Combinatorics and Arithmetic of Lissajous 3-braids

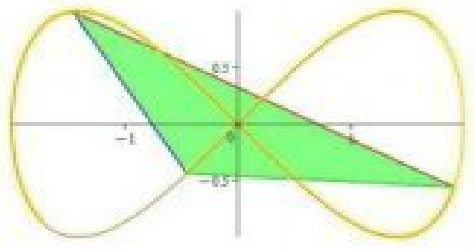
Hiroaki Nakamura (The Univ. of Osaka)

Online talk at the workshop

”Combinatorics and Arithmetic for Physics”

Organisers: Gérard H.E. DUCHAMP, Maxim KONTSEVICHI, Gleb KOSHEVOY, Sergei NECHAEV, Karol A. PENSON

@IHES, 19-21 Nov. 2025.



My talk corresponds to the arxiv paper https://arxiv.org/pdf/2008.00585
with some subsequent progresses and it is a **joint work with Eiko Kin and Hiroyuki Ogawa**.

This set of slides has been edited into a print–friendly form with some correction

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Talk Plan

1. Lissajous Curves, 3-body motions
2. Toward good representatives
3. Geometric Background: Shape Sphere
4. Motivation from knots

1. Lissajous Curves

Consider the curve on $\mathbb{C} = \mathbb{R} + \sqrt{-1}\mathbb{R}$:

$$L(t) = A\sin\bigl(2\pi m t\bigr) + \sqrt{-1}B\sin\bigl(2\pi n t + \phi\bigr) \quad (t \in \mathbb{R})$$

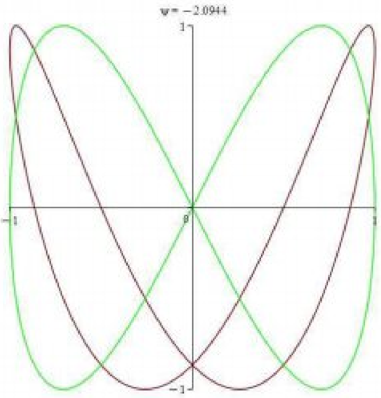
where $A, B > 0$,

$m \in \mathbb{Z}$: horizontal angular frequency,

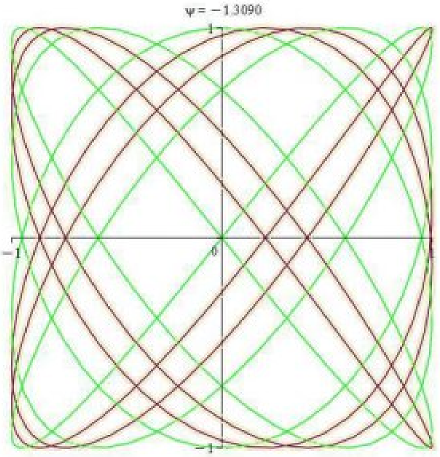
$n \in \mathbb{Z}$: vertical angular frequency,

$\phi \in \mathbb{R}$: phase difference.

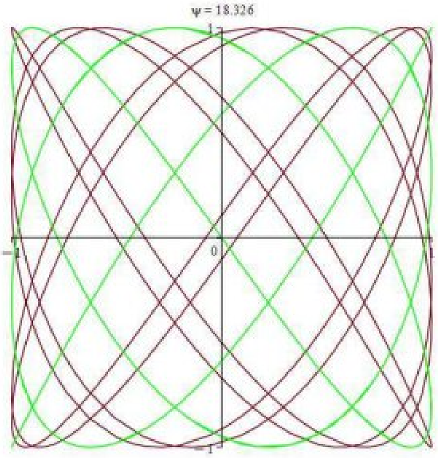
Right click a pin-mark to view the attached video.



m=1, n=-2



m=4, n=-5



m=-5, n=7

Consider the curve on $\mathbb{C} = \mathbb{R} + \sqrt{-1}\mathbb{R}$:

$$L(t) = A\sin(2\pi m t) + \sqrt{-1}B\sin(2\pi n t + \phi) \quad (t \in \mathbb{R})$$

where $A, B > 0$,

$m \in \mathbb{Z}$: horizontal angular frequency,

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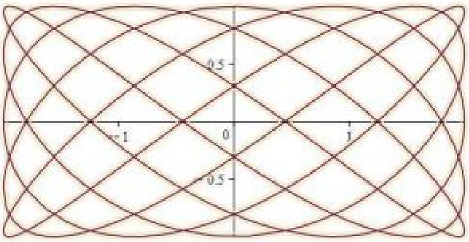
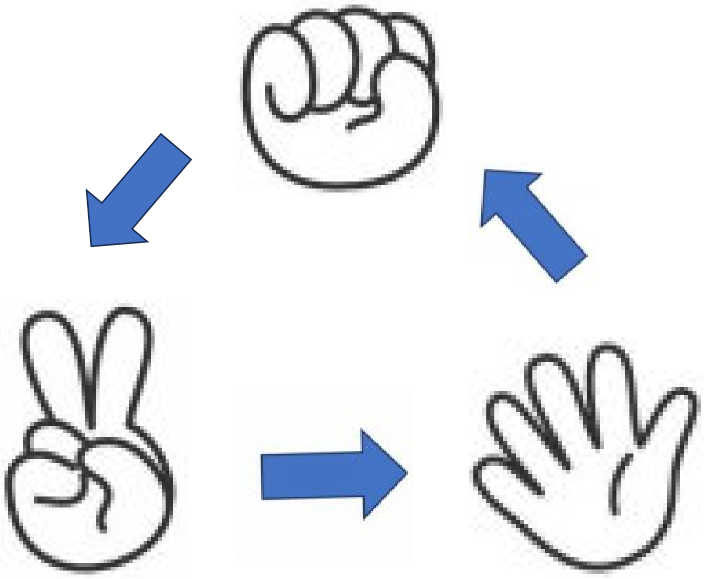


☞ $L(t)=L(t+1)$ a closed curve.

A=2, B=1,
m=-5, n=7,
Φ=11π/10



$L(t)=L(t+n)$

$L(t+\frac{1}{3})=L(t+\frac{1}{3}+n)$

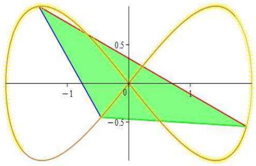
$L(t-\frac{1}{3})=L(t-\frac{1}{3}+n)$

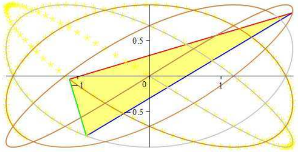
Consider 3-body motion
a(t)=L(t-$\frac{1}{3}$), b(t)=L(t), c(t)=L(t+$\frac{1}{3}$)

✋ ✊ ✌

on the Lissajous curve
(Choreographic Motion of a triangle)

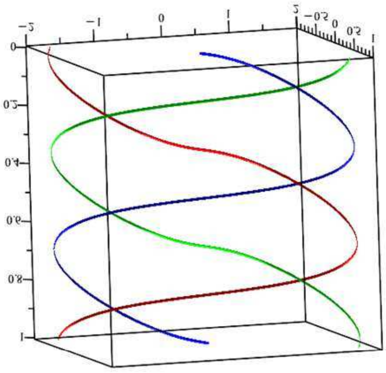
…→paper→rock→scissors
→paper→rock→scissors→paper
→rock→scissors→…

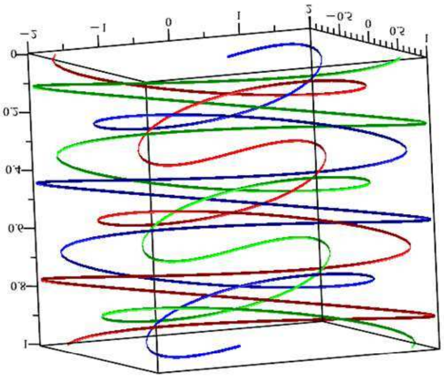




Cf. Moeckel-Montgomery: realization in celestial mechanics

Drawing the motion of {a(t),b(t),c(t)} along the timeline t: 0→1 produces a pure braid with 3 stands, as long as the 3 bodies have no collisions:

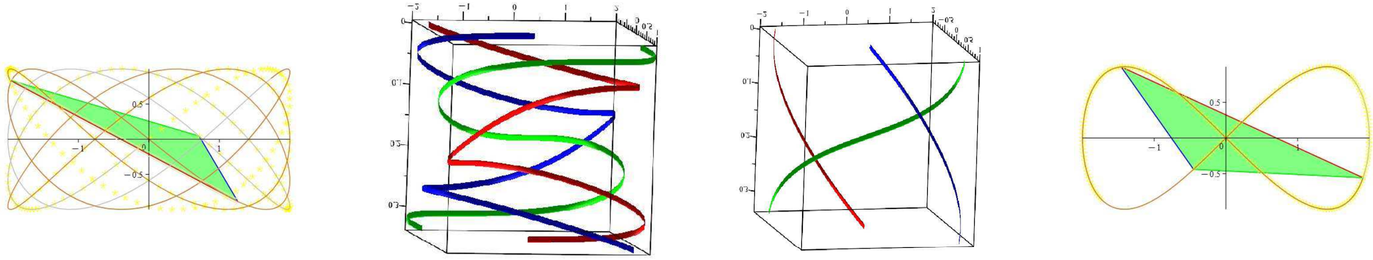




Restricting the timeline to t:0→$\frac{1}{3}$ gives a 3-braid $\beta = \beta$(m,n,ϕ) cyclically permuting {a,b,c}
AIM: Give a classification of the conjugacy classes of those “Lissajous 3-braids” in "B_3/center” and give good illustration of representatives in each class.

AIM 1 : To give a classification of the conjugacy classes of those “Lissajous 3-braids” $\beta = \beta\,(\mathrm{m,n},\phi)$ in B_3 modulo center (⇔modulo perspectives).

AIM 2 ：To give good illustration of a representative in each class.



$\beta\,(-5,-8,0) \cong \beta\,(1,-2,0)$

In regards of “collision-free” 3-body motions, for simple reasons of symmetry, we don’t lose much information by restricting the frequency pair (m,n) to those satisfying

Frequencies Hypothesis
(1)m and n are coprime,
(2)m≡n≡1（mod 3）

Collision-free condition ⇔ $\frac{m-n}{6} - \frac{\phi}{\pi} m \notin \mathbb{Z}$

Lemma (Eliminating continuous parameter ϕ).
For any fixed horizontal and vertical frequencies m and n, changing the phase difference ϕ amounts to β(m,n,ϕ) lying in the same conjugacy class or its mirror reflected class.

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Mirror reflection:

$$\sigma_i \leftrightarrow \sigma_i^{-1} \ (i = 1, 2)$$

Recall Frequencies Hypothesis
(1)m and n are coprime,
(2)m≡n≡1 (mod 3)

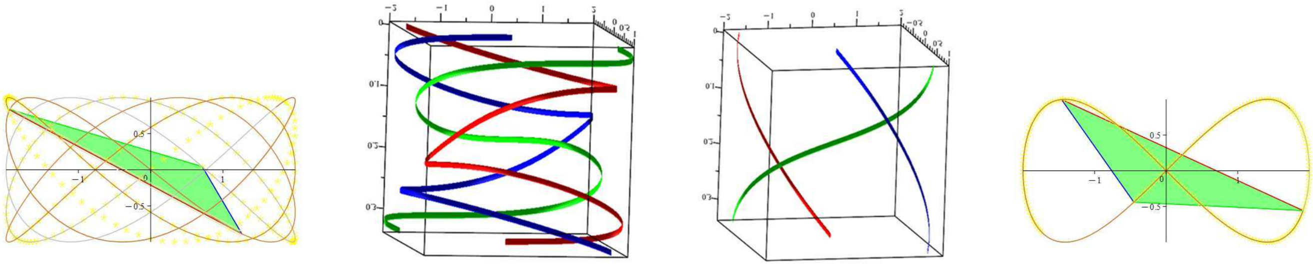
Collision-free condition
⇔ $\frac{m-n}{6} - \frac{\phi}{\pi} m \notin \mathbb{Z}$

Definition: Given (m,n) as above left, define C{m,n} to be the extended conjugacy class of B_3/<center> that contains all Lissajous 3-braids β(m,n,ϕ) for ϕ with the collision-free condition.
We call C{m,n} the **Lissajous (extended) class of type (m,n).**

Theorem (Bijective Reduction of standard Frequency Pairs)
The Lissajous 3-braid classes C{m,n} are exhausted by those (m,n) from the set

$$\mathcal{P} := \left\{ (m,n) \in \mathbb{Z}^2 \,\middle|\, \begin{array}{c} \gcd(m,n) = 1, m \equiv n \equiv 1 \bmod 3 \\ mn < 0,\ |m| < |n| \leq 2|m| \end{array} \right\}.$$

Moreover, each (m,n) of *P* gives distinct classes.



β (-5,-8,0) ≅ β (1,-2,0),
where (1,-2) belongs to *P*
but (-5,-8) does not.

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2. Toward a good representative of C{m,n}

Goal:

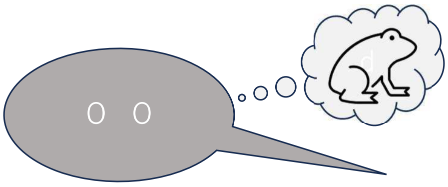
To the initial data (m,n, ϕ), associate:

・a sequence (frieze pattern) of tadpole symbols {b,d,p,q}

・a 3-braid in $B_3/center \cong PSL_2(\{Z\})$

representing the Lissajous motion by (m,n, ϕ).

b	d
p	q



Example (m,n) = (-5,7) slope=1, level=1
Enhanced Christoffel word cw(1,1)=c(1)c(1)=0101 → qbdbqbdb → ⋯.

Algorithm consists of 4 steps:

1. To standard pair (m,n) in P, associate a positive integer N (level) and a positive rational ξ(slope).

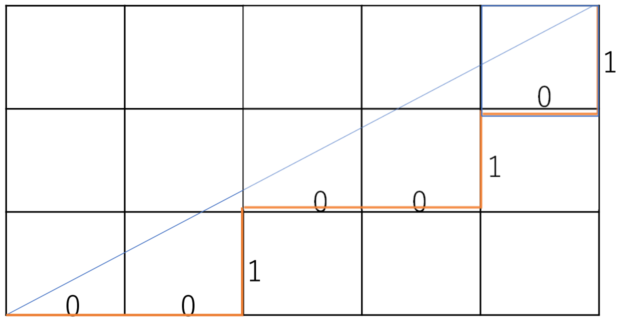
2. From the slope ξ, produce the Christoffel word c(ξ) in two symbols {0,1}.

3. Enhance doubled c(ξ)c(ξ) to cw(ξ,N) by a symbolic mapping at level N with the following rules
   - ☞ 0→ either one of $\{(pq)^{N-1}p, (qp)^{N-1}q,\ (bd)^{N-1}\,d, (db)^{N-1}d\}$;
   - ☞ 1→ either one of $\{(pq)^{N}p,\quad (qp)^{N}q,\ (bd)^{N}\,d, (db)^{N}d\}$;
   - ☞ If m>0 then begin by p, if m<0 then begin by q;
   - ☞ Every word from $\{(pq)^{N-1}p\,, (pq)^{N}p\}$ is adjacent to a word from $\{(db)^{N-1}d, (db)^{N}d\}$;
   - ☞ Every word from $\{(qp)^{N-1}p\,, (qp)^{N}p\}$ is adjacent to a word from $\{(bd)^{N-1}b, (bd)^{N}b\}$;

4. Obtain a 3-braid in C{m,n} from cw(ξ,N) by sending the symbols b,d,p,q to braids by
   ☞ d→$\sigma_1^{-1}\sigma_2^{-1}$, ☞ b→$\sigma_2^{-1}\sigma_1^{-1}$, ☞ q→$\sigma_2\sigma_1$, ☞ p→ $\sigma_1\sigma_2$

Review of Christoffel word:

c(q/p)=the word reading 01-symbols along the lattice path closest under the diagonal slope on any p×q-rectangle.



C(3/5)=00100101

Step 1: Given (m,n) in the set **of standard frequency pairs**

$$\mathcal{P} := \left\{ (m,n) \in \mathbb{Z}^2 \,\middle|\, \begin{matrix} \gcd(m,n)=1, m \equiv n \equiv 1 \bmod 3 \\ mn < 0,\ \ |m| < |n| \le 2|m| \end{matrix} \right\}.$$

the ratio |n/m| is in the semi-closed interval (1,2] which is a disjoint union of the smaller intervals (u_{N+1},u_N], where u_N=(3N-1)/(3N-2) forms a decreasing sequence 2>5/4>8/7>11/10>14/13>⋯>1

☞The level N of (m,n) is defined so that |n/m| is contained in (u_{N+1},u_N].

☞The slope ξ =q/p (with GCD(p+q,6)=1) is then defined by the equations

$$\begin{cases} |m| & = p(3N-2)+q(3N+1), \\ |\frac{m-n}{3}| & = p(2N-1)+q(2N+1). \end{cases}$$

Note: The above correspondence gives a bijection between *P* and the following level-slope set

$$\mathcal{L} := \left\{ (N,\xi) \in \mathbb{N} \times \mathbb{Q}_{\ge 0} \,\middle|\, \xi = \frac{q}{p},\ \gcd(p,q) = \gcd(p+q,3) = 1 \right\}.$$

Step 3. Enhance doubled c(ξ)c(ξ) to cw(ξ,N) by a symbolic mapping at level N with the following rules

- ☞ 0→ either one of $\{(pq)^{N-1}p, (qp)^{N-1}q,\ (bd)^{N-1}\, d, (db)^{N-1}d\}$;
- ☞ 1→ either one of $\{(pq)^{N}p,\ \ (qp)^{N}q,\ (bd)^{N}\, d, (db)^{N}d\}$;
- ☞ If m>0 then begin by p, if m<0 then begin by q;
- ☞ Every word from $\{(pq)^{N-1}p\,, (pq)^{N}p\}$ is adjacent to a word from $\{(db)^{N-1}d, (db)^{N}d\}$;
- ☞ Every word from $\{(qp)^{N-1}p\,, (qp)^{N}p\}$ is adjacent to a word from $\{(bd)^{N-1}b, (bd)^{N}b\}$;

Step4. Obtain a 3-braid in C{m,n} from cw(ξ,N) by sending the symbols b,d,p,q to braids by

☞ d→$\sigma_1^{-1}\sigma_2^{-1}$, ☞ b→$\sigma_2^{-1}\sigma_1^{-1}$, ☞ q→$\sigma_2\sigma_1$, ☞ p→ $\sigma_1\sigma_2$

Example (m,n) = (1,-2) slope=0, level=1
Enhanced Christoffel word cw(0,1)=c(0)c(0)=00 → pd
→ $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$

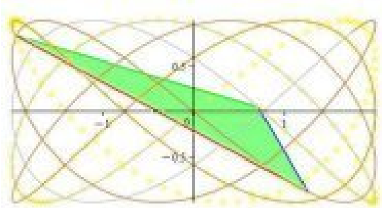
Example (m,n) = (-5,7) slope=1, level=1
Enhanced Christoffel word cw(1,1)=c(1)c(1)=0101 → qbdbqbdb → ⋯

Example (m,n)=(-23,28), level N=2, slope ¼
Christoffel word c(1/4)=00001
Enhanced Christoffel word
cw(1/4,2)=qpqbdbqpqbdbqpqpq-bdbqpqbdbqpqbdbdb

Example (m,n)=(-17,25), Level N=1, Slope=3/5
Christoffel word c(3/5)=00100101
Enhanced cw(3/5,1)=qbqpqbqbdbqbdb-qbqpqbqbdbqbdb

Observation：
mn=even, pq=even ⇒cw(q/p)=half+mirror (type I) ,
mn=odd, pq=odd ⇒ cw(q/p)=half+half (type II)

3. Geometric Background: Shape Sphere (Moduli space of the plane triangles)



Translate a motion of triangles
along Lissajous curve into
a closed curve on the configuration space

$$Conf^3(\mathbb{C}) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{C}^3 \middle| a \neq b \neq c \neq a \right\}$$

The Lissajous triangles $\Delta(t) = \left\{ \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} \right\}_{0 \leq t \leq \frac{1}{3}}$ forms a closed curve

giving an element of $\pi_1(Conf^3(C)/S_3)=B_3$

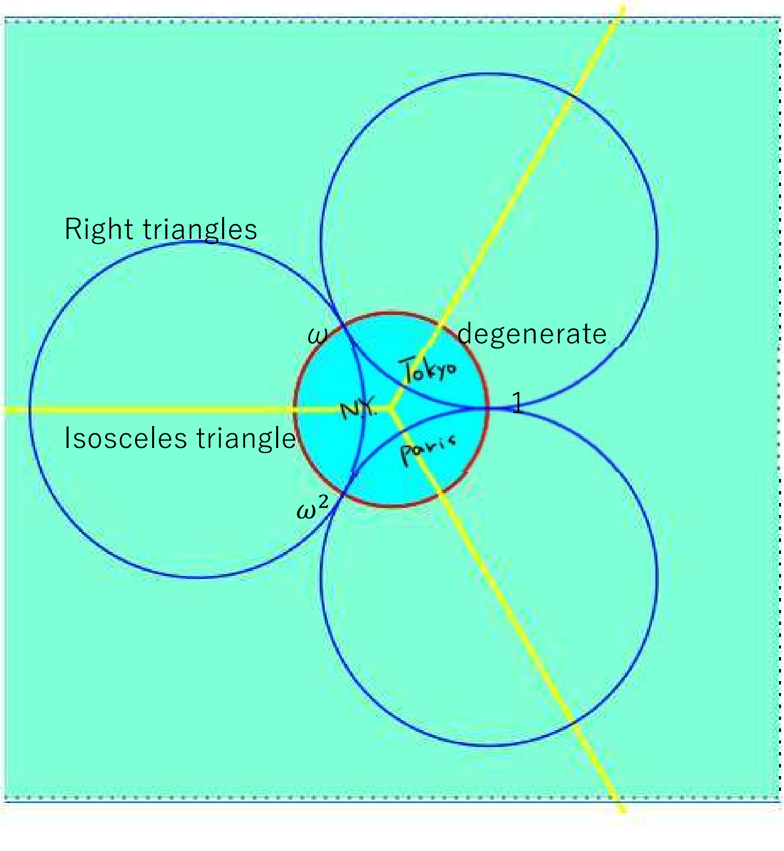
Shape function

$$\psi : Conf^3 \rightarrow S^2 - 3pt := P^1(C) - \{1, \omega, \omega^2\}$$

We employ the ratio of finite Fourier transforms of triangle vertices (ω:=exp(2πi/3))

$$\psi([a(t), b(t), c(t)]) = \frac{a(t) + \omega\ b(t) + \omega^2 c(t)}{a(t) + \omega^2 b(t) + \omega\ c(t)}$$

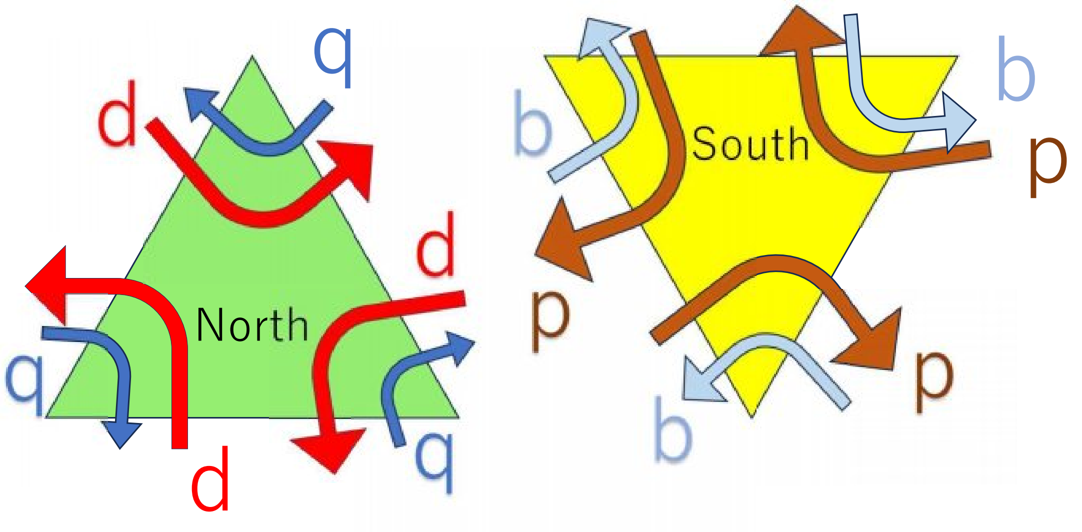
And trace their image in $\pi_1\left(\frac{P^1(C)-\{1,\omega,\omega^2\}}{S_3}\right) = B_3/center = PSL_2(Z)$



$S^2 - 3pt$
$:= P^1(C) - \{1, \omega, \omega^2\}$

Regard
the north hemisphere as an ideal triangle with vertices (punctures) 1, ω, ω^2
the south hemisphere as an ideal triangle with vertices (punctures) 1, ω, ω^2
Read the pattern of the orbit curve of Δ(t) cutting these triangle in terms of {b,d,p,q}-words

☞ p→$\sigma_1^{-1}\sigma_2^{-1}$, ☞ q→$\sigma_2^{-1}\sigma_1^{-1}$,
☞ b→$\sigma_2\sigma_1$, ☞ d→ $\sigma_1\sigma_2$



$$1 \to \langle(\sigma_1\sigma_2)^3\rangle(= \text{center}) \to B_3 \to \mathrm{PSL}_2(\mathbb{Z}) \to 1.$$

$$\left.\begin{aligned}
\mathbf{b} &= \sigma_2^{-1}\sigma_1^{-1}(\text{south}, \text{left}) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \\
\mathbf{d} &= \sigma_1^{-1}\sigma_2^{-1}(\text{north}, \text{left}) \mapsto \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\
\mathbf{p} &= \sigma_1\sigma_2(\text{south}, \text{right}) \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\
\mathbf{q} &= \sigma_2\sigma_1(\text{north}, \text{right}) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}
\end{aligned}\right\}$$

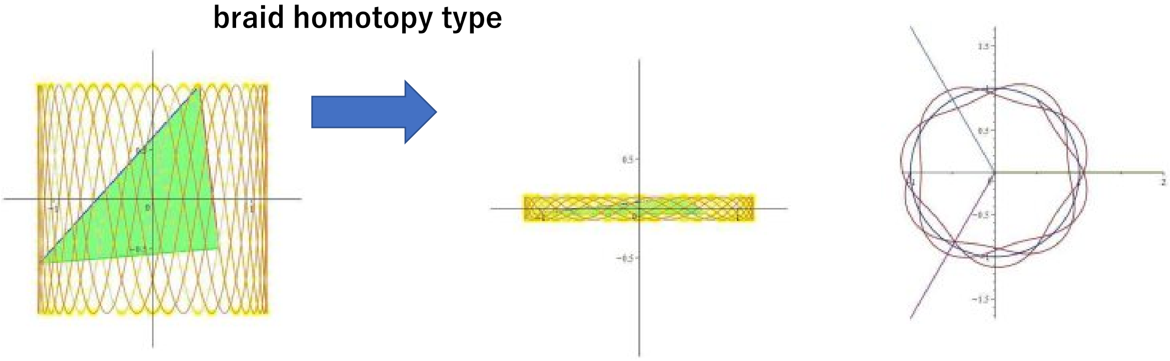
{b,d,p,q} generates an index 2 subgroup

$$\bar{\Gamma}^2 \ (\subset \mathrm{PSL}_2(\mathbb{Z}))$$
$$\|$$
$$(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$$

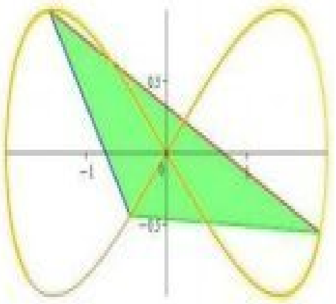
How to translate the behavior of {orbit $\psi(\Delta(t))$} into braid words?

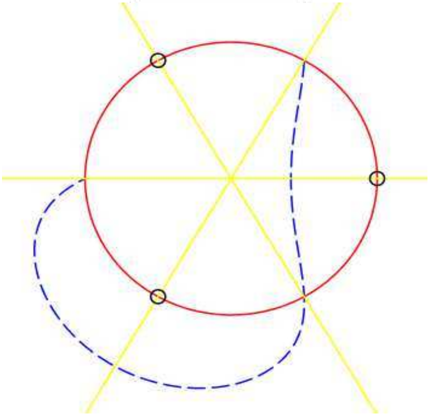
$$\psi(\Delta_{m,n,\phi}(t)) = \underbrace{e^{-4\pi imt+\frac{\pi}{3}i}}_{\text{Global flight along the equator}} \underbrace{\left(\frac{1+\frac{Bi}{A}e^{2\pi i(m-n)t-\phi i}}{1+\frac{Bi}{A}e^{-2\pi i(m-n)t+\phi i}}\right)}_{\text{Small figure 8 vibration}}$$

Deformation by A>>B keeps braid homotopy type

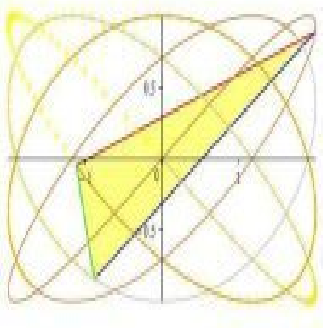


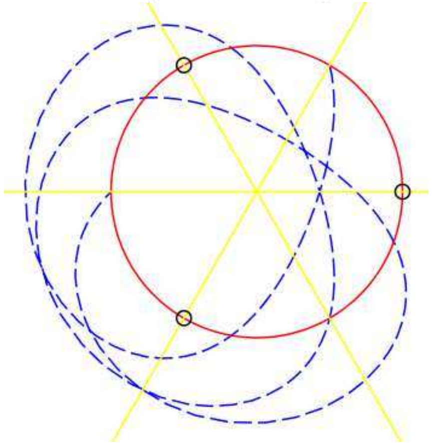
The case [m = 1, n=-2]





The case [m = 4, n=-5]



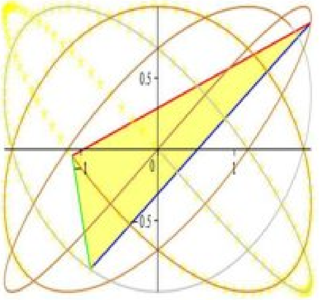


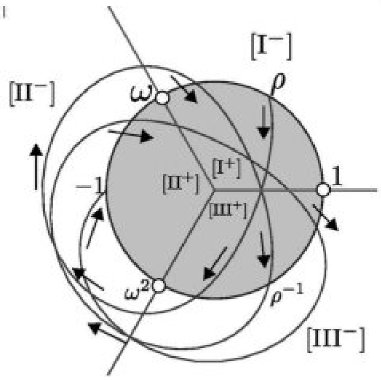
C.Series (1985) geodesic on modular curve H/PSL(2,Z)
→ a geodesic on H
→Cutting sequence on the Farey Tesselation
→Continued fraction expansion of an end point of the geodesic on H
(works of E.Artin, R.Moekkel)

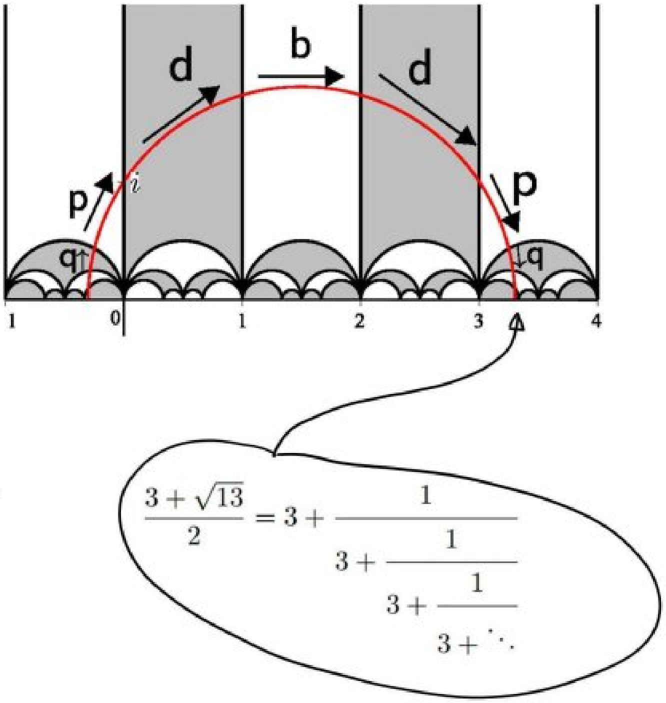
F.Boca. C.Merriman (2018) H/ Γ (2) version
→Cutting sequence on the checkered Farey Tesselation
→Continued fractions by odd terms of an end point of the geodesic on H.

(m,n)=(4,-5), Level N=2, Slope ξ=0

$W_{4,-5} = dbd\text{-}pqp = \sigma_2^3\sigma_1^{-3} = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}$

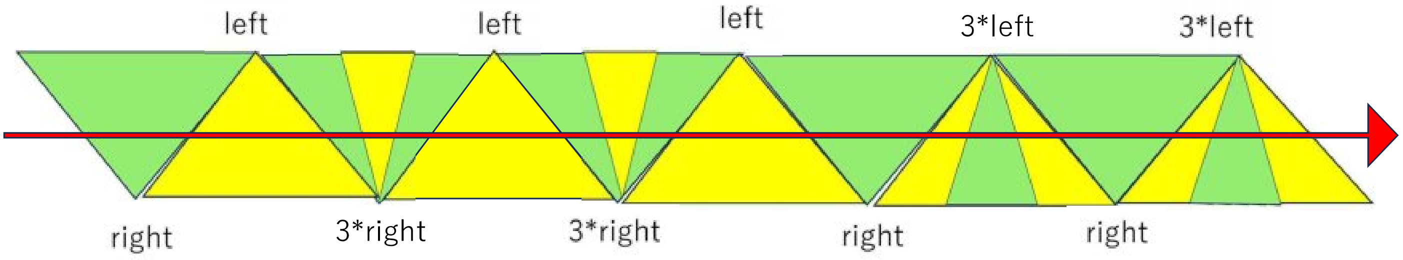






$$\frac{3+\sqrt{13}}{2} = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ddots}}}$$

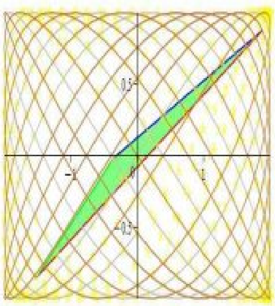
Frequency (m,n)=(-11,16) ⇔ Slope&Level (ξ,N)=(2/3, 1)
C(2/3)=00101
☞ cw(2/3,1)=qb**qpq**b**qpq**-bq**bdb**q**bdb**

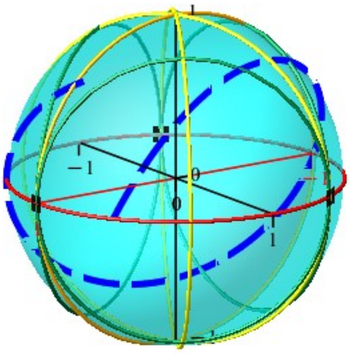


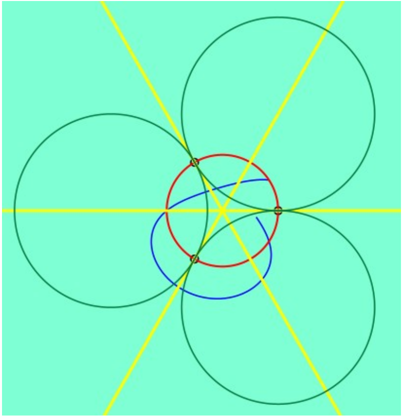
The word can be realized exactly from the motion

$$(m,n) = (-11,16),\ \phi = \frac{13\pi}{11},\ 0 \le t \le \frac{1}{3}$$

$$(m,n)=(-11,16),\ \phi=\frac{13\pi}{11},\ 0\le t\le\frac{1}{3}$$







$$(m,n)=(-11,16),\ \boxed{\phi=0},\ 0\le t\le\frac{1}{3}$$

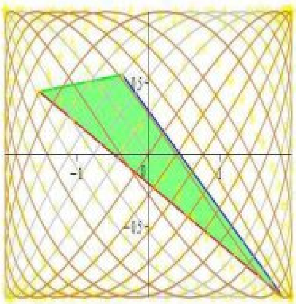
This gives a more symmetric sequence in {b,d,p,q}

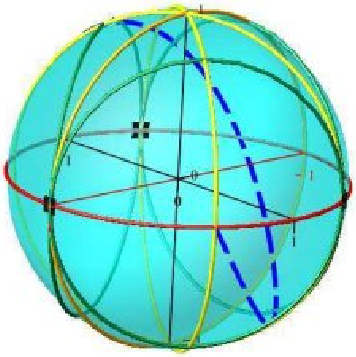
The above enhanced Chirstoffel word

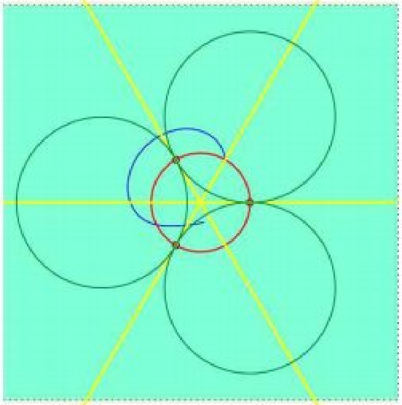
☞ cw(2/3,1)=qb**qpq**b**qpq**-bq**bdb**q**bdb**

We can shift it to “Double palindromic word”

☞ pw(2/3,1)=b**qpq**b**qpqb**-q**bdb**q**bdbq** (=:$\boldsymbol{W_{m,n}}$)







Observation：
(type I)
mn=even, pq=even ⇒ ϕ =0 stays “collision-free”
→ cw(q/p)=half+mirror
→ pw(q/p)=doubly-palindromic sequence $W_{m,n}$

E.Kin, H.N., H.Ogawa "Lissajous 3-braids“ Journal of Math. Soc. Japan 75(1)(2023), 195--228.

(type II)
mn=odd,　pq=odd ⇒ ϕ =0 has collision
→cw(q/p)=half+half

E.Kin, H.N., H.Ogawa “Lissajous 3 braids with phase differences” (in preparation)

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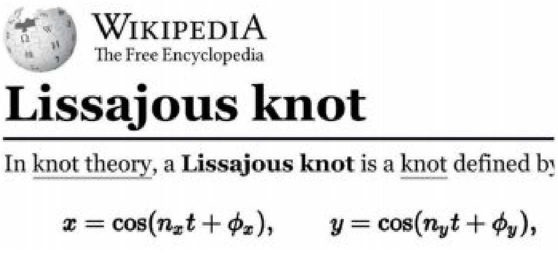
4. Knots obtained by closing Lissajous 3-braids are special cases of Lissajous toric knots

[1]C. Lamm: *There are infinitely many Lissajous knots*, Manuscripta Math. **93**, 29–37 (1997)
[2] C. Lamm and D. Obermeyer: *Billiard knots in a cylinder*, J. Knot Theory Ramifications **8**, 353–366 (1999)
[3] C. Lamm: *Deformation of cylinder knots*, 4th chapter of Ph.D. thesis, ‘Zylinder-Knoten und symmetrische Vereinigungen‘, Bonner Mathematische Schriften **321** (1999), available since 2012 as arXiv:1210.6639
[4] C. Lamm: *Symmetric unions and ribbon knots*, Osaka J. Math. **37** (2000), 537–550
[5] M. Soret and M. Ville: *Singularity knots of minimal surfaces in R4*, J. Knot Theory Ramifications **20** (2011), 513–546
[6] M. Soret and M. Ville: *Lissajous-toric knots*, J. Knot Theory Ramifications **29**, 2050003 (2020)

Lissajous knots and Lissajous toric knots are different classes of knots.



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Lissajous knot

In knot theory, a Lissajous knot is a knot defined by

$$x = \cos(n_x t + \phi_x), \qquad y = \cos(n_y t + \phi_y),$$

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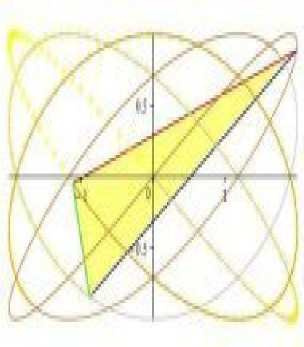
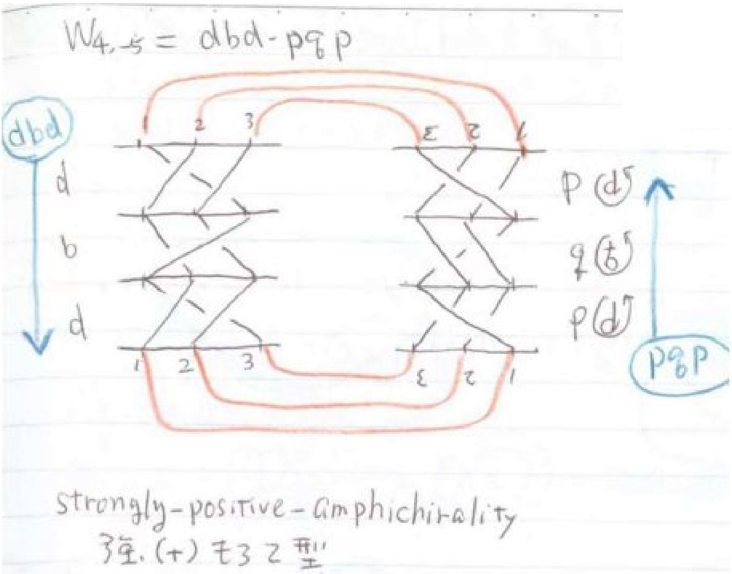
Lissajous-toric knot

In knot theory, a Lissajous-toric knot is a knot defined by parametric equations of the form:

m=4, n=-5, Φ=0

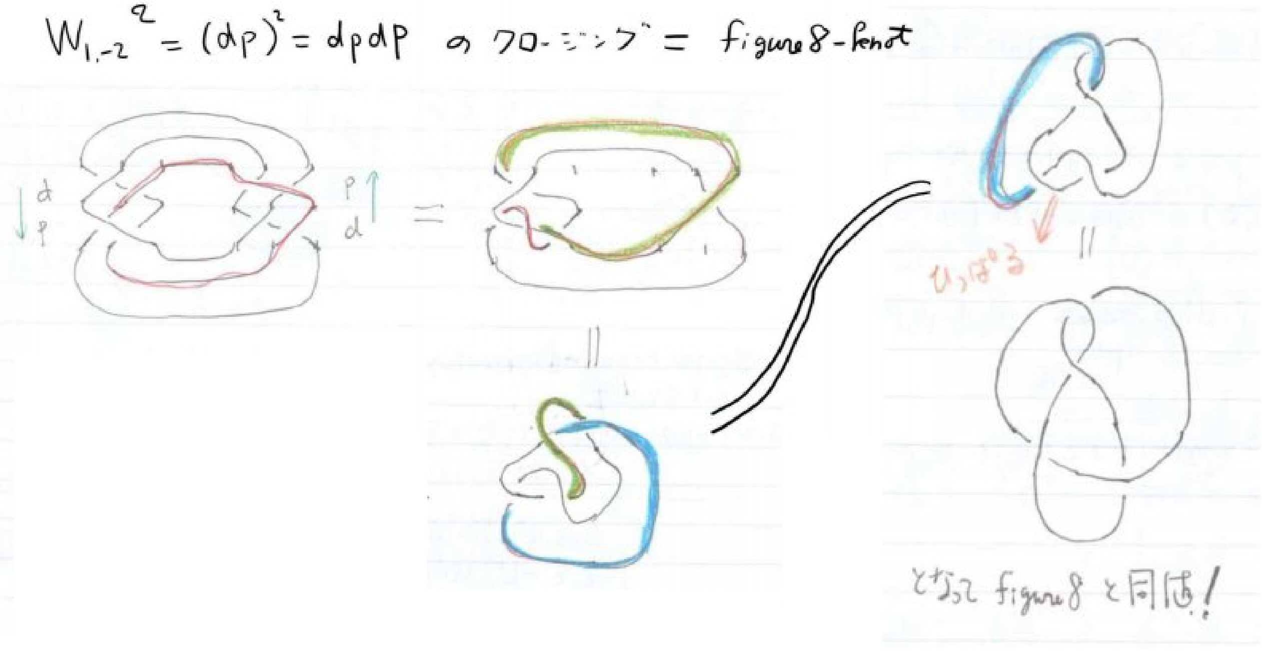




$W_{4,-5}$ = dbd-pqp

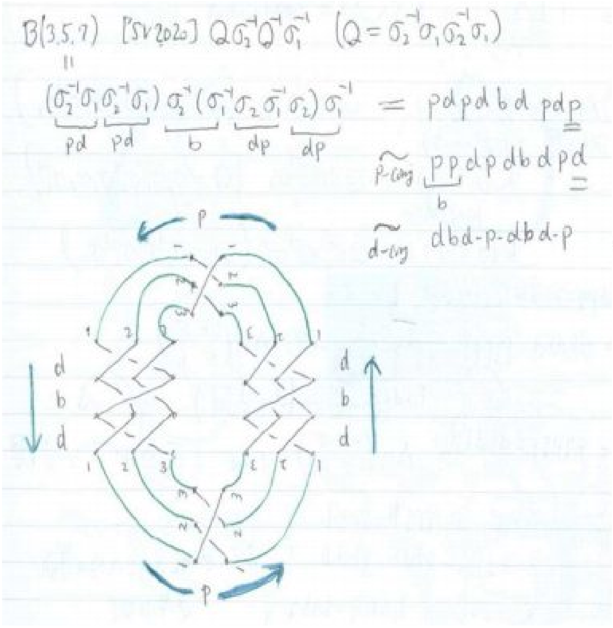
strongly-positive-amphichirality
強.(+)モろこ型

$W_{1,-2}{}^{2} = (dp)^{2} = dpdp$ のクロージング = figure8-knot



となって figure8 と同値!

This example motivated us to study Lissajous braids with general phase differences ϕ



B(3,5,7) [SV2020] $Q\sigma_2^{-1}Q^{-1}\sigma_1^{-1}$ $(Q=\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1)$

$$\| $$

$$(\underbrace{\sigma_2^{-1}\sigma_1}_{pd}\underbrace{\sigma_2^{-1}\sigma_1}_{pd})\underbrace{\sigma_2^{-1}(\sigma_1^{-1}}_{b}\underbrace{\sigma_2\sigma_1^{-1}}_{dp}\underbrace{\sigma_2)\sigma_1^{-1}}_{dp} = pdpdbdpd\underline{\underline{p}}$$

$$\underset{p\text{-conj}}{\sim} \underbrace{pp}_{b}dpdbdp\underline{\underline{d}}$$

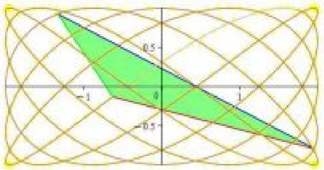
$$\underset{d\text{-conj}}{\sim} dbd\text{-}p\text{-}dbd\text{-}p$$

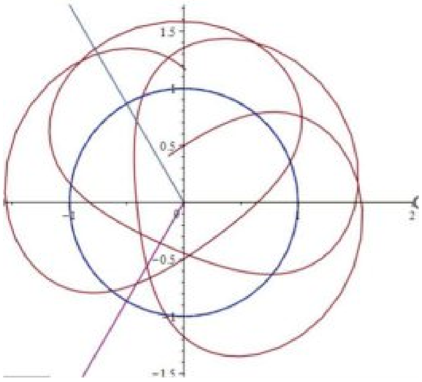
$m=-5$, $n=7$, 位相差 $\phi=-\frac{\pi}{10}$

のとき

$$W_{-5,7,\phi} = bdbq\text{-}bdbq$$

$(\sigma_1\sigma_2\sigma_1)$-conj.





Thank you very much for attentions!