On recent conjectures concerning free Jordan algebras and free alternative algebras

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Plan of the talk

1. Recalling the definitions
2. Conjectures of Kashuba–Mathieu and of Shang
3. Exploring the conjectures
4. Some open questions

Alternative algebras and Jordan algebras

- Alternative algebra: nonassociative algebra in which the associator $(a,b,c) := (ab)c - a(bc)$ is a skew-symmetric function of its arguments.
- Motivated by an example: Cayley's octonions. First systematic study by Zorn (1931).
- Jordan algebra: nonassociative commutative algebra in which $[L_a, L_{a^2}] = 0$, that is, $a^2(ax) = a(a^2x)$ for all a, x.
- Motivated by an example: Hermitian matrices with the operation $A \circ B := \frac{1}{2}(AB + BA)$, named after Jordan who (together with von Neumann and Wigner) proposed this structure to generalize the quantum mechanical formalism.

Throughout this talk, I shall only discuss algebras over a field $\Bbbk$ of zero characteristic (for example, $\mathbb{Q}$).

Free algebras

For both of these classes of algebras, free algebras are not well understood.

- (folklore) $\mathrm{Alt}(x) = \mathrm{Jord}(x)$ is the polynomial algebra $\Bbbk[x]$.
- (Artin, circa 1931) $\mathrm{Alt}(x,y) = \Bbbk\langle x,y\rangle$.
- (Cohn 1954, Shirshov 1956) $\mathrm{Jord}(x,y) = \Bbbk\langle x,y\rangle^{\sigma}$ where σ is the standard anti-automorphism (word reversal).
- (Iltyakov, 1984) $\mathrm{Alt}(x,y,z)$ has an explicit (complicated) vector space basis.

Beyond these cases, not much is known.

Moreover, for large number of generators these algebras become somewhat pathological (zero divisors, nilpotent elements etc.).

Obtaining new information on free alternative algebras and free Jordan algebras is an interesting open problem.

The Tits–Kantor–Koecher construction

For a Jordan algebra J, denote $D_{a,b} := [L_a, L_b]$. It is known that each $D_{a,b}$ is a derivation of J; such derivations are called *inner*. Using these, can define a Lie algebra (Tits 1962, Kantor 1964, Koecher 1967)

$$\mathsf{TKK}(J) := \mathfrak{sl}_2 \otimes J \oplus \mathrm{Inner}(J)$$

with the Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + \frac{1}{2}\mathrm{tr}(xy) D_{a,b},$$
$$[D_{a,b}, x \otimes c] = x \otimes D_{a,b}(c),$$
$$[D_{a,b}, D_{c,d}] = D_{D_{a,b}(c),d} + D_{c,D_{a,b}(d)}.$$

A functorial version of the TKK construction

Inner derivations are not functorial, which can be remedied by passing to the universal central extension of TKK(J). One associates to the given Jordan algebra J the vector space

$$\mathscr{B}(J) := \Lambda^2(J)/(a \wedge a^2 \colon a \in J),$$

which has a canonical surjection onto $\mathrm{Inner}(J)$, $a \wedge b \mapsto D_{a,b}$. Furthermore, one defines on the vector space

$$\mathrm{TAG}(J) := \mathfrak{sl}_2 \otimes J \oplus \mathscr{B}(J)$$

a bilinear operation by

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab + \frac{1}{2}\mathrm{tr}(xy) a \wedge b,$$
$$[a \wedge b, x \otimes c] = x \otimes D_{a,b}(c),$$
$$[a \wedge b, c \wedge d] = D_{a,b}(c) \wedge d + c \wedge D_{a,b}(d).$$

One can show that this operation makes TAG(J) a Lie algebra.

The Berman–Moody construction

For an alternative algebra A, denote $D_{a,b} := [L_a, L_b] + [R_a, R_b] + [L_a, R_b]$. It is known that for each $a, b \in A$ this is a derivation of A; such derivations are called *inner*. Using these, can define a Lie algebra (Berman–Moody 1992)

$$\mathrm{BM}(A) := \mathfrak{sl}_3 \otimes A \oplus \mathrm{Inner}(A)$$

with the Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes a \cdot b + \{x, y\} \otimes [a, b] + \frac{1}{3}\mathrm{tr}(xy) D_{a,b},$$
$$[D_{a,b}, x \otimes c] = x \otimes D_{a,b}(c),$$
$$[D_{a,b}, D_{c,d}] = D_{D_{a,b}(c),d} + D_{c,D_{a,b}(d)}.$$

(Here and below we denote by $\{x, y\} := \frac{1}{2}(xy + yx) - \frac{1}{3}\mathrm{tr}(xy) I_3$ the standard $\mathfrak{sl}_3$-module projection $S^2\mathfrak{sl}_3 \twoheadrightarrow \mathfrak{sl}_3$.)

A functorial version of the BM construction

Inner derivations are not functorial, which can be remedied by passing to the universal central extension of BM(A). One associates to the given alternative algebra A the vector space

$$\mathscr{C}(A) := \Lambda^2(A)/(ab \wedge c + bc \wedge a + ca \wedge b \colon a, b, c \in A),$$

which has a canonical surjection onto $\mathrm{Inner}(A)$, $a \wedge b \mapsto D_{a,b}$. Furthermore,

$$\mathrm{ABG}(A) := \mathfrak{sl}_3 \otimes A \oplus \mathscr{C}(A)$$

has a bilinear operation

$$[x \otimes a, y \otimes b] = [x, y] \otimes a \cdot b + \{x, y\} \otimes [a, b] + \frac{\mathrm{tr}(xy)}{3} a \wedge b,$$
$$[a \wedge b, x \otimes c] = x \otimes D_{a,b}(c),$$
$$[a \wedge b, c \wedge d] = D_{a,b}(c) \wedge d + c \wedge D_{a,b}(d).$$

One can show that this operation makes ABG(A) a Lie algebra.

Homology of Lie algebras

For a Lie algebra $\mathfrak{g}$, one can compute an important invariant, $H_\bullet(\mathfrak{g},\Bbbk)$, the homology of $\mathfrak{g}$ with trivial coefficients. It is the homology of the Chevalley–Eilenberg complex $C_\bullet(\mathfrak{g}) = (\Lambda(\mathfrak{g}), d)$, where

$$d(x_1\wedge\cdots\wedge x_n) = \sum_{1\le i<j\le n} (-1)^{i+j-1}[x_i,x_j]\wedge x_1\wedge\cdots\widehat{x_i}\cdots\widehat{x_j}\cdots\wedge x_n.$$

Also has some other equivalent definitions (via abelian and non-abelian derived functors).

The conjecture of Kashuba and Mathieu

Note that $\mathfrak{sl}_2$ acts on $\mathrm{TAG}(J) := \mathfrak{sl}_2 \otimes J \oplus \mathscr{B}(J)$; the first component is a multiple of the adjoint module, and the second is multiple of the trivial module. In fact, $\mathrm{TAG}(\mathrm{Jord}(V))$ is a *free* Lie algebra generated by $\mathfrak{sl}_2 \otimes V$ in the category of $\mathfrak{sl}_2$-modules that only have trivial and adjoint components.

Conjecture (Kashuba, Mathieu, 2019)

The $\mathfrak{sl}_2$-module

$$H_k(\mathrm{TAG}(\mathrm{Jord}(V)), \Bbbk)$$

has no trivial or adjoint component for $k > 1$.

Note: if we look at all homology groups, $H_0(\mathrm{TAG}(\mathrm{Jord}(V)), \Bbbk) \cong \Bbbk$ is a trivial module, and $H_1(\mathrm{TAG}(\mathrm{Jord}(V)), \Bbbk) \cong \mathfrak{sl}_2 \otimes V$ is a sum of adjoint modules.

Thus, if this conjecture were true, we would have formulas for dimensions of components of the free Jordan algebra, its $GL(V)$-module structure etc.: write down the Euler characteristics for trivial/adjoint isotypic components of $H_\bullet(\mathrm{TAG}(\mathrm{Jord}(V)), \Bbbk)$ in two different ways.

The conjecture of Shang

Motivated by the Kashuba–Mathieu conjecture, Shang noticed that one do the same for $\mathfrak{sl}_3$, and proposed the following

Conjecture (Shang 2023)

The $\mathfrak{sl}_3$-module

$$H_k(\mathrm{ABG}(\mathrm{Alt}(V)),\Bbbk)$$

has no trivial or adjoint component for $k > 1$.

Similarly, if this conjecture were true, we would have formulas for dimensions of components of the free alternative algebra, its $GL(V)$-module structure etc.

The conjecture of Shang

Shang only checked the prediction of his conjecture for the free alternative algebra on one generator, and for elements of degree at most 3 in free alternative algebras on any number of generators.

Proposition (D.)

The prediction of the Shang conjecture gives correct dimensions and module structures for $\mathrm{Alt}(V)$ *in degrees at most* 6*, for any* V*.*

The module structure for $n = 5,6$ was not previously known, it was computed using the `albert` program and rechecked using Gröbner bases for operads.

Theorem (D.)

For $\mathrm{Alt}(x,y,z)$*, the prediction of the Shang conjecture gives a wrong result in degree* 7*:* 2373 *instead of* 2388*.*

(The correct dimension was computed in two different ways: using the basis of Iltyakov and using the `albert` program.)

The conjecture of Kashuba and Mathieu

For 2 generators, Kashuba and Mathieu verified that prediction in degree at most 15.

For 3 generators, they verified that prediction in degree at most 8. This is quite significant, since for 3 generators, there are particular Jordan polynomials in 3 variables of degree 8 ("Glennie elements") that vanish on all Jordan algebras obtained from associative algebras by the $\frac{1}{2}(ab+ba)$ operation.

Using the `albert` program for computations with nonassociative algebras, we checked the prediction for 3 generators in degree at most 12.

The conjecture of Kashuba and Mathieu

Proposition (D.–Hentzel)

The prediction of the Kashuba–Mathieu conjecture gives correct dimensions and module structures for $\mathrm{Jord}(V)$ *in degree at most* 10*, for any* V*.*

To prove for any V, enough to understand the S_n-module structure of multilinear elements with $n \leq 10$. This required some new computational tricks; even the dimension, let alone the module structure, was not previously known for $n = 9, 10$.

In particular, the dimensions of spaces of multilinear elements are

$$1, 1, 3, 11, 55, 330, 2345, 19089, 175203, 1785840$$

The module structure for degrees 8, 9, 10

$$\mathrm{Jord}(8) \cong V_{2,1^6} \oplus V^{6}_{2^2,1^4} \oplus V^{11}_{2^3,1^2} \oplus V^{10}_{2^4} \oplus V^{5}_{3,1^5} \oplus V^{26}_{3,2,1^3} \oplus V^{34}_{3,2^2,1}$$
$$\oplus V^{30}_{3^2,1^2} \oplus V^{19}_{3^2,2} \oplus V^{14}_{4,1^4} \oplus V^{41}_{4,2,1^2} \oplus V^{32}_{4,2^2} \oplus V^{34}_{4,3,1} \oplus V^{10}_{4^2} \oplus V^{16}_{5,1^3}$$
$$\oplus V^{32}_{5,2,1} \oplus V^{12}_{5,3} \oplus V^{9}_{6,1^2} \oplus V^{12}_{6,2} \oplus V^{3}_{7,1} \oplus V_{8}$$

$$\mathrm{Jord}(9) \cong V_{2,1^7} \oplus V^{7}_{2^2,1^5} \oplus V^{18}_{2^3,1^3} \oplus V^{22}_{2^4,1} \oplus V^{6}_{3,1^6} \oplus V^{38}_{3,2,1^4} \oplus V^{74}_{3,2^2,1^2} \oplus V^{44}_{3,2^3}$$
$$\oplus V^{58}_{3^2,1^3} \oplus V^{85}_{3^2,2,1} \oplus V^{20}_{3^3} \oplus V^{20}_{4,1^5} \oplus V^{84}_{4,2,1^3} \oplus V^{109}_{4,2^2,1} \oplus V^{107}_{4,3,1^2} \oplus V^{86}_{4,3,2} \oplus V^{44}_{4^2,1} \oplus V^{31}_{5,1^4}$$
$$\oplus V^{91}_{5,2,1^2} \oplus V^{64}_{5,2^2} \oplus V^{78}_{5,3,1} \oplus V^{22}_{5,4} \oplus V^{25}_{6,1^3} \oplus V^{53}_{6,2,1} \oplus V^{24}_{6,3} \oplus V^{12}_{7,1^2} \oplus V^{15}_{7,2} \oplus V^{4}_{8,1} \oplus V_{9}$$

$$\mathrm{Jord}(10) \cong V_{2,1^8} \oplus V^{7}_{2^2,1^6} \oplus V^{26}_{2^3,1^4} \oplus V^{38}_{2^4,1^2} \oplus V^{26}_{2^5} \oplus V^{8}_{3,1^7} \oplus V^{53}_{3,2,1^5}$$
$$\oplus V^{139}_{3,2^2,1^3} \oplus V^{144}_{3,2^3,1} \oplus V^{93}_{3^2,1^4} \oplus V^{226}_{3^2,2,1^2} \oplus V^{122}_{3^2,2^2} \oplus V^{114}_{3^3,1} \oplus V^{26}_{4,1^6} \oplus V^{151}_{4,2,1^4} \oplus V^{272}_{4,2^2,1^2}$$
$$\oplus V^{162}_{4,2^3} \oplus V^{257}_{4,3,1^3} \oplus V^{394}_{4,3,2,1} \oplus V^{105}_{4,3^2} \oplus V^{143}_{4^2,1^2} \oplus V^{138}_{4^2,2} \oplus V^{50}_{5,1^5} \oplus V^{212}_{5,2,1^3} \oplus V^{263}_{5,2^2,1}$$
$$\oplus V^{289}_{5,3,1^2} \oplus V^{224}_{5,3,2} \oplus V^{144}_{5,4,1} \oplus V^{16}_{5,5} \oplus V^{58}_{6,1^4} \oplus V^{168}_{6,2,1^2} \oplus V^{120}_{6,2^2} \oplus V^{155}_{6,3,1} \oplus V^{50}_{6,4}$$
$$\oplus V^{40}_{7,1^3} \oplus V^{80}_{7,2,1} \oplus V^{35}_{7,3} \oplus V^{16}_{8,1^2} \oplus V^{20}_{8,2} \oplus V^{4}_{9,1} \oplus V_{10}$$

The conjecture of Kashuba and Mathieu

However...

Theorem (D.–Hentzel)

For $\mathrm{Jord}(x, y)$, *the prediction of the Kashuba–Mathieu conjecture is correct in degrees at most* 18 *but gives a wrong result in degree* 19*:* 262658 *instead of* 262656*.*

Schur positivity

Each of the conjectures predicts the $GL(V)$-module structure of the free algebra, that is, the multiplicities of all irreducible modules $S^\lambda(V)$, where λ is a partition.

For alternative algebras, some predicted multiplicities are negative for $|\lambda| = 10$ (for example, the predicted multiplicity for $\lambda = (2,2,2,2,2)$ is -11), giving another indication of why the conjecture is false.

By contrast, for Jordan algebras, all predicted multiplicities are non-negative for as far as we computed them (for $|\lambda| \leq 25$), and even more so, the corresponding modules are restrictions of S_{n+1}-modules, so the conjecture is false in a much more subtle way.

Some open questions

- The original question remains: obtain information about the free alternative algebras with at least 4 generators, and free Jordan algebras with at least 3 generators.
- Kashuba–Mathieu: if the conjecture is true, we have $\mathsf{TAG}(\mathrm{Jord}(V)) = \mathsf{TKK}(\mathrm{Jord}(V))$. Perhaps this is nevertheless the case? (True for $\dim(V) = 2$, even though the conjecture is false.)
- Shang: if the conjecture is true, we have $\mathsf{ABG}(\mathrm{Alt}(V)) = \mathsf{BM}(\mathrm{Alt}(V))$. Perhaps this is nevertheless the case? (True for $\dim(V) = 2$, even though the conjecture is false.)
- Is the Schur positivity phenomenon observed in the Jordan case true? Is there a conceptual explanation?

An unrelated puzzle

Yesterday, Anastasia told us about Motzkin-type numbers:

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2}.$$

Poincaré: keep only the leading terms of the polynomial coefficients:

$$m'_n = 2m'_{n-1} + 3m'_{n-2}.$$

The vector space of "approximate solutions" $\{m'_n\}$ has a basis $\{3^n\}$, $\{(-1)^n\}$, so we expect a generic $\{m_n\}$ to resemble the first one, and a one-dimensional vector space of exceptional slowly growing $\{m_n\}$ that resemble the second one.

Conjecture: exceptional sequences $\{m_n\}$ are proportional to the one with

$$m_0 = 1, \quad m_1 = \frac{9\sqrt{3}-4\pi}{4\pi-3\sqrt{3}}$$

(guessed using the "Inverse Equation Solver" of Robert Munafo, `https://mrob.com/pub/ries/index.html`).