambda(x|q, t)$ monic polynomials $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0\}$
integer partition

common eigenfunctions of the $\{D_a\}_{a=1}^N$, symmetric in $(x_1, \dots, x_N)$

$$D_a P_\lambda(x|q, t) = e_a(s) P_\lambda(x|q, t)$$

↑
elementary symmetric functions of

$$S = (s_i)_{i=1}^N; s_i = t^{N-i} q^{\lambda_i}$$

lots of combinatorics, still not quite there yet...

(DUAL) PIERI RULES

$$e_a(\times) P_\lambda(\times|q, t) = H_a P_\lambda(\times|q, t)$$

↑
operator acting by shifts of λ (or $s = (t^{N-i} q^{s_i})$)

$$H_a = \sum_{\substack{I \subset \{1, 2, \dots, N\} \\ |I| = a}} \prod_{\substack{i \in I \\ j \notin I \\ i > j}} \frac{s_j - t s_i}{s_j - s_i} \frac{s_j - \frac{q}{t} s_i}{s_j - q s_i} \prod_{i \in I} \frac{1}{q^{s_i}}$$

Pieri Operators, AKA Commuting Hamiltonians

UNIVERSAL (Harish-Chandra) SERIES

SOLUTION

look for series solutions of Macdonald eigenvalue eqn,
avg in sector where $\left| \frac{x_{i+1}}{x_i} \right| < 1 \quad i=1, 2, \dots, N-1$, s_i generic:

$$= e^{\frac{\log x \log(s)}{\log q}} f(x, s | q, t) ; f = 1 + O\left(\left\{ \frac{x_{i+1}}{x_i} \right\}_{i=1}^{N-1}\right)$$

NB: f not symmetric of x !

$$\sum_{\substack{I \subset \{1, 2, \dots, N\} \\ |I| = a}} \prod_{m \in I} s_m \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \cdot \prod_{\ell \in I} T_{q, x_\ell} f = e_a(s) f$$

$D_a(x, s)$ ↙

what have we gained?

Facts:

- $a=1$ is enough : determines f uniquely via a triangular non-singular system
- f specializes to Macdonald polynomials $P_\lambda(x)$ when $s_i = t^{N-i} q^{\lambda_i}$ $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n \geq 0$ integer partition

- f has a very nice expression found by [Shiraishi-Noumi 2011] of q -hypergeometric flavor

- Also $\rightarrow$ related to Nekrasov Susy Gauge theory function [Shiraishi et al, last decade]

- Duality (ies!)

Pieri Rules for $f(x, s | q, t)$

$$H_a = \sum_{\substack{I \subset \{1, 2, \dots, N\} \\ |I| = a}} \prod_{\substack{i \in I \\ j \notin I \\ i > j}} \frac{s_j - t s_i}{s_j - s_i} \frac{s_j - \frac{q}{t} s_i}{s_j - q s_i} \prod_{i \in I} \prod_{j, s_i} t$$

$$H_a(s, x) f(x, s | q, t) = e_a(x) f(x, s | q, t)$$

$$H_a(s, x) = \sum_{\substack{I \subset \{1, 2, \dots, N\} \\ |I| = a}} \prod_{i \in I} x_i \prod_{\substack{i \in I \\ j \notin I \\ i > j}} \frac{s_j - t s_i}{s_j - s_i} \frac{s_j - \frac{q}{t} s_i}{s_j - q s_i} \prod_{i \in I} \prod_{j, s_i} t$$

Example of Sl_2

$$D_1(x) = \frac{t^{x_1-x_2}}{x_1-x_2} \Gamma_1 + \frac{t^{x_2-x_1}}{x_2-x_1} \Gamma_2 \quad \Gamma_i: x_i \rightarrow qx_i$$

$$\left(t - \frac{x_2}{x_1}\right) F(qx_1, x_2) - \left(t \frac{x_2}{x_1} - 1\right) F(x_1, qx_2) = (s_1 + s_2) \left(1 - \frac{x_2}{x_1}\right) F(x_1, x_2)$$

$$F(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} \sum_{n=0}^{\infty} C_n(s_1, s_2) \left(\frac{x_2}{x_1}\right)^n \quad (s_1 = tq^{\lambda_1}; s_2 = q^{\lambda_2})$$

coefft of $\left(\frac{x_2}{x_1}\right)^n$: $\phi(x, s)$

$$(tq^{\lambda_1} C_n - C_{n-1} q^{\lambda_1+1}) q^{-n} + (q^{\lambda_2} C_n - tq^{\lambda_2-1} C_{n-1}) q^n = (s_1 + s_2) (C_n - C_{n-1})$$

$$[s_1(1 - q^{-n}) + s_2(1 - q^n)] C_n = [s_1(1 - \frac{q^{1-n}}{t}) + s_2(1 - tq^{n-1})] C_{n-1}$$

$$C_m = \prod_{n=1}^m \left(\frac{1 - tq^{n-1}}{1 - q^n} \frac{1 - tq^{n-1} s_2/s_1}{1 - q^n s_2/s_1} \right) = \frac{(t)_n}{(q)_n} \frac{(ts_2/s_1)_n}{(qs_2/s_1)_n} \left(\frac{q}{t}\right)^n$$

Example of Sl_2 (continued)

$$f(x, s | q, t) = \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n} \frac{(ts)_n}{(qs)_n} \left(\frac{q}{t} x \right)^n$$

$$= {}_2\phi_1 \left(\begin{matrix} t & ts \\ qs \end{matrix} \middle| \frac{q}{t} x \right)$$

$\left(\begin{matrix} x = \frac{x_2}{x_1} \\ s = s_2/s_1 \end{matrix} \right)$

$s_1 = tq^{\lambda_1}$
 $s_2 = q^{\lambda_2}$

$s = \frac{s_2}{s_1} = t^{-1} q^{\lambda_2 - \lambda_1} \geq 0$

q -hypergeometric series !

Here comes the quantum dilogarithm

$$f(x, s | q, t) = \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n} \frac{(ts)_n}{(qs)_n} \left(\frac{q}{t} x \right)^n$$

$$= \frac{E\left(qx \frac{1-ts}{1-qs} T\right)}{E\left(\frac{q}{t}x \frac{1-ts}{1-qs} T\right)} \cdot 1$$

Mellin-Barnes

$$T = T_{q,s} \quad ; \quad Ts = qsT$$

Proof:

$$\text{use } \frac{E(ab)}{E(a)} = \sum_{n=0}^{\infty} \frac{(b)_n}{(q)_n} a^n, \text{ with } \begin{cases} a = \frac{q}{t} \times \frac{1-ts}{1-qs} T \\ b = t \end{cases}$$

$$\begin{aligned} \frac{E\left(\frac{q}{t} \times \frac{1-ts}{1-qs} T\right)}{E\left(\frac{q}{t} \times \frac{1-ts}{1-qs} T\right)} \cdot 1 &= \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n} \left(\frac{q}{t} \times \frac{1-ts}{1-qs} T\right)^n \cdot 1 \\ &= \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n} \frac{(ts)_n}{(qs)_n} \left(\frac{q}{t}\right)^n \end{aligned} \quad \square$$

Use this to prove properties of f ?

DUALITIES

$$f(x, s | q, t) = \frac{E(qx)}{E(\frac{q}{t}x)} \frac{E(\frac{q}{t}s)}{E(qs)} f(s, x | q, t)$$

BISPECTRAL DUALITY

interchange of x, s variables

$$f(x, s | q, t) = \frac{E(tx)}{E(\frac{q}{t}x)} f(x, s | q, \frac{q}{t})$$

POINCARÉ DUALITY

$t \leftrightarrow \frac{q}{t}$ (cf. Ha)

Ex Proof of Bispectral+Poincaré duality $\Gamma := T_{q,x}$

$$\frac{E(qx \frac{1-ts}{1-qs} T)}{E(\frac{q}{t}x \frac{1-ts}{1-qs} T)} \cdot 1 = \frac{E(qx)}{E(\frac{q}{t}x)} \frac{E(ts)}{E(qs)} \frac{E(qs \frac{1-\frac{q}{t}x}{1-qx} \Gamma)}{E(ts \frac{1-\frac{q}{t}x}{1-qx} \Gamma)} \cdot 1$$

$$\Leftrightarrow \frac{E(qx \frac{1-ts}{1-qs} T)}{E(\frac{q}{t}x \frac{1-ts}{1-qs} T)} \cdot \frac{E(\frac{q}{t}x)}{E(qx)} = \frac{E(qs \frac{1-\frac{q}{t}x}{1-qx} \Gamma)}{E(\frac{q}{t}s \frac{1-\frac{q}{t}x}{1-qx} \Gamma)} \cdot \frac{E(ts)}{E(qs)}$$



$$[x T, s \Gamma] = 0$$

Commute both actions!

$$\frac{E(q^x \frac{1-t^s}{1-q^s} T)}{E(\frac{q}{t}^x \frac{1-t^s}{1-q^s} T)} \cdot \frac{E(\frac{q}{t}^x)}{E(q^x)} = \frac{E(q^s \frac{1-\frac{q}{t}^x}{1-q^x} \Gamma)}{E(t^s \frac{1-\frac{q}{t}^x}{1-q^x} \Gamma)} \cdot \frac{E(t^s)}{E(q^s)}$$



$$\frac{E(t^s \frac{1-\frac{q}{t}^x}{1-q^x} \Gamma)}{E(q^s \frac{1-\frac{q}{t}^x}{1-q^x} \Gamma)} \cdot \frac{E(\frac{q}{t}^x)}{E(q^x)} = \frac{E(\frac{q}{t}^x \frac{1-t^s}{1-q^s} T)}{E(q^x \frac{1-t^s}{1-q^s} T)} \cdot \frac{E(t^s)}{E(q^s)}$$

$$\frac{1-\frac{q}{t}^x}{1-q^x} \Gamma \frac{E(\frac{q}{t}^x)}{E(q^x)} = \frac{E(\frac{q}{t}^x)}{E(q^x)} \Gamma ; \quad \frac{1-t^s}{1-q^s} T \frac{E(t^s)}{E(q^s)} = \frac{E(t^s)}{E(q^s)} T$$

suppression of all rational fractions!

$$\frac{E\left(t_s \frac{1-q_t^x}{1-q_x} \Gamma\right) \cdot \frac{E(q_t x)}{E(q x)}}{E\left(q_s \frac{1-q_t^x}{1-q_x} \Gamma\right) E(q x)} = \frac{E\left(\frac{q_x}{t} \frac{1-t_s}{1-q_s} T\right) \cdot \frac{E(t \cdot s)}{E(q s)}}{E\left(q x \frac{1-t_s}{1-q_s} T\right) E(q s)}$$

$$\Rightarrow \frac{E(q_t x) E(t_s \Gamma)}{E(q x) E(q_s \Gamma)} \cdot 1 = \frac{E(t \cdot s) E\left(\frac{q_x}{t} T\right)}{E(q s) E(q x T)} \cdot 1$$

$$\Leftrightarrow \frac{E(q_t x) E(t_s)}{E(q x) E(q_s)} = \frac{E(t \cdot s) E\left(\frac{q_x}{t}\right)}{E(q s) E(q x)}$$

qed $\square$

ALTERNATIVE (USING q-hypergeometric theory)

Heine Transformations

$${}_2\phi_1\left(\begin{matrix} a & b \\ c \end{matrix} \middle| z\right) = \frac{E(a)E(bz)}{E(c)E(z)} {}_2\phi_1\left(\begin{matrix} \frac{c}{a} & z \\ bz \end{matrix} \middle| a\right) \quad (H1)$$

$$= \frac{E(c/a)E(az)}{E(c)E(z)} {}_2\phi_1\left(\begin{matrix} \frac{abz}{c} & a \\ az \end{matrix} \middle| \frac{c}{a}\right) \quad (H2)$$

$$= \frac{E(abz/c)}{E(z)} {}_2\phi_1\left(\begin{matrix} \frac{c}{b} & \frac{c}{a} \\ c \end{matrix} \middle| \frac{abz}{c}\right) \quad (H3)$$

(H1) $\Rightarrow$ Bispectral duality

(H3) $\Rightarrow$ Poincaré duality

NB: order 6

q-WHITTAKER LIMIT

$$s_i = t^{N-i} q^{\lambda_i}$$

$\lambda_1 \geq \lambda_2 \dots \geq \lambda_n \geq 0$ integer partition

$t \rightarrow \infty \Rightarrow s_i$ dominates

$$t^{-\frac{a(a-1)}{2}} D_a(x, s) \xrightarrow{t \rightarrow \infty}$$

$$D_a^\infty(x, q^\lambda) = \sum_{\substack{I \subset \{1, \dots, N\} \\ |I| = a}} q^{\sum_{i \in I} \lambda_i} \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i}{x_i - x_j} \prod_{i \in I} T_{q, x_i}$$

$$t^{-\frac{a(a-1)}{2}} e_a(s) \xrightarrow{t \rightarrow \infty} q^{\sum_{i=1}^a \lambda_i}$$

q-Whittaker universal series

$$g(x, q^\lambda) \propto \lim_{t \rightarrow \infty} f(x, s_i = t^{N-i} q^{\lambda_i} | q, t)$$

$$D_a^\infty(x, q^\lambda) \cdot g(x, q^\lambda) = q^{\sum_{i=1}^a \lambda_i} g(x, q^\lambda)$$

$$g(x, q^\lambda) = 1 + O\left(\left\{\frac{x_{i+1}}{x_i}\right\}_{i=1}^{N-1}\right)$$

Why the name?

Pieri Rules = q-Toda equation

$$H_a^t(s, x) \rightarrow H_a^\infty(q^\lambda, x) \quad \text{keep } T_i = T_{q, q}^{\lambda_i} = \text{shift } \lambda_i \rightarrow \lambda_i + 1$$

$$H_a^\infty(q^\lambda, x) = \sum_{\substack{I \subset \{1, \dots, N\} \\ |I| = a}} \prod_{i \in I} x_i \prod_{\substack{\bar{i} \in I \\ i-1 \notin I}} (1 - q^{\lambda_{\bar{i}} - \lambda_{i-1}}) \prod_{i \in I} T_i$$

$a=1$ Open q Toda Hamiltonian for SL_N [eg Etingof]

$$H_1^\infty(q^\lambda, x) = \sum_{\bar{i}=1}^N x_i T_i - \sum_{\bar{i}=2}^N x_i q^{\lambda_{\bar{i}} - \lambda_{i-1}} T_i$$

$H_a^\infty =$ commuting family of higher Hamiltonians

Example of SL_2 (keep $\lambda_i \in \mathbb{C}$)

$$f(x, s|q, t) = \sum_{n=0}^{\infty} \frac{(t)_n}{(q)_n} \frac{(ts)_n}{(qs)_n} \left(\frac{q}{t}x\right)^n$$

q -Whittaker limit: $\begin{cases} s_1 = tq^{\lambda_1} \\ s_2 = q^{\lambda_2} \end{cases} \Rightarrow s = \frac{s_2}{s_1} = t^{-1} q^{\lambda_2 - \lambda_1} = t^{-1} \sigma$
 $T = T_{q, \sigma}$

$$\lim_{t \rightarrow \infty} f(x, s|q, t)$$

$$= g(x, \sigma|q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n} (\sigma)_n \cdot x^n$$

NB: no longer a q -hypergeometric

Properties:

- When $\sigma = q^{-n}$ $n \in \mathbb{Z}_+$, the series truncates to the q -Whittaker polynomials

$$\begin{aligned} W_n(x) &= \sum_{k=0}^n \frac{(-1)^k q^{k(k+1)/2}}{(q)_k} (q^{-n})_k x^k \\ &= \sum_{k=0}^n \binom{n}{k}_{q^{-1}} x^k \end{aligned}$$

usually $W_{\lambda_1, \lambda_2}(x) = x_1^{\lambda_1} x_2^{\lambda_2} W_{n=\lambda_1-\lambda_2}(x)$

Quantum Dilogarithm expression:

$$\begin{aligned} g(x, \sigma | q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n} (\sigma)_n \cdot x^n \\ &= E(qx(1-\sigma)T) \cdot 1 \end{aligned}$$

Note: from $E(ab)/E(a)$ take $a \rightarrow 0$ $ab = x$ fixed

$$\Rightarrow E(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q)_n} x^n$$

Open q-Toda equation:

$$x = \frac{x_2}{x_1} \quad \left\{ \begin{array}{l} T_1 = T_{q,\sigma}^{-1} \\ T_2 = T_{q,\sigma} \end{array} \right.$$

$$\{ T^{-1} + x(1-\sigma)T \} g(x, \sigma | q) = (1+x) g(x, \sigma | q)$$

Direct Proof: $g = E(q x (1-\sigma)T) \cdot 1$

$$T^{-1} E(x(1-\sigma)T) \cdot 1 = E(xT) E(-q^{-1} x \sigma T) \cdot 1$$

$$E(x(1-\sigma)T) \cdot 1 = E(xT) E(-x \sigma T) \cdot 1$$

$$(T^{-1} - 1) E(x(1-\sigma)T) \cdot 1 = E(xT) \underbrace{(q^{-1} x \sigma T)}_{\leftarrow} \underbrace{E(-x \sigma T)}_{\rightarrow} \cdot 1$$

$$= (q^{-1} \sigma) E(q x T) E(-q x \sigma T) \cdot 1$$

$$= q^{-1} \sigma E(q x (1-\sigma)T) \cdot 1$$

$$(T^{-1}-1)E(x(1-\sigma)T) \cdot 1 = q^{-1}\sigma E(qx(1-\sigma)T) \cdot 1$$

$$E(x(1-\sigma)T) = [1-x(1-\sigma)T]E(qx(1-\sigma)T)$$

$$\Leftrightarrow (T^{-1}-1)(1-x(1-\sigma)T)g = q^{-1}\sigma g$$

$$\Leftrightarrow [T^{-1} + x(1-\sigma)T]g = (1+x)g \quad \square$$

lesson 2 = E allows to formulate purely algebraic proofs !

GENERAL SL_N CASE

[Shiraishi -
Noumi 11]

$$f_N(\mathbf{x}, \mathbf{s} | q, t) = \sum_{\substack{\theta_{i,j} \geq 0 \\ 1 \leq i < j \leq N}} c(\{\theta_{i,j}\}) \prod_{1 \leq i < j \leq N} \left(\frac{x_j}{x_i} \right)^{\theta_{i,j}}$$

$$c(\{\theta_{i,j}\}) = \prod_{a=1}^N \prod_{i=1}^{a-1} \prod_{j=i+1}^a \frac{(q^{\sum_{b>a}(\theta_{i,b}-\theta_{j,b})} t s_j / s_i)_{\theta_{i,a}}}{(q^{\sum_{b>a}(\theta_{i,b}-\theta_{j,b})} q s_j / s_i)_{\theta_{i,a}}} \prod_{j=i}^{a-1} \frac{(q^{-\theta_{j,a} + \sum_{b>a}(\theta_{i,b}-\theta_{j,b})} \frac{q}{t} s_j / s_i)_{\theta_{i,a}}}{(q^{-\theta_{j,a} + \sum_{b>a}(\theta_{i,b}-\theta_{j,b})} s_j / s_i)_{\theta_{i,a}}}$$

THM $f_N(\mathbf{x}, \mathbf{s} | q, t)$ is the HC series solution to
the Macdonald eqs for SL_N

q -Hypergeometric steroids!

DUALITIES

[Shiraishi -
Noumi 11]

$$\text{let } \Delta(x) := \prod_{1 \leq i < j \leq N} \frac{E(q \frac{x_i}{x_j})}{E(\frac{q}{t} \frac{x_i}{x_j})}$$

$$\Delta(s) f_N(x, s | q, t) = \Delta(x) f_N(s, x | q, t)$$

BISPECTRAL

$$\varphi_N(x, s | q, t) := \frac{f_N(x, s | q, t)}{\Delta(x)} = \varphi_N(x, s | q, \frac{q}{t})$$

POINCARÉ

$$t \leftrightarrow \frac{q}{t}$$

Remark:

$$\begin{aligned}
 c(\{\theta_{i,j}\}) &= \frac{(q^{-\theta_{j,a}} q/t)_{\theta_{i,a}}}{(q^{-\theta_{j,a}})_{\theta_{i,a}}} \prod_{a=1}^N \prod_{i=1}^{a-1} \prod_{j=i+1}^a \frac{(q^{\sum_{b>a}(\theta_{i,b}-\theta_{j,b})} t s_j/s_i)_{\theta_{i,a}}}{(q^{\sum_{b>a}(\theta_{i,b}-\theta_{j,b})} q s_j/s_i)_{\theta_{i,a}}} \prod_{j=i+1}^{a-1} \frac{(q^{-\theta_{j,a} + \sum_{b>a}(\theta_{i,b}-\theta_{j,b})} \frac{q}{t} s_j/s_i)_{\theta_{i,a}}}{(q^{-\theta_{j,a} + \sum_{b>a}(\theta_{i,b}-\theta_{j,b})} s_j/s_i)_{\theta_{i,a}}} \\
 &= \underbrace{\left(\frac{q}{t}\right)^{\theta_{i,a}} \frac{(t)_{\theta_{i,a}}}{(q)_{\theta_{i,a}}}}_{\text{}} \prod_{a=1}^N \prod_{i=1}^{a-1} \prod_{j=i+1}^a \frac{(q^{\sum_{b>a}(\theta_{i,b}-\theta_{j,b})} t s_j/s_i)_{\theta_{i,a}}}{(q^{\sum_{b>a}(\theta_{i,b}-\theta_{j,b})} q s_j/s_i)_{\theta_{i,a}}} \prod_{j=i+1}^{a-1} \frac{(q^{-\theta_{j,a} + \sum_{b>a}(\theta_{i,b}-\theta_{j,b})} \frac{q}{t} s_j/s_i)_{\theta_{i,a}}}{(q^{-\theta_{j,a} + \sum_{b>a}(\theta_{i,b}-\theta_{j,b})} s_j/s_i)_{\theta_{i,a}}}
 \end{aligned}$$

$$\frac{E(q^x)}{E(\frac{q}{t} x)} = \sum_{n \geq 0} \frac{(t)_n}{(q)_n} \left(\frac{q}{t} x\right)^n \Rightarrow \text{each } \theta_{i,a} = \text{ratio of diloqs!}$$

$$\prod_{a=N}^2 \prod_{i=1}^{a-1} \frac{E\left(q \frac{x_a}{x_i} \prod_{j=i+1}^a \frac{1-t s_j/s_i}{1-q s_j/s_i} \prod_{j=i+1}^{a-1} \frac{1-\frac{q}{t} \frac{s'_j}{s_i}}{1-\frac{s'_j}{s_i}} T_{s_i}^{-1}\right)}{E\left(\frac{q}{t} \frac{x_a}{x_i} \prod_{j=i+1}^a \frac{1-t s_j/s_i}{1-q s_j/s_i} \prod_{j=i+1}^{a-1} \frac{1-\frac{q}{t} \frac{s'_j}{s_i}}{1-\frac{s'_j}{s_i}} T_{s_i}^{-1}\right)} \cdot 1?$$

$$= \sum_{\theta_{i,a} \geq 0} \left(\frac{q}{t}\right)^{\theta_{i,a}} \frac{(t)_{\theta_{i,a}}}{(q)_{\theta_{i,a}}} (\dots)^{\theta_{i,a}}$$

THM
[DF]

$$f_N(x, s | q, t) =$$

$$\prod_{a=N}^2 \int ds' \prod_{i=1}^{a-1} \frac{E \left(q \frac{x_a}{x_i} \prod_{j=i+1}^a \frac{1-ts_j/s_i}{1-qs_j/s_i} \prod_{j=i+1}^{a-1} \frac{1-\frac{q}{t} s'_j/s_i}{1-s'_j/s_i} T_{s_i}^{-1} \right)}{E \left(\frac{q}{t} \frac{x_a}{x_i} \prod_{j=i+1}^a \frac{1-ts_j/s_i}{1-qs_j/s_i} \prod_{j=i+1}^{a-1} \frac{1-\frac{q}{t} s'_j/s_i}{1-s'_j/s_i} T_{s_i}^{-1} \right)} \prod_{\bar{i}=2}^{N-1} \delta(s'_i - s_{\bar{i}})$$

Proof: the integral implements "retroaction"

$$\int ds' \phi(s') T_{q,s} \delta(s' - s)$$

$$= \phi(qs) T_{q,s}$$

A FIRST CONSEQUENCE: RANK RECURSION

$$f_N(x, s | q, t) =$$

$$\int ds' \prod_{i=1}^{N-1} \frac{E \left(q \frac{x_N}{x_i} \prod_{j=i+1}^N \frac{1-ts_j/s_i}{1-qs_j/s_i} \prod_{j=i+1}^{N-1} \frac{1-\frac{q}{t}s'_j/s_i}{1-s'_j/s_i} T_{s_i}^{-1} \right)}{E \left(\frac{q}{t} \frac{x_N}{x_i} \prod_{j=i+1}^N \frac{1-ts_j/s_i}{1-qs_j/s_i} \prod_{j=i+1}^{N-1} \frac{1-\frac{q}{t}s'_j/s_i}{1-s'_j/s_i} T_{s_i}^{-1} \right)} \prod_{i=2}^{N-1} \delta(s'_i - s_i)$$

$$\times f_{N-1}(x_1, \dots, x_{N-1}; s_1, \dots, s_{N-1} | q, t)$$

EXAMPLE OF SL_3

$$\begin{aligned}
 f_3(\mathbf{x}, \mathbf{s} | q, t) &= \sum_{n_1, n_2, n_3 \geq 0} \frac{(t)_{n_1}}{(q)_{n_1}} \frac{(\frac{q}{t} q^{-n_2} s_2 / s_1)_{n_1}}{(q^{-n_2} s_2 / s_1)_{n_1}} \frac{(ts_2 / s_1)_{n_1}}{(qs_2 / s_1)_{n_1}} \frac{(ts_3 / s_1)_{n_1}}{(qs_3 / s_1)_{n_1}} \\
 &\quad \times \frac{(t)_{n_2}}{(q)_{n_2}} \frac{(ts_3 / s_2)_{n_2}}{(qs_3 / s_2)_{n_2}} \frac{(t)_{n_3}}{(q)_{n_3}} \frac{(\frac{q}{t} q^{n_1 - n_2} s_2 / s_1)_{n_3}}{(q^{n_1 - n_2} s_2 / s_1)_{n_3}} \left(\frac{q}{t} \frac{x_3}{x_1} \right)^{n_1} \left(\frac{q}{t} \frac{x_3}{x_2} \right)^{n_2} \left(\frac{q}{t} \frac{x_2}{x_1} \right)^{n_3} \\
 &= \int ds'_2 \frac{E \left(q \frac{x_3}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} \frac{1-ts_3/s_1}{1-qs_3/s_1} \frac{1-q/ts'_2/s_1}{1-s'_2/s_1} T_{s_1}^{-1} \right)}{E \left(\frac{q}{t} \frac{x_3}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} \frac{1-ts_3/s_1}{1-qs_3/s_1} \frac{1-q/ts'_2/s_1}{1-s'_2/s_1} T_{s_1}^{-1} \right)} \frac{E \left(q \frac{x_3}{x_2} \frac{1-ts_3/s_2}{1-qs_3/s_2} T_{s_2}^{-1} \right)}{E \left(\frac{q}{t} \frac{x_3}{x_2} \frac{1-ts_3/s_2}{1-qs_3/s_2} T_{s_2}^{-1} \right)} \delta(s'_2 - s_2) \underbrace{\frac{E \left(q \frac{x_2}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} T_{s_1}^{-1} \right)}{E \left(\frac{q}{t} \frac{x_2}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} T_{s_1}^{-1} \right)}}_{SL_2} \cdot 1
 \end{aligned}$$

RELATION TO MACDONALD POLYNOMIALS

Specialize $(s_1 = t^2 q^{\lambda_1}, s_2 = t q^{\lambda_2}, s_3 = q^{\lambda_3})$
 $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ integer partition

$$\Rightarrow P_{\lambda_1, \lambda_2, \lambda_3}(x) = x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} f_3(x, s | q, t)$$

① finite truncation of the series easy

② $P_{\text{symmetric}}$ non-trivial

q-Whittaker limit $t \rightarrow \infty$

$$\int ds'_2 \frac{E\left(q \frac{x_3}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} \frac{1-ts_3/s_1}{1-qs_3/s_1} \frac{1-q/ts'_2/s_1}{1-s'_2/s_1} T_{s_1}^{-1}\right) E\left(q \frac{x_3}{x_2} \frac{1-ts_3/s_2}{1-qs_3/s_2} T_{s_2}^{-1}\right)}{E\left(\frac{q}{t} \frac{x_3}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} \frac{1-ts_3/s_1}{1-qs_3/s_1} \frac{1-q/ts'_2/s_1}{1-s'_2/s_1} T_{s_1}^{-1}\right) E\left(\frac{q}{t} \frac{x_3}{x_2} \frac{1-ts_3/s_2}{1-qs_3/s_2} T_{s_2}^{-1}\right)} \delta(s'_2 - s_2) \frac{E\left(q \frac{x_2}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} T_{s_1}^{-1}\right)}{E\left(\frac{q}{t} \frac{x_2}{x_1} \frac{1-ts_2/s_1}{1-qs_2/s_1} T_{s_1}^{-1}\right)} \cdot 1$$

$$(s_1 = t^2 \sigma_1, s_2 = t \sigma_2, s_3 = \sigma_3) \quad \sigma_i = q^{\lambda_i} \quad (t \rightarrow \infty)$$

$$\rightarrow g(x, \sigma | q) =$$

$$E\left(q \frac{x_3}{x_1} \left(1 - \frac{\sigma_2}{\sigma_1}\right) T_1^{-1}\right) E\left(q \frac{x_3}{x_2} \left(1 - \frac{\sigma_3}{\sigma_2}\right) T_2^{-1}\right) E\left(q \frac{x_2}{x_1} \left(1 - \frac{\sigma_2}{\sigma_1}\right) T_1^{-1}\right) \cdot 1$$

Open Toda eqn: (Pieri) $\hookrightarrow$ cf [Schroder-Shapiro '18]

$$\{x_1 T_1^{-1} + x_2 \left(1 - \frac{\sigma_2}{\sigma_1}\right) T_2^{-1} + x_3 \left(1 - \frac{\sigma_3}{\sigma_2}\right) T_3^{-1}\} g = (x_1 + x_2 + x_3) g$$

$\rightarrow$ Direct Proof (same as before).

General case of q -Whittaker limit (SL_N)

- all integrals become trivial, pure product.
- Rank recursion:

$$g_N(x, \sigma | q, t) =$$

$$\left\{ \prod_{i=1}^{N-1} E\left(q \frac{x_N}{x_i} (1 - \frac{\sigma_{i+1}}{\sigma_i}) T_i^{-1}\right) \right\} g_{N-1}(x_1, \dots, x_{N-1}, \sigma_1, \dots, \sigma_{N-1} | q, t)$$

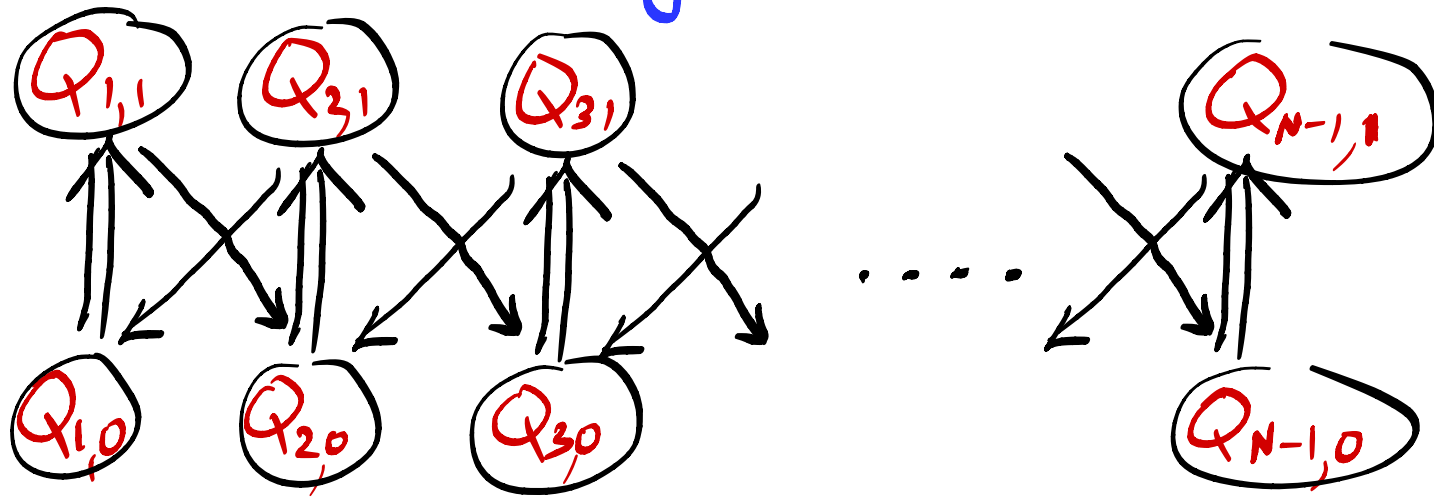
- Open Toda:

cf [Schrader-Shapiro '18]
[DF + Kedem '21]

$$\left\{ x_1 T_1^{-1} + \sum_{i=2}^N x_i (1 - \frac{\sigma_i}{\sigma_{i-1}}) T_i^{-1} \right\} g_N = \left(\sum_{i=1}^N x_i \right) g_N$$

EPILOGUE: Quantum Dilogarithm and Cluster Algebra

- q Toda operators and their Whittaker duals D_a^∞ are elements in the quantum Cluster Algebra of the A_{N-1} Q-system.



Mutation relation:
(A-variables)

$$Q_{i,n+1} Q_{i,n-1} = Q_{i,n}^2 - Q_{i+1,n} Q_{i-1,n}$$

Thm [DF+Kedem 15]

There is a representation of the solutions $Q_{a,n}$ of the quantum Q -system for A_{N-1} as difference operators:

$$Q_{a,n} = \sum_{\substack{I \subset \{1, \dots, N\} \\ |I|=a}} \left(\prod_{i \in I} x_i \right)^n \prod_{\substack{i \in I \\ j \notin I}} \frac{x_i}{x_i - x_j} \prod_{i \in I} T_{q, x_i}$$

$$q^a Q_{a,n+1} Q_{a,n-1} = Q_{a,n}^2 - Q_{a+1,n} Q_{a-1,n} \\ (a=1, 2, \dots, N-1, n \in \mathbb{Z})$$

Remark: $Q_{a,0} = D_a^\infty$ Whittaker limit of Macdonald operators!

Remark (bis):

$$Q_{a,n} = q^{-\frac{n}{2}} \gamma^{-n} D_a^\infty \gamma^n$$

where $\gamma := e^{\frac{\sum_{i=1}^n (\log x_i)^2}{2 \log q}}$ "Gaussian".

Proof: $\gamma^{-1} \Gamma \gamma = q^{\frac{1}{2}} \times \Gamma$ "dressing".

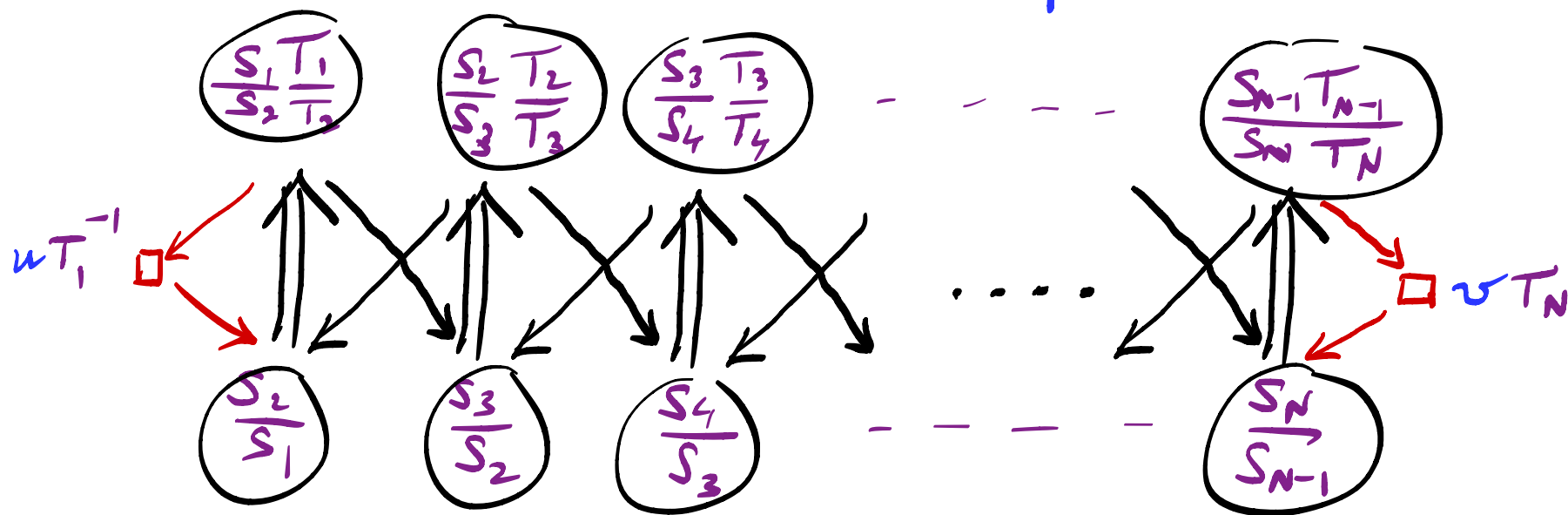
These are actually the "translates" of D_a^∞ under the $SL(2, \mathbb{Z})$ action on nDAHA by

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \text{Ad}_{\gamma^{-1}}$
in the functional representation.

x-variables:

$$X_i = \prod_j A_j^{B_{ij}^{-1}} \leftarrow \text{quiver adjacency}^{\text{inverse}}$$

$$X_i X_j = q^{B_{ij}} X_j X_i$$

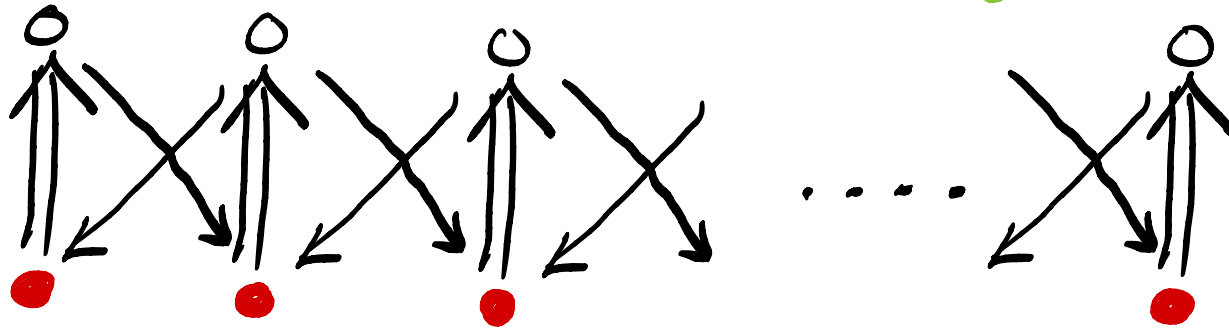


mutations are driven by $\text{Ad}_{\frac{1}{E(X_i)}}$

Cluster Groupoid = sequence of mutations preserving the quiver.

Example 1. Dehn Twist $\sim T \in SL(2, \mathbb{Z})$

[Schrader-Shapiro '18]
[DF-Kedem 21]



Corresponds to a product of dilogarithms :

$$\prod_{i=1}^{N-1} \frac{1}{L\left(\frac{s_{i+1}}{s_i}\right)}$$

followed by a permutation top $\leftrightarrow$ bottom vertices

$$\gamma(T) = e^{\frac{\sum_{i=1}^N (\log T_i)^2}{2 \log q}}$$

Fourier transform: (uses Whittaker series).

$$(\mathcal{Q}(\{x, T_{q,x}\})) g_N(x, s) = \hat{\mathcal{Q}}(\{s, T_{q,s}\}) g_N(x, s)$$

$$eg: \bullet (\hat{D}_a^\infty(x, s)) = s_1 s_2 \dots s_a$$

q -Whittaker operator

eigenvalue

$$\bullet (\hat{e}_a(x)) = H_a^\infty(s, x)$$

elementary sym
fctn

Toda Hamiltonian

(Thm)

[DF-Kedem 19]

$$\hat{\gamma} = \gamma(xT) \prod_{i=1}^{N-1} \frac{1}{E(s_{i+1}/s_i)}$$

Note: the Gaussian acts like a theta function

$$\begin{cases} \gamma^{-1} T_{q,x} \gamma = q^{\frac{1}{2}} x T_{q,x} \\ \gamma^{-1} x \gamma = x \end{cases}$$

$$\theta(x) := E(-x q^{\frac{1}{2}}) E(-x^{-1} q^{\frac{1}{2}}) E(q) \quad \text{Jacobi Triple product!}$$

$$\text{Then } \begin{cases} \theta(x) T_{q,x} \theta(x)^{-1} = q^{\frac{1}{2}} x T_{q,x} \\ \theta(x) x \theta(x)^{-1} = x \end{cases}$$

$\Rightarrow$ all dilogarithm identities!

- Macdonald theory fits in this framework
[DF, Kedem, Khoroshkin, Schrader, Shapiro '24]

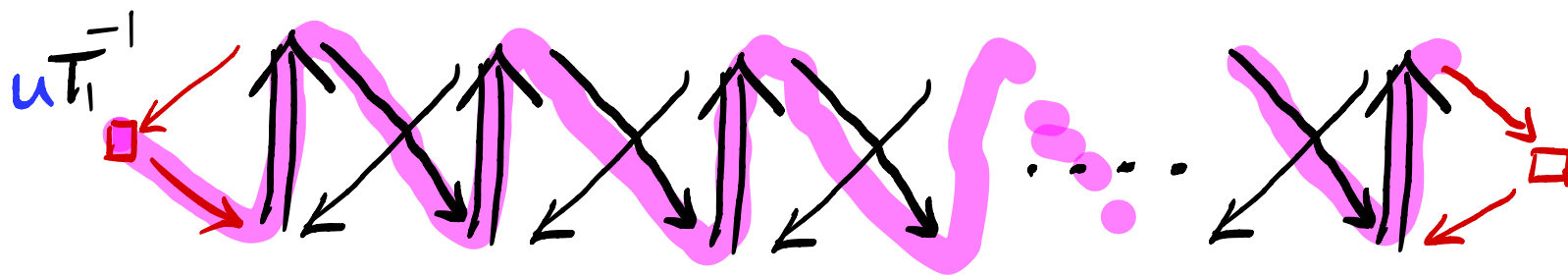
$$D_a^t = \sum_{k=0}^{N-1} \frac{1-t^{N-k}}{1-t} e_k(x) \cdot D_{a,-k}^\infty$$

↑
polynomial of t w/ coefficients in the Cluster Algebra
of the quantum Q -system.

universally Laurent under mutations

Example 2.

BAXTER Q-OPERATOR



This sequence of mutations produces a Baxter Q-operator for open q Toda. [Schrader-Shepino '18]

$$Q(u) = \prod_{i=1}^N E\left(u \left(1 - \frac{s_i}{s_{i-1}}\right) T_i^{-1}\right) \quad (s_0 = \infty)$$

$$\widehat{Q}(u) = \prod_{i=1}^N E(u x_i^{-1}) \quad \leftarrow \text{eigenvalue on } g_N.$$

another pure dilog identity!

CONCLUSION

- importance of quantum dilogarithm
- $\sigma\text{-gin} = q$ Cluster Algebra of q -Whittaker limit

- generalization to Macdonald case

a cluster Algebra for SL_2 Macdonald

[PdF, Hedem, Schrader, Shapiro, Khavoshkin 24]

conjectures for Q -operator for SL_N Macdonald [in progress]

continuing f_N to other sectors $x_i \leftrightarrow x_{i+1}$ [in progress]

other types, Koornwinder?

Merci!

P. Di Francesco and R. Kedem " Duality and Macdonald difference operators" [arXiv 2303.04276]

P. Di Francesco and R. Kedem " Macdonald duality and the proof of the quantum Q -system conjecture" [arXiv 2112.09798]
(Selecta)