On the integrality of some P-recursive sequences

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Joint work with Alin Bostan

Outline

Key definitions

Definition

A sequence $(s_n)_{n\in\mathbb{N}}$ of rational numbers is called *P-recursive* if it satisfies a linear (homogeneous) recurrence relation with coefficients in $\mathbb{Q}[n]$.

$$P_0(n)s_n + P_1(n)s_{n-1} + \ldots + P_k(n)s_{n-k} = 0, \quad P_0 \neq 0,\ n \geq k.$$

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Examples:

- *Fibonacci numbers*: $F_n = F_{n-1} + F_{n-2}, \quad F_0 = F_1 = 1.$
- *Catalan numbers*: $(n+1)C_n = 2(2n-1)C_{n-1}, \quad C_0 = 1.$
- *Apéry numbers*:
  $$n^3 A_n = (34n^3 - 51n^2 + 27n - 5)A_{n-1} - (n-1)^3 A_{n-2}, \quad A_0 = 1, A_1 = 5.$$

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A power series $f(x) \in \mathbb{Q}[[x]]$ is called *D-finite* if it satisfies a linear (homogeneous) differential equation with coefficients in $\mathbb{Q}[x]$.

$$a_r(x)f^{(r)}(x) + \ldots + a_1(x)f'(x) + a_0(x)f(x) = 0, \quad a_r(x) \neq 0.$$

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Theorem [Stanley, 1980]

A power series is D-finite if and only if its coefficient sequence is P-recursive.

Problem setup

Let $(s_n)_n$ be a P-recursive sequence of rational numbers.

Let $S(x) = \sum_{n \geq 0} s_n x^n$ be the generating function of $(s_n)_n$.

- When are $(s_n)_n$ and $S(x)$ *integral*?

  $\rightarrow$ $s_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

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  Motivations:

  - "I know numbers are beautiful. If they aren't beautiful, nothing is."
    — *Paul Erdős*.
  - Connections with combinatorial objects.
  - Questions of irrationality and transcendence: the quest for integer sequences.

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Let $S(x) = \sum_{n \geq 0} s_n x^n$ be the generating function of $(s_n)_n$.

- When are $(s_n)_n$ and $S(x)$ *integral*?
  - $\rightarrow$ $s_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$.
- When is $S(x)$ *globally bounded*?
  - $\rightarrow$ $(s_n)_n$ is *almost integral*: $\exists C \in \mathbb{Z}^*$ such that $C^n s_n \in \mathbb{Z}$ for all $n \geq 1$.
  - $\rightarrow$ $S(x)$ has a nonzero radius of convergence.

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  $\rightarrow$ $S(x)$ has a nonzero radius of convergence.

- When is $S(x)$ *algebraic*?

  $\rightarrow$ $\exists P(x, y) \in \mathbb{Q}[x, y] \setminus \{0\}$ such that $P(x, S(x)) = 0$.

Outline

1. Introduction and definitions
2. Motivating example
3. Generalization
4. Algorithm
5. Conclusion

Motzkin-type sequences

A sequence $(m_n)_n$ that satisfies the recurrence relation

$$(n+2)m_n = (2n+1)m_{n-1} + (3n-3)m_{n-2}, \quad n \geq 2 \tag{1}$$

with some $m_0, m_1 \in \mathbb{Q}$ is called a *Motzkin-type sequence*.

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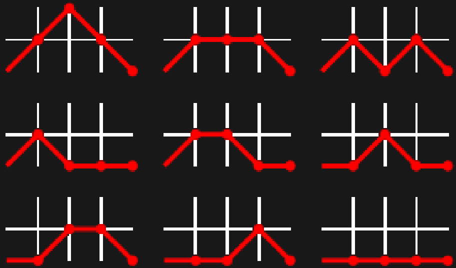
$m_0 = m_1 = 1 \rightsquigarrow$ *Motzkin numbers*: 1, 1, 2, 4, 9, 21, 51, 127, ...



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For which (m_0, m_1) is $\sum_{n\geq 0} m_n x^n$ algebraic/globally bounded/integral?

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For which (m_0, m_1) *is* $\sum_{n\geq 0} m_n x^n$ *algebraic/globally bounded/integral?*

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- Eisenstein's theorem/criterion. [Eisenstein 1852], [Heine 1853, 1854]
- The converse is not true! For example, $\sum_{n\geq 0} \binom{2n}{n}^2 x^n$ is transcendental.

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For which (m_0, m_1) is $\sum_{n\geq 0} m_n x^n$ algebraic/globally bounded/integral?

Theorem [Klazar, Luca, 2005]

A Motzkin-type sequence is integral if and only if $m_0 = m_1 \in \mathbb{Z}$. If $m_0 \neq m_1$, the sequence is not globally bounded.

The original proof of Klazar-Luca theorem

A sequence $(m_n)_n$ that satisfies the recurrence relation

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with some $m_0, m_1 \in \mathbb{Q}$ is called a *Motzkin-type sequence*.

- $m_0 = m_1$
  - $m_0 = m_1 = 1 \implies$ Motzkin numbers.
  - $m_0 = m_1 \in \mathbb{Z} \implies$ Motzkin numbers, scaled.
- $m_0 \neq m_1$
  - Convert (1) to a first-order inhomogeneous ODE satisfied by $\sum_{n\geq 0} m_n x^n$.
  - Study one cleverly chosen pair (m_0, m_1).
  - Show that global boundedness fails.

The original proof of Klazar-Luca theorem

"Study one cleverly chosen pair (m_0, m_1) and show that global boundedness fails".

1. Find a, b, m_0, m_1 such that $S(x) := a + bx + \sum_{n\geq 0} m_n x^{n+2}$ satisfies $gS'(x) - \frac{1}{2}g' S(x) = g,\ \ S(0) = 0$, where $g := 1 - 2x - 3x^2$.

2. Consequently, $S(x) = \sqrt{g}\int \frac{dx}{\sqrt{g}}$. Observe that $\sqrt{g}$ is globally bounded.

3. Write $\int \frac{dx}{\sqrt{g}}$ as a power series $\sum_{n\geq 1} \frac{d_{n-1}}{n} x^n$, where $\sum_{n\geq 0} d_n x^n := \frac{1}{\sqrt{g}}$.

4. Show that $p \nmid d_{p-1}$ for prime $p > 3$ since $\binom{2i}{i} \not\equiv 0 \pmod{p}$ iff $i \leq \frac{p-1}{2}$, and
$$d_n := [x^n](1 - 2x - 3x^2)^{-1/2} = \frac{(-1)^n}{4^n}\sum_{i=0}^{n}(-3)^i\binom{2i}{i}\binom{2n-2i}{n-i}.$$

5. Conclude by using the fact that $[x^n]S(x) = m_{n-2}$ for $n \geq 2$.

The new proof of Klazar-Luca theorem

$> rec := \{(n+2)m(n)-(2n+1)m(n-1)-(3n-3)m(n-2), m(0) = m_0, m(1) = m_1\} :$

$> dsolve(gfun:\text{-}\ rectodiffeq(rec, m(n), M(x)), M(x));$

$$M(x) = \frac{\left(\int \frac{x(3xm_0-3xm_1-2m_0)}{(3x^2+2x-1)^{3/2}}\,dx + c_1\right)\sqrt{3x^2+2x-1}}{x^2}$$

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Theorem [André, 1989]

A power series y is algebraic iff y is globally bounded and $\frac{dy}{dx}$ is algebraic.

Corollary of André's theorem

The primitive of an algebraic power series is globally bounded iff it is algebraic.

The new proof of Klazar-Luca theorem

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$$\int \frac{x(3xm_0 - 3xm_1 - 2m_0)}{(3x^2+2x-1)^{3/2}} dx = \frac{m_0(3-7x) + m_1(5x-1)}{4\sqrt{3x^2+2x-1}} + \frac{(m_0 - m_1)\log(\sqrt{9x^2+6x-3} + 3x + 1)}{\sqrt{3}} + const$$

$\implies M(x)$ is algebraic iff $m_0 = m_1 \implies M(x)$ is globally bounded iff $m_0 = m_1$.

Corollary of André's theorem

The primitive of an algebraic power series is globally bounded iff it is algebraic.

The new proof of Klazar-Luca theorem

$M(x)$ is globally bounded if and only if $m_0 = m_1$. But when is it integral?

If $m_1 = m_0$, then

$$M(x) = \frac{m_0\left(1 - x - \sqrt{1 - 2x - 3x^2}\right)}{2x^2}.$$

Need: $m_n \in \mathbb{Z}$ for all n. Hence, $m_0 \in \mathbb{Z}$ is *necessary*. It is also *sufficient* because

$$\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + \ldots \in \mathbb{Z}[[x]]$$

Proofs: Motzkin numbers, inductive coefficient matching, criteria for the integrality of n^{th} roots of power series [Pomerat, Straub, 2024]...

Diagonals of multivariate rational functions (Christol, Deligne, Lipschitz...)

Definition

Let $R = \sum_{n_1,\ldots,n_k \geq 0} c(n_1, \ldots, n_k) x_1^{n_1} \ldots x_k^{n_k} \in \mathbb{Q}[[x_1, \ldots, x_k]] \cap \mathbb{Q}(x_1, \ldots, x_k)$.

The *diagonal* of R is a univariate power series defined by

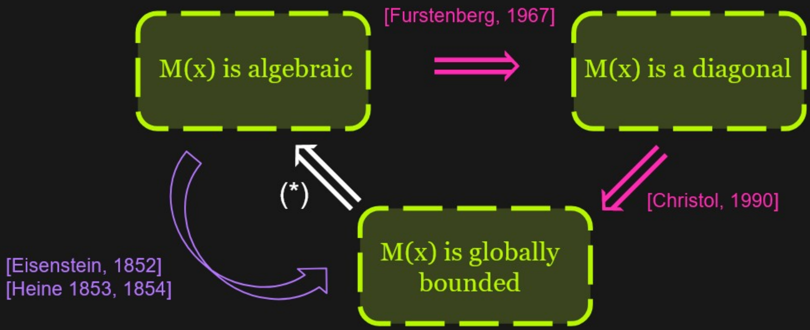
$$\mathrm{Diag}(R) = \sum_{n \geq 0} c(n, \ldots, n) t^n \in \mathbb{Q}[[t]].$$

Example:

$$R = \frac{1}{1 - x_1 - x_2} = \sum_{i,j \geq 0} \binom{i+j}{i} x_1^i x_2^j,$$

$$\mathrm{Diag}(R) = \sum_{n \geq 0} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}.$$

Klazar-Luca theorem for Motzkin-type sequences: refinement



Pink and purple implications hold for any functions! But (*) is only true because $M(x)$ is a product of algebraic functions and a primitive of an algebraic function.

Outline

Generalization

Conjecture

Let $(s_n)_n$ be a sequence of rational numbers that satisfies a linear homogeneous recurrence relation with polynomial coefficients of degree 1:

$$\sum_{k=0}^{d}(a_k n + b_k)s_{n-k} = 0, \quad a_0 = 1,\ a_i \in \mathbb{Q} \setminus \{0\} \text{ for } i = 1, \dots, d \quad d \in \mathbb{N}.$$

Let $S(x) = \sum_{n\geq 0} s_n x^n$ be its generating function. The following are equivalent:

1. $S(x)$ is algebraic.
2. $S(x)$ is a diagonal.
3. $S(x)$ is globally bounded.

Hypergeometric functions

The *hypergeometric function* with parameters

$a_1, \ldots, a_p \in \mathbb{C}, \quad b_1, \ldots, b_q \in \mathbb{C} \setminus -\mathbb{N}$ for some $p, q \in \mathbb{N}$ is defined as:

$$ {}_pF_q\left[\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix} ; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!}, $$

where $(k)_n$ is the rising factorial (also known as *Pochhammer symbol*):

$$ (k)_n = \begin{cases} 1 & \text{if } n = 0, \\ k(k+1)\ldots(k+n-1) & \text{if } n > 0. \end{cases} $$

Gaussian hypergeometric functions correspond to $p = 2, \; q = 1$:

$$ {}_2F_1\left[\begin{matrix} a_1 \quad a_2 \\ b_1 \end{matrix} ; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n} \frac{x^n}{n!}. $$

Proved cases of the conjecture

Recurrences of **order 1** (with linear polynomial coefficients):

$$(n + b_0)s_n + (a_1 n + b_1)s_{n-1} = 0, \quad a_1 \neq 0.$$

Closed-form expression of the generating function is:

$$S(x) = s_0 \, {}_2F_1\left[\begin{matrix} 1 + \frac{b_1}{a_1} & 1 \\ 1 + b_0 & \end{matrix} ; -a_1 x\right].$$

Apply Euler transformation ${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z)$:

$$S(x) = s_0(1 + a_1 x)^{b_0 - 1 - \frac{b_1}{a_1}} \, {}_2F_1\left[\begin{matrix} b_0 - \frac{b_1}{a_1} & b_0 \\ 1 + b_0 & \end{matrix} ; -a_1 x\right].$$

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One can show that the hypergeometric function ${}_2F_1$ in (2) is:

- either a polynomial
- or of "height 1" (equal number of integer top and bottom parameters).

Proved cases of the conjecture

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- either a polynomial
- or of "height 1" (equal number of integer top and bottom parameters).

Theorem [Christol, 1986]

A hypergeometric function of height 1 is globally bounded iff it is algebraic.

Corollary

$S(x)$ is globally bounded iff it is algebraic, iff it is a diagonal.

Proved cases of the conjecture

Theorem [Bostan, M., 2025]

Let $(s_n)_n$ be a sequence of rational numbers that satisfies:

$$(n + b_0)s_n + (a_1 n + b_1)s_{n-1} + (a_2 n + b_2)s_{n-2} = 0, \quad a_1, a_2 \neq 0.$$

Let $S(x) = \sum_{n\geq 0} s_n x^n$ be its generating function. The following are equivalent:

1. $S(x)$ is algebraic,
2. $S(x)$ is a diagonal,
3. $S(x)$ is globally bounded,

provided that $b_2 = (2a_2 b_1 - a_1 a_2 b_0)/a_1$.

Proof of “algebraic $\iff$ globally bounded $\iff$ diagonal”

The only direction we *really* need to prove is globally bounded $\implies$ algebraic.

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$$> rec := \{(n+b_0)s(n)+(a_1 n+b_1)s(n-1)+(a_2 n+b_2)s(n-2), s(0)=s_0, s(1)=s_1\} :$$

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$$S(x) = (1+a_1x+a_2x^2)^{-\frac{b_2}{2a_2}-1+\frac{b_0}{2}} \left(\int (a_1 s_0 x + b_0 s_1 x + b_1 s_0 x + b_0 s_0 + s_1 x)(1+a_1x+a_2x^2)^{-\frac{a_2 b_0 - b_2}{2a_2}} x^{b_0-1} e^{\frac{\operatorname{artanh}\left(\frac{2a_2x+a_1}{\sqrt{a_1^2-4a_2}}\right)(a_1a_2b_0+a_1b_2-2a_2b_1)}{a_2\sqrt{a_1^2-4a_2}}} dx + c_1 \right) x^{-b_0} e^{-\frac{\operatorname{artanh}\left(\frac{2a_2x+a_1}{\sqrt{a_1^2-4a_2}}\right)(a_1a_2b_0+a_1b_2-2a_2b_1)}{a_2\sqrt{a_1^2-4a_2}}}.$$

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$> simplify(subs(b_2 = \frac{2a_2 b_1 - a_1 a_2 b_0}{a_1}, \%));$

$$S(x) = (1 + a_1 x + a_2 x^2)^{b_0 - 1 - \frac{b_1}{a_1}} \left(\int x^{b_0} (1 + a_1 x + a_2 x^2)^{-b_0 + \frac{b_1}{a_1}} (a_1 s_0 + b_0 s_1 + b_1 s_0 + \frac{b_0 s_0}{x} + s_1)\, dx + c_1 \right) x^{-b_0}.$$

Proof of "algebraic $\iff$ globally bounded $\iff$ diagonal"

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Corollary of André's theorem

The primitive of an algebraic power series is globally bounded iff it is algebraic.

The integrand is algebraic $\implies$ this special subcase of the conjecture is proved.

Linearity of the coefficients: why it matters

Consider the recurrence relation for *Apéry numbers*:

$$n^3 A_n = (34n^3 - 51n^2 + 27n - 5)A_{n-1} - (n-1)^3 A_{n-2}.$$

- $A_0 = 1, A_1 = 5 \rightsquigarrow$ integral sequence of Apéry numbers:

$$A_0 = 1,\ A_1 = 5 \implies A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z}.$$

  Hence, the generating function of Apéry numbers is globally bounded. [Mimura, 1983]: $A_1 = 5A_0 \iff \sum_{n\geq 0} A_n x^n$ is globally bounded. [Mimura, 1983]: $A_1 = 5A_0 \iff \sum_{n\geq 0} A_n x^n$ is globally bounded.

- At the same time, the generating function of Apéry numbers is transcendental.

  $\rightarrow$ Many proofs, e.g. via "minimization" [Bostan, Rivoal, Salvy, 2024].

A needle in a haystack

> There exist 2-order linear recurrence relations with (nonlinear) polynomial coefficients that admit a *basis* of globally bounded yet transcendental solutions.

(Conjecturally) not possible if "linear recurrence relations" is replaced by "linear differential equations" (corollary of the Grothendieck–Katz p-curvature conjecture).

A needle in a haystack

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Example found by Armin Straub:

$$(n+2)^2 u_{n+2} - 3(3n^2+9n-2)u_{n+1} + 27(n+4)(n-2)u_n = 0, \quad n \geq 0.$$

The recurrence on $(u_n)_n$ comes from the DE satisfied by $y(x) = \sum_{n\geq 0} u_n x^n$:

$$((c+d)x-b)y(x) + (3x^2c - 2ax + 1)\frac{dy(x)}{dx} + (cx^3 - ax^2 + x)\frac{d^2y(x)}{dx^2} = 24u_0 + u_1$$

with $a = 9,\ b = -24,\ c = 27,\ d = -243.$

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There exist 2-order linear recurrence relations with (nonlinear) polynomial coefficients that admit a *basis* of globally bounded yet transcendental solutions.

Example found by Armin Straub:

$$(n+2)^2 u_{n+2} - 3(3n^2 + 9n - 2)u_{n+1} + 27(n+4)(n-2)u_n = 0, \quad n \geq 0.$$

$$u_0 = 1,\ u_1 = 0 \quad \rightsquigarrow \quad 1, 0, 54, 180, 945, 4536, 19656, 74520, 227205\dots$$

$$u_0 = 0,\ u_1 = 1 \quad \rightsquigarrow \quad 0, 1, -\frac{3}{2}, 10, \frac{105}{2}, 252, 1092, 4140, \frac{25245}{2}, 21340\dots$$

It can be shown that the first sequence is integral, and the second one takes values in $\frac{1}{2}\mathbb{Z}$!

A needle in a haystack

There exist 2-order linear recurrence relations with (nonlinear) polynomial coefficients that admit a *basis* of globally bounded yet transcendental solutions.

$$u_0 = 1,\ u_1 = 0 \rightsquigarrow \sum_{n\geq 0} u_n x^n = 4(9x-1)^2 - 3\ {}_2F_1\left[\begin{matrix} -2/3 & 4/3 \\ & 1 \end{matrix}; 27x(1-9x+27x^2)\right],$$

$$u_0 = 0,\ u_1 = 1 \rightsquigarrow \sum_{n\geq 0} u_n x^n = \frac{(9x-1)^2}{6} - \frac{{}_2F_1\left[\begin{matrix} -2/3 & 4/3 \\ & 1 \end{matrix}; 27x(1-9x+27x^2)\right]}{6}.$$

One can show that this ${}_2F_1$ lies in $1 + 3x\mathbb{Z}[[x]]$ and is transcendental.

Algebraicity of ${}_2F_1$: [Schwarz, 1873], [Beukers, Heckman, 1989], [Fürnsinn, Yurkevich, 2024].

A needle in a haystack

There exist 2-order linear recurrence relations with (nonlinear) polynomial coefficients that admit a *basis* of globally bounded yet transcendental solutions.

In fact, Armin Straub's example

$$(n+2)^2 u_{n+2} - 3(3n^2+9n-2)u_{n+1} + 27(n+4)(n-2)u_n = 0, \quad n \geq 0.$$

can be used to obtain an *infinite* family of recurrences with the boxed property:

$$(n+2)^2 \tilde{u}_{n+2} - 3a(3n^2+9n-2)\tilde{u}_{n+1} + 27a^2(n+4)(n-2)\tilde{u}_n = 0, \quad a \in \mathbb{Q}, \ n \geq 0.$$

because $\tilde{u}_n = a^n u^n$ for all n.

Outline

Effective version

In practice, how to decide for which values of s_0, s_1 the generating function $S(x)$ of

$$(n + b_0)s_n + (a_1 n + b_1)s_{n-1} + (a_2 n + b_2)s_{n-2} = 0$$

is algebraic/globally bounded?

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Possible cases:

`Alg`: $S(x)$ is algebraic for all $(s_0, s_1) \in \mathbb{Q}^2$.

`Transc`: $S(x)$ is transcendental for all $(s_0, s_1) \in \mathbb{Q}^2 \setminus \{(0,0)\}$.

`Mixed`: The set of pairs (s_0, s_1) such that $S(x)$ is algebraic forms a one-dimensional $\mathbb{Q}$-vector subspace of $\mathbb{Q}^2$.

Effective version

In practice, how to decide for which values of s_0, s_1 the generating function $S(x)$ of

$$(n+b_0)s_n + (a_1 n + b_1)s_{n-1} + (a_2 n + b_2)s_{n-2} = 0$$

is algebraic/globally bounded?

⇝ **Algorithm:**

Input: b_0, a_1, b_1, a_2, b_2 such that $a_1, a_2 \neq 0,\ b_2 = (2a_2b_1 - a_1a_2b_0)/a_1,\ b_0 \in \mathbb{N}$.

Output: `Alg`; `Transc`; or `Mixed`, (s_0, s_1).

The condition $b_2 = (2a_2b_1 - a_1a_2b_0)/a_1$ guarantees algebraicity $\iff$ global boundedness. So we found **all** algebraic and **all** globally bounded solutions!

Key idea: linear combination of certain integrals

The generating function $S(x) = \sum_{n \geq 0} s_n x^n$ writes as:

$$S(x) = \underbrace{x^{-b_0}(1 + a_1 x + a_2 x^2)^{\frac{(b_0-1)a_1 - b_1}{a_1}}}_{\text{algebraic}} (c_1 + b_0 s_0 I_1 + (a_1 s_0 + b_0 s_1 + b_1 s_0 + s_1) I_2)$$

with

$$I_1 = \int x^{b_0 - 1}(1 + a_1 x + a_2 x^2)^{-b_0 + \frac{b_1}{a_1}} dx,$$

$$I_2 = \int x^{b_0}(1 + a_1 x + a_2 x^2)^{-b_0 + \frac{b_1}{a_1}} dx.$$

General shape of I_1, I_2 (for $b_0 \neq 0$):

$$\int x^n (1 + a_1 x + a_2 x^2)^q dx, \quad a_1, a_2 \in \mathbb{Q}^*, \; n \in \mathbb{N}, \; q \in \mathbb{Q}.$$

Algorithm outline

1. Handle $a_1^2 - 4a_2 = 0$ or $b_0 = 0$ separately.
2. Decide algebraicity of l_1 and l_2.
3. Proceed according to the table below:

l_1	l_2	**Conclusion**
Algebraic	Algebraic	`Alg`
Algebraic	Transcendental	`Mixed`, $(b_0 + 1, -a_1 - b_1)$
Transcendental	Algebraic	`Mixed`, $(0, 1)$
Transcendental	Transcendental	Check if $b_0 s_0 l_1 + (a_1 s_0 + b_0 s_1 + b_1 s_0 + s_1) l_2$ is algebraic for some $(\tilde{s}_0, \tilde{s}_1) \in \mathbb{Q}^2 \setminus \{(0,0)\}$. Yes $\implies$ `Mixed`, $(\tilde{s}_0, \tilde{s}_1)$ No $\implies$ `Transc`

Algorithm outline

$$S(x) = \mathcal{A}(x)(c_1 + b_0 s_0 I_1 + (a_1 s_0 + b_0 s_1 + b_1 s_0 + s_1) I_2), \quad \mathcal{A}(x) \text{ algebraic}$$

I_1	I_2	**Conclusion**
Algebraic	Algebraic	`Alg`
Algebraic	Transcendental	`Mixed`, $(b_0 + 1, -a_1 - b_1)$
Transcendental	Algebraic	`Mixed`, $(0, 1)$
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Demonstration: a déjà vu moment

The recurrence for *Motzkin-type sequences*:

$$(n+2)m_n - (2n+1)m_{n-1} - (3n-3)m_{n-2} = 0$$

corresponds to

$$b_0 = 2,\ a_1 = -2,\ b_1 = -1,\ a_2 = -3,\ b_2 = 3.$$

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corresponds to

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$$I_1 = \underbrace{\int x(1-2x-3x^2)^{-\frac{3}{2}}\,dx}_{\text{algebraic}}, \qquad I_2 = \underbrace{\int x^2(1-2x-3x^2)^{-\frac{3}{2}}\,dx}_{\text{transcendental}}.$$

The algorithm returns `Mixed` with $(b_0+1, -(a_1+b_1)) = (3,3)$.

Another example

The recurrence for *large Schröder numbers*:

$$(n+1)s_n - (6n-3)s_{n-1} + (n-2)s_{n-2} = 0$$

corresponds to

$$b_0 = 1,\ a_1 = -6,\ b_1 = 3,\ a_2 = 1,\ b_2 = -2.$$

$$I_1 = \underbrace{\int (1-6x+x^2)^{-\frac{3}{2}}\,dx}_{\text{algebraic}}, \qquad I_2 = \underbrace{\int x(1-6x+x^2)^{-\frac{3}{2}}\,dx}_{\text{algebraic}}.$$

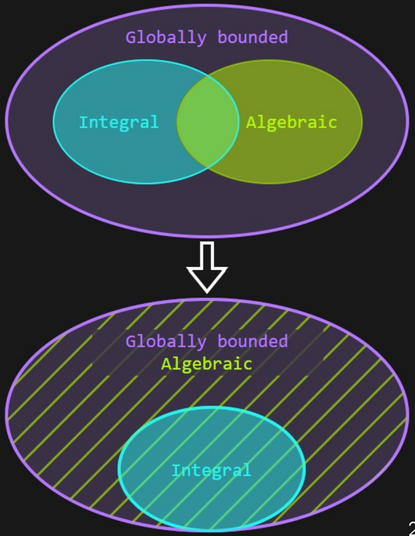
The algorithm returns `Alg`, proving that $S(x) = \sum_{n\geq 0} s_n x^n$ is **always algebraic**.

Outline

Back to the integrality problem. Are we in $\mathbb{Z}[[x]]$?

Motivation: reduce the search for integral solutions to studying algebraic ones.

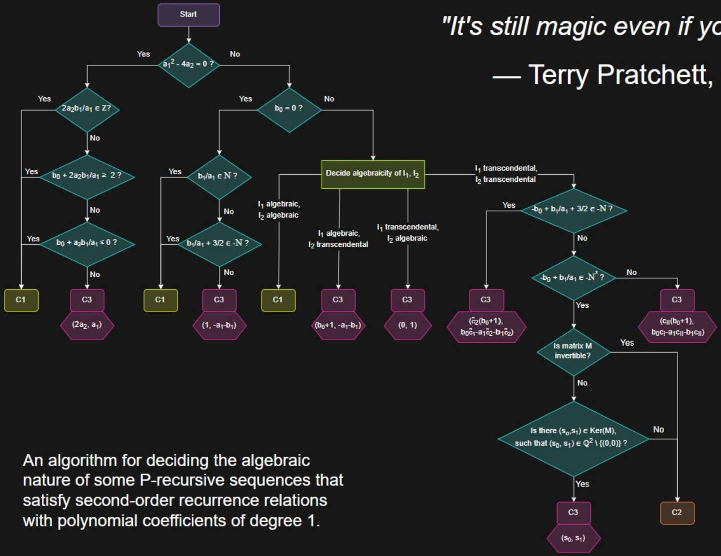
- [Fatou, 1906]:
  $\rightsquigarrow$ Integrality criteria for rational functions.
- [Pomerat, Straub, 2024]:
  $\rightsquigarrow$ Integrality criteria for n^{th} roots of power series of type $1 + x\mathbb{Z}[[x]]$.
- Work in progress by Bostan and M.:
  $\rightsquigarrow$ Integrality criteria for algebraic power series.



The road ahead

- Case $b_2 \neq \frac{2a_2b_1 - a_1a_2b_0}{a_1} \implies$ exponentials raised to nonzero powers in the generating function.
- Case $b_0 \notin \mathbb{N}$ is not currently treated by the algorithm.
- Higher-order recurrence relations.
- Nonlinear polynomial coefficients.
- Deciding when an algebraic function belongs to $\mathbb{Z}[[x]]$.

Thank you for your attention!



"It's still magic even if you know how it's done."

— Terry Pratchett, *A Hat Full of Sky*.

An algorithm for deciding the algebraic nature of some P-recursive sequences that satisfy second-order recurrence relations with polynomial coefficients of degree 1.

Outline

Deciding the algebraic nature of I_1, I_2.

$$I_1 = \int x^{b_0-1}(1+a_1x+a_2x^2)^{-b_0+\frac{b_1}{a_1}}\,dx, \qquad I_2 = \int x^{b_0}(1+a_1x+a_2x^2)^{-b_0+\frac{b_1}{a_1}}\,dx.$$

General shape of I_1, I_2 (for $b_0 \neq 0$):

$$I(n,q) := \int x^n(1+a_1x+a_2x^2)^q\,dx, \quad a_1, a_2 \in \mathbb{Q}^*, \; n \in \mathbb{N}, \; q \in \mathbb{Q}.$$

$I(n,q)$ decomposes as a combination of:

- Algebraic functions and $I(0,q)$, or
- Algebraic functions, $I(0,q)$ and $I(-2q-1,q)$, or
- Algebraic functions and logs multiplied by (computable) constants that depend on the recurrence parameters.

Integrality analysis: example

Consider a recurrence relation

$$n s_n + (2n+3) s_{n-1} + 9(n+3) s_{n-2} = 0.$$

Apply the algorithm $\implies$ $S(x) = \sum_{n \geq 0} s_n x^n$ is globally bounded if and only if $s_1 = -5 s_0$. In that case, one has

$$S(x) = s_0 (1 + 2x + 9x^2)^{-\frac{5}{2}}.$$

For which values of $s_0 \in \mathbb{Z}$ does $s_0 (1 + 2x + 9x^2)^{-\frac{5}{2}}$ lie in $\mathbb{Z}[[x]]$?

Integrality analysis: example

For which values of $s_0 \in \mathbb{Z}$ does $s_0(1+2x+9x^2)^{-\frac{5}{2}}$ lie in $\mathbb{Z}[[x]]$?

Lemma [Pomerat, Straub, 2025]

Let $a, b \in \mathbb{Z}$, $\lambda \in \mathbb{Q}$. Let k be the denominator of λ brought to the lowest terms. Then $(1+ax+bx^2)^\lambda \in \mathbb{Z}[[x]]$ if and only if

- $a, b \in k\ rad(k)\mathbb{Z}$, or
- $k = 2\kappa$ and $a, b \in \kappa\ rad(\kappa)\mathbb{Z}$ as well as $(a, b) \equiv (2, 1) \pmod 4$.

Here $rad(k)$ denotes the largest squarefree integer dividing k.
For example, $rad(4) = 2$, $rad(24) = 6$.

Integrality analysis: example

For which values of $s_0 \in \mathbb{Z}$ does $s_0(1+2x+9x^2)^{-\frac{5}{2}}$ lie in $\mathbb{Z}[[x]]$?

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- $k = 2\kappa$ and $a, b \in \kappa\ rad(\kappa)\mathbb{Z}$ as well as $(a, b) \equiv (2, 1) \pmod 4$.

In our case $a = 2, b = 9, \lambda = -\frac{5}{2}, k = 2, \kappa = 1 \implies (1+2x+9x^2)^{-\frac{5}{2}} \in \mathbb{Z}[[x]]$.
Hence, $s_0(1+2x+9x^2)^{-\frac{5}{2}} \in \mathbb{Z}[[x]]$ for all $s_0 \in \mathbb{Z}$.

What if $b_2 \neq \frac{2a_1b_1 - a_1a_2b_0}{a_1}$?..

- No counterexample to the claim "globally bounded $\implies$ algebraic" found.
- There could still be algebraic and globally bounded solutions! Consider the following recurrence:

$$ns_n + (3n+2)s_{n-1} + (2n+2)s_{n-2} = 0,$$

so that

$$b_0 = 0, \; a_1 = 3, \; b_1 = 2, \; a_2 = 2, \; b_2 = 2.$$

One has $b_2 = 2 \neq 4 = (2a_1b_1 - a_1a_2b_0)/a_1$. At the same time, the generating function is rational ($\implies$ algebraic $\implies$ globally bounded):

$$\sum_{n \geq 0} s_n x^n = \frac{(5s_0 + s_1)(x^2 + x) + s_0}{(x+1)(2x+1)^2}.$$