Characterization of the unit object in the category $R\text{-}\widetilde{\mathrm{gmod}}$

(Joint work with K. Matsuura)

— Combinatorics and Arithmetic for Physics —

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Preliminaries

• $A = (a_{ij})_{i,j\in I:=\{1,2,\cdots,n\}}$: symmetrizable generalized Cartan matrix for Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

• $\{\alpha_i : i \in I\} \subset \mathfrak{h}^*$: set of simple roots, $\{h_i : i \in I\} \subset \mathfrak{h}$: set of simple coroots, $\{\Lambda_i : i \in I\} \subset \mathfrak{h}^*$: set of fundamental weights s.t. $a_{ij} = \alpha_j(h_i)$ and $\Lambda_j(h_i) = \delta_{ij}$.

• Root lattice : $Q := \bigoplus_i \mathbb{Z}\alpha_i \supset Q_+ := \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k} \in Q_+$, define the height of β by $|\beta| = k$.

• $(\ ,\ )$: symm.bilinear form on $\mathfrak{h}^*$ s.t. $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\lambda(h_i) = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{h}^*$ (We shall use the notation $\langle h_i, \lambda\rangle$ for $\lambda(h_i)$.)

• $P := \{\lambda \in \mathfrak{h}^* \,|\, \langle h_i, \lambda\rangle \in \mathbb{Z}(\forall i \in I)\}$: weight lattice$\supset P_+$: dominant weights

• $P^* := \{h \in \mathfrak{h} \,|\, \langle h, P\rangle \subset \mathbb{Z}\}$: dual weight lattice

• $W = \langle s_i \mid i \in I\rangle$: Weyl group

• $U_q(\mathfrak{g}) := \langle q^h, e_i, f_i \mid h \in P^*, \ i \in I\rangle/\mathbb{Q}(q)$: Quantum algebra ass. A.

• $U_q^-(\mathfrak{g}) := \langle f_i \mid i \in I\rangle$, $U_q^+(\mathfrak{g}) \overset{dual}{\longleftrightarrow} \mathcal{A}_q(\mathfrak{n}_+)$: Quantum coordinate algebra

Crystal Base I

Definition

Let $\mathbb{A} \subset \mathbb{Q}(q)$ *be the subring regular at* $q = 0$. *A pair* (L, B) *is a* *crystal base* *of* $M = \oplus_\lambda M_\lambda \in \mathcal{O}_{\mathrm{int}}(\mathfrak{g})$*(resp.* $U_q^-(\mathfrak{g})$*), if it satisfies:*

1. L *is a free* $\mathbb{A}$*-submodule of* M *(resp.* $U_q^-(\mathfrak{g})$*) such that*
$$M \cong \mathbb{Q}(q) \otimes_{\mathbb{A}} L \quad \text{(resp. } U_q^-(\mathfrak{g}) \cong \mathbb{Q}(q) \otimes_{\mathbb{A}} L\text{)}$$
$$L = \oplus_\lambda L_\lambda \ (L_\lambda := L \cap M_\lambda).$$
2. B *is a basis of the* $\mathbb{Q}$*-vector space* L/qL *and*
$$B = \sqcup_\lambda B_\lambda \ (B_\lambda := B \cap L_\lambda / qL_\lambda).$$
3. $\tilde{e}_i L \subset L$ *and* $\tilde{f}_i L \subset L$. *(*$\tilde{e}_i, \tilde{f}_i \in \mathrm{End}_{\mathbb{Q}(q)}(M)$ *Kashiwara operator)*
4. $\tilde{e}_i B \subset B \sqcup \{0\}$ *and* $\tilde{f}_i B \subset B \sqcup \{0\}$.
5. *For* $u, v \in B$, $\tilde{f}_i u = v \Longleftrightarrow \tilde{e}_i v = u$.

Crystal Base II

By ④ of the definition, B holds a colored oriented graph structure, called crystal graph:

Definition

The crystal graph of a crystal B is a colored oriented graph given by the rule:

$$b_1 \xrightarrow{i} b_2 \quad \Longleftrightarrow \quad b_2 = \tilde{f}_i b_1 \qquad (b_1, b_2 \in B).$$

Let $V(\lambda)$ (resp. $U_q^-(\mathfrak{g})$) be the integrable simple h.w.module (resp. nilp. negative subalg. of $U_q(\mathfrak{g})$) with the h.w.v u_λ ($\lambda \in P_+$) (resp. $1 := u_\infty$). Define

$$L(\lambda) := \sum_{i_j \in I, l \geq 0} \mathbb{A} \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\lambda, \quad B(\lambda) := \{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\lambda \bmod qL(\lambda) \,|\, i_j \in I, l \geq 0\} \setminus \{0\},$$

$$L(\infty) := \sum_{i_j \in I, l \geq 0} \mathbb{A} \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty, \quad B(\infty) := \{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty \bmod qL(\infty) \,|\, i_j \in I, l \geq 0\} \setminus \{0\}.$$

Theorem (Kashiwara)

The pair $(L(\lambda), B(\lambda))$ (resp. $(L(\infty), B(\infty))$) is a crystal base of $V(\lambda)$. (resp. $U_q^-(\mathfrak{g})$).

Crystals

Definition (Crystal)

A 6-tuple $(B, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i\in I}$ *is a crystal if* B *is a set and* $\exists 0 \notin B$ *and maps:*

$$\mathrm{wt} : B \to P, \quad \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \quad (i \in I)$$
$$\tilde{e}_i : B \sqcup \{0\} \to B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \ (i \in I),$$

satisfying :

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$.
2. *If* $b, \tilde{e}_i b \in B$, *then* $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
3. *If* $b, \tilde{f}_i b \in B$, *then* $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.
4. *For* $b, b' \in B$ *and* $i \in I$, *one has* $\tilde{f}_i b = b' \Longleftrightarrow b = \tilde{e}_i b'$.
5. *If* $\varphi_i(b) = -\infty$ *for* $b \in B$, *then* $\tilde{e}_i b = \tilde{f}_i b = 0$ *and* $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Example

For $i \in I$, set $B_i := \{(n)_i \mid n \in \mathbb{Z}\}$ and its crystal structure is given by

$$
\begin{aligned}
&\mathrm{wt}((n)_i) = n\alpha_i,\ \ \varepsilon_i((n)_i) = -n,\ \ \varphi_i((n)_i) = n,\\
&\varepsilon_j((n)_i) = \varphi_j((n)_i) = -\infty \ \ (i \neq j),\\
&\tilde{e}_i((n)_i) = (n+1)_i, \quad \tilde{f}_i((n)_i) = (n-1)_i,\\
&\tilde{e}_j((n)_i) = \tilde{f}_j((n)_i) = 0 \ \ (i \neq j)
\end{aligned}
$$

Crystal graph of B_i:

$$\cdots \xrightarrow{\ i\ } 1 \xrightarrow{\ i\ } 0 \xrightarrow{\ i\ } -1 \xrightarrow{\ i\ } -2 \longrightarrow$$

We define a morphism of crystals.

Definition

For crystals $\mathcal{B}_1, \mathcal{B}_2$, a crystal morphism $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that

1. $\Psi(0) = 0$,
2. *if $b \in \mathcal{B}_1$ and $\Psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for all $i \in I$,*
3. *if $b, b' \in \mathcal{B}_1$, $\Psi(b), \Psi(b') \in \mathcal{B}_2$ and $\tilde{f}_i b = b' (\Leftrightarrow b = \tilde{e}_i b')$, then $\tilde{f}_i \Psi(b) = \Psi(b')$ and $\Psi(b) = \tilde{e}_i \Psi(b')$ for all $i \in I$.*

$\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ is called an embedding if $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ is an injection and called an isomorphism if a bijection.

Crystals also hold a tensor product structure, which is obtained by the tensor product rule of crystal bases of $U_q(\mathfrak{g})$-modules.

Definition

The tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ of crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined to be the set $\mathcal{B}_1 \times \mathcal{B}_2$ whose crystal structure is defined by

1. $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,
2. $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle\}$,
3. $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle\}$,
4. $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & if\ \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & if\ \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$
5. $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & if\ \varphi(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & if\ \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$

where we write $b_1 \otimes b_2$ for $(b_1, b_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, and we set $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

Braid-type isomorphisms

For tensor products of the crystals B_i, there exist isomorphisms.

Proposition (Braid-type isomorphisms [N2])

For $i \in I$, set $B_i = \{(z)_i \mid z \in \mathbb{Z}\}$ and let $A = (a_{ij})_{i,j\in I}$ be a Cartan matrix. Then, there exist the following isomorphisms of crystals $\phi_{ij}^{(k)}$ $(k = 0, 1, 2, 3)$:

$$\phi_{ij}^{(0)} : B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i,\ (z_1)_i \otimes (z_2)_j \longmapsto (z_2)_j \otimes (z_1)_i, \qquad a_{ij} = a_{ji} = 0,$$

$$\phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j, \qquad a_{ij} = a_{ji} = -1,$$

$$(z_1)_i \otimes (z_2)_j \otimes (z_3)_j \longmapsto (\max(z_3, z_2 - z_1))_j \otimes (z_1 + z_3)_i \otimes (\max(-z_1, z_3 - z_2))_j,$$

$$\phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i, \qquad a_{ji} = -1, a_{ij} = -2,$$

$$(z_1)_i \otimes (z_2)_j \otimes (z_3)_i \otimes (z_4)_j$$

$$\longmapsto (\max(z_4, z_2 - 2z_1, 2z_3 - z_2))_j \otimes (\max(z_1 + z_4, z_2, z_1 - z_2 + 2z_3))_i$$

$$\otimes (-\max(-z_3, -z_4 - 2z_1, -2z_2 + 2z_3 - z_4))_j \otimes (-\max(-z_3 + z_4, -z_1, z_3 - z_2))_i.$$

Proposition

$$\phi^{(2)}_{ji} : B_j \otimes B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_i \otimes B_j \otimes B_i \otimes B_j, \quad a_{ji} = -1, a_{ij} = -2,$$

$$(z_1)_j \otimes (z_2)_i \otimes (z_3)_j \otimes (z_4)_i$$
$$\longmapsto (\max(-z_2 + z_3, -z_1 + z_2, z_4))_i \otimes (\max(z_1 - 2z_2 + 2z_3, z_3, z_1 + z_1 + 2z_4))_j$$
$$\otimes (-\max(-2z_2 + z_3 - z_4, -z_1 - z_4, -z_2))_j \otimes (-\max(-2z_2 + z_3, -z_1, z_3 - 2z_4))_i.$$

We omit the expression of $\phi^{(3)}_{ij}$. We call these $\phi^{(k)}_{ij}$ $(k = 0, 1, 2, 3)$ the braid-type isomorphisms of B_i's. Note that $\phi^{(k)}_{ij}$ and $\phi^{(k)}_{ji}$ are inverse to each other.

Cellular Crystal $\mathbb{B}_{\underline{w}} = \mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes \cdots \otimes B_{i_k}$

By braid-type isomorphisms, for any $w \in W$ and its reduced words $i_1 \cdots i_l$ and $j_1 \cdots j_l$, we obtain

$$B_{i_1} \otimes \cdots \otimes B_{i_l} \cong B_{j_1} \otimes \cdots \otimes B_{j_l}$$

Definition (cellular crystal)

For a reduced word $\underline{w} = i_1 i_2 \cdots i_k$ of $w \in W$, we call the crystal $\mathbb{B}_{\underline{w}} := B_{i_1} \otimes \cdots \otimes B_{i_k}$ a **cellular crystal** *associated with $\underline{w}$.*

Quiver Hecke Algebra I

For a finite index set I and a field $\mathbf{k}$, let $(Q_{i,j}(u,v))_{i,j\in I} \subset \mathbf{k}[u,v]$ be a family of polynomials satisfying: $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, $Q_{i,i}(u,v) = 0$ for any $i, j \in I$ and certain conditions. For $\beta = \sum_i m_i\alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$.

Definition

For $\beta \in Q_+$, the quiver Hecke algebra $R(\beta)$ associated with a Cartan matrix $A = (a_{ij})_{i,j\in I}$ and polynomials $(Q_{ij}(u,v))_{i,j\in I}$ is the algebra generated by

$$\{e(\nu)|\nu \in I^\beta := \{((\nu_1,\cdots,\nu_m) \mid \sum_{k=1}^{m} \alpha_{\nu_k} = \beta\}\}, \quad \{x_k|1 \le k \le n\}, \quad \{\tau_i|1 \le i \le n-1\}$$

with the relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1, \quad e(\nu)x_k = x_k e(\nu), \quad x_k x_l = x_l x_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \ \text{ if } |k-l| > 1,$$

Quiver Hecke Algebra II

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \overline{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\overline{Q}_{i,j}(u,v,w) = \frac{Q_{i,j}(u,v)-Q_{i,j}(w,v)}{u-w} \in \mathbf{k}[u,v,w]$ and set $R := \bigoplus_{\beta\in Q_+} R(\beta)$.

Grading

The relations above are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a $\mathbb{Z}$-graded algebra (N.B.: x_k and τ_l are inhomogeneous)

R-modules

R-modules

1. Define a *grading shift functor* q on a $\mathbb{Z}$-graded $R(\beta)$-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$ by:
$$qM := \bigoplus_{k\in\mathbb{Z}} (qM)_k, \quad \text{where } (qM)_k = M_{k-1}.$$
2. For $M \in R(\beta)$-Mod and $N \in R(\beta')$-Mod, define the convolution product by
$$M \circ N := R(\beta+\beta')e(\beta,\beta') \otimes_{R(\beta)\otimes R(\beta')} (M \otimes N) \quad \Big(e(\beta,\beta') := \sum_{\nu\in I^\beta,\, \nu'\in I^{\beta'}} e(\nu,\nu')\Big)$$
3. $M \nabla N := \mathrm{hd}(M \circ N)$ (head), $M \Delta N := \mathrm{soc}(M \circ N)$ (socle), where the head of a module is the quotient by its radical and the socle of a module is the sum of all simple submodules.
4. If $M \cong M^*$, we say M is *self-dual*.

$R(\beta)\text{-gmod}$: Category of finite-dimensional graded $R(\beta)$-modules.

$R(\beta)\text{-gproj}$: Category of finitely generated graded projective $R(\beta)$-modules.

$*$ N.B. These categories are monoidal categories by the convolution product.

Categorification of $U_q^-(\mathfrak{g})$ and $\mathcal{A}_q(\mathfrak{n})$

Theorem (Khovanov-Lauda, Rouquier)

Let $\mathcal{K}(R\text{-gmod})$ (resp. $\mathcal{K}(R\text{-gproj})$) be the Grothendieck ring of the monoidal category R-gmod (resp. R-gproj). Then we obtain

$$\mathcal{K}(R\text{-gproj}) \cong U_q^-(\mathfrak{g})_{\mathbb{Z}}, \qquad \mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}}$$

$$\mathcal{A}_q(\mathfrak{n}) := \bigoplus_{\beta \in Q_-} \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q))$$ *is called a quantum coordinate ring*

Define the functors

$$\begin{aligned} E_i \quad &: \quad R(\beta)\text{-gmod} \to R(\beta-\alpha_i)\text{-gmod by} \quad E_i(M) := e(\alpha_i, \beta-\alpha_i)M \\ F_i \quad &: \quad R(\beta)\text{-gmod} \to R(\beta+\alpha_i)\text{-gmod by} \quad F_i(M) = \langle i \rangle \circ M, \end{aligned}$$

where $e(\alpha_i, \beta-\alpha_i) := \sum_{\nu \in I^\beta, \nu_1 = i} e(\nu)$ and $\langle i \rangle$ is a 1-dim. simple $R(\alpha_i)$-module.
We also define E_i^* and F_i^* in the opposite manner.

Categorfication of $B(\infty)$ by Lauda and Vazirani

For a simple module $M \in R(\beta)$-gmod, define

$$\mathrm{wt}(M) = -\beta,\ \varepsilon_i(M) = \max\{n \in \mathbb{Z} \mid E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle,$$
$$\widetilde{E}_i M := q_i^{1-\varepsilon_i(M)}\mathrm{soc}(E_i M) \cong q_i^{\varepsilon_i(M)-1}\mathrm{hd}(E_i M) \quad (q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}),$$
$$\widetilde{F}_i M := q_i^{\varepsilon_i(M)}\langle i\rangle\nabla M := q_i^{\varepsilon_i(M)}\mathrm{hd}(\langle i\rangle \circ M) = q_i^{\varepsilon_i(M)}\mathrm{hd}(F_i M).$$

Theorem (Lauda-Vazirani)

Define $\mathbb{B}(R\text{-gmod})$ to be a set of self-dual simple modules in R-gmod. The 6-tuple, $(\mathbb{B}(R\text{-gmod}), \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\})$ holds a crystal structure and there exists the following isomorphism of crystals:

$$\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty) \ \ (= \text{crystal base of } U_q^-(\mathfrak{g})).$$

Definition of $\mathscr{C}_w$

We recall the subcategory $\mathscr{C}_w \subset R$-gmod and its localization $\widetilde{\mathscr{C}_w}$. For $M \in R(\beta)$-gMod, we define

$$W(M) := \{\gamma \in Q_+ \cap (\beta - Q_+) \mid e(\gamma, \beta - \gamma)M \neq 0\},$$

For $w \in W$, we define the monoidal full subcategory $\mathscr{C}_w$ of R-gmod by

$$\mathscr{C}_w := \{M \in R\text{-gmod} \mid W(M) \subset Q_+ \cap wQ_-\}$$

When $w = w_0 \in W$ is the longest element, $\mathscr{C}_{w_0} = R$-gmod.

Definition

Let $s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in W$. For a dominant weight $\Lambda \in P_+$ and $w \in W$, we define the determinantial module associated with w and Λ by

$$\mathbf{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l} \mathbf{1},$$

where $\mathbf{1}$ is a trivial $R(0)$-module and $m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l}\Lambda \rangle \quad (k = 1, \cdots, l)$.

Braiders and Real Commuting Family I

Let Λ be $\mathbb{Z}$-lattice and $\mathcal{T} = \oplus_{\lambda\in\Lambda}\mathcal{T}_\lambda$ be a $\mathbf{k}$-linear Λ-graded monoidal category with $1 \in \mathcal{T}_0$ and the bifunctor $\otimes : \mathcal{T}_\lambda \times \mathcal{T}_\mu \to \mathcal{T}_{\lambda+\mu}$. (Later Λ will be the root lattice Q)

Definition (Kashiwara-Kim-Oh-Park)

A graded braider is a triple (C, R_C, ϕ), where $C \in \mathcal{T}$, $\mathbb{Z}$-linear map $\phi : \Lambda \to \mathbb{Z}$ and a morphism

$$R_C : C \otimes X \to q^{\phi(\lambda)} X \otimes C \quad (X \in \mathcal{T}_\lambda),$$

which is functorial in $X \in \mathcal{T}$ such that satisfying the following commutative diagram:

$$\begin{tikzcd} C \otimes X \otimes Y \arrow[r, "R_C(X)\otimes Y"] \arrow[dr, "R_C(X\otimes Y)"'] & q^{\phi(\lambda)} \otimes X \otimes C \otimes Y \arrow[d, "X\otimes R_C(Y)"] \\ & q^{\phi(\lambda+\mu)}(X \otimes Y) \otimes C, \end{tikzcd} \qquad \begin{tikzcd} C \otimes \mathbf{1} \arrow[r, "R_C(\mathbf{1})"] \arrow[dr, "\cong"'] & \mathbf{1} \otimes C \arrow[d, "\cong"] \\ & C. \end{tikzcd}$$

Localization

There exists a real commuting family of graded braiders in R-gmod.

$$\{M(w\Lambda_i, \Lambda_i), R_{M(w\Lambda_i,\Lambda_i)}, \phi_{M(w\Lambda_i,\Lambda_i)}\}_{i\in I}$$

Let $\widetilde{\mathscr{C}_w} := \mathscr{C}_w[M(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$ denote the localization of $\mathscr{C}_w$ by the graded braider above. We also let

$$\widetilde{R\text{-gmod}}[w] := R\text{-gmod}[M(w\Lambda_i, \Lambda_i)^{\circ -1}; i \in I]$$

denote the localization of R-gmod by the same graded braider. We set $C_\Lambda := M(w\Lambda, \Lambda)$, in particuler, $C_i := M(w\Lambda_i, \Lambda_i)$. Note that for dominant weights λ, μ, we have

$$C_\lambda \circ C_\mu \cong C_{\lambda+\mu} \text{ up to grading shifts}$$

Then, $\{C_\Lambda \mid \Lambda \in P_+\}$ serves an analogus property like a multiplicative set in commutative ring theory.

Properties of $\widetilde{\mathscr{C}}_w$

We summarize propeties of $\widetilde{\mathscr{C}}_w$. Let $Q_w : \mathscr{C}_w \mapsto \widetilde{\mathscr{C}}_w$ be a canonical functor

Proposition (KKOP)

1. *$\widetilde{\mathscr{C}}_w$ is an abelian monoidal category and the functor Q_w is exact.*
2. *For any simple object $S \in \mathscr{C}_w$, $Q_w(S)$ is simple in $\widetilde{\mathscr{C}}_w$.*
3. *$\widetilde{C}_i := Q_w(C_i)$ ($i \in I$) is invertible central graded braider in $\widetilde{\mathscr{C}}_w$, where in a monoidal category $\mathcal{T}$, an object X is invertible if the functors $X \circ -$ and $- \circ X$ are category equivalences.*
4. *For $\lambda \in P$, let $\mu, \eta \in P_+$ such that $\lambda = \mu - \eta$. Then, we obtain $\widetilde{C}_\lambda := \widetilde{C}_\eta^{-1} \circ \widetilde{C}_\mu$.*
5. *Any simple object in $\widetilde{\mathscr{C}}_w$ is isomorphic to $\widetilde{C}_\lambda \circ Q_w(S)$ for some simple module $S \in \mathscr{C}_w$ and $\lambda \in P$ ($\lambda \in P$ and $S \in \mathscr{C}_w$ are not necessarily unique.).*

Crystal Structure on $\widetilde{\mathscr{C}_w}$ I

We recall notions of rigidity and an invariant Λ.

Definition

Let X, Y be objects in a monoidal category $\mathcal{T}$, and $\varepsilon : X \otimes Y \to 1$ and $\eta : 1 \to Y \otimes X$ morphisms in $\mathcal{T}$. We say that a pair (X, Y) is dual pair or X is a left dual to Y or Y is a right dual to X if the following compositions are identities:

$$X \simeq X \otimes 1 \xrightarrow{\mathrm{id}\otimes\eta} X \otimes Y \otimes X \xrightarrow{\varepsilon\otimes\mathrm{id}} 1 \otimes X \simeq X, \;\; Y \simeq 1 \otimes Y \xrightarrow{\eta\otimes\mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}\otimes\varepsilon} Y \otimes 1 \simeq Y$$

We denote a right dual to X by $\mathscr{D}(X)$ and a left dual to X by $\mathscr{D}^{-1}(X)$.

Theorem (KKOP)

$\widetilde{\mathscr{C}_w}$ is rigid, i.e., every object in $\widetilde{\mathscr{C}_w}$ has a left dual and a right dual.

Crystal Structure on $\widetilde{\mathscr{C}_w}$ II

Definition

Let $(\mathscr{C}, \otimes)$ be a graded monoidal category and $M, N \in \mathscr{C}$ be simple objects. In case

$$\dim \mathrm{HOM}(M \otimes N, N \otimes M) = 1$$

the pair (M, N) is said to be Λ-definable. A non-zero morphism $\mathbf{r}_{M,N}$ in this space is called a R-matrix, and we set

$$\Lambda(M, N) := \deg(\mathbf{r}_{M,N}).$$

Let $\mathrm{Irr}(\widetilde{\mathscr{C}_w})$ be the set of equilavence classes of simple objects in $\widetilde{\mathscr{C}_w}$ up to grading shifts. We define a crystal structure on $\mathrm{Irr}(\widetilde{\mathscr{C}_w})$.

Theorem (Kashiwara-N)

For $X \in \mathrm{Irr}(\widetilde{\mathscr{C}_w})$, we define as follows:

$$\begin{aligned}
\varepsilon_i(X) &= d_i^{-1}\widetilde{\Lambda}(Q_w(\langle i\rangle), X), & \varepsilon_i^*(X) &= d_i^{-1}\widetilde{\Lambda}(X, Q_w(\langle i\rangle)),\\
\varphi_i(X) &= \varepsilon_i(X) + \langle h_i, \mathrm{wt}(X)\rangle, & \varphi_i^*(X) &= \varepsilon_i^*(X) + \langle h_i, \mathrm{wt}(X)\rangle,\\
\widetilde{F}_i X &= q_i^{\varepsilon_i(X)} Q_w(\langle i\rangle)\nabla X, & \widetilde{F}_i^* X &= q_i^{\varepsilon_i^*(X)} X\nabla Q_w(\langle i\rangle),\\
\widetilde{E}_i X &= q_i^{\varphi_i(X)+1} X\nabla \mathscr{D} Q_w(\langle i\rangle), & \widetilde{E}_i^* X &= q_i^{\varphi_i^*(X)+1} \mathscr{D}^{-1} Q_w(\langle i\rangle)\nabla X,\\
\textit{If } i \in I \setminus I_w &\Longrightarrow & \widetilde{F}_i(X) &= \widetilde{E}_i(X) = 0,\ \varepsilon_i(X) = \varepsilon_i^*(X) = -\infty.
\end{aligned}$$

where $d_i = (\alpha_i, \alpha_i)/2$ and $\widetilde{\Lambda}(M, N) := (\Lambda(M, N) + (\mathrm{wt}(M), \mathrm{wt}(N)))/2$. Then these operators define a crystal structure on $\mathrm{Irr}(\widetilde{\mathscr{C}_w})$.

Crystal Structure on $\widetilde{\mathscr{C}_w}$ IV

Let $\underline{w} = i_1 i_2 \cdots i_l$ be a reduced word of $w \in W$ and $\mathbb{B}_{\underline{w}}$ be the cellular crystal associated to $\underline{w}$. For $M \in \mathrm{Irr}(\mathscr{C}_w)$, we define $K_{\underline{w}}(M) \in \mathbb{Z}^l$ by

$$M_l = M,\ c_k = \varepsilon^*_{i_k}(M_k) \text{ for } 1 \le k \le l,$$
$$M_{k-1} = (\widetilde{E}^*_{i_k})^{c_k}(M_k) \implies K_{\underline{w}}(M) := (c_1, \ldots, c_l).$$

We regard $K_{\underline{w}}(M)$ as an element of $\mathbb{B}_{\underline{w}}$ by $(c_1, \ldots, c_l) \mapsto \widetilde{f}_{i_1}^{c_1}(0)_{i_1} \otimes \cdots \otimes \widetilde{f}_{i_l}^{c_l}(0)_{i_l}$. We extend $K_{\underline{w}} : \mathrm{Irr}(\mathscr{C}_w) \to \mathbb{Z}^l$ to $K_{\underline{w}} : \mathrm{Irr}(\widetilde{\mathscr{C}_w}) \to \mathbb{Z}^l$ by

$$\widetilde{C}_\Lambda^{-1} \circ Q_w(M) \mapsto K_w(M) - K_w(C_\Lambda)$$

for $\Lambda \in P_+$.

Theorem (Kashiwara-N)

$K_{\underline{w}} : \mathrm{Irr}(\widetilde{\mathscr{C}_w}) \to \mathbb{Z}^l \simeq \mathbb{B}_{\underline{w}}$ *is an isomorphism of crystals.*

The following thorem is neccesary to define ε_i^* on $\mathbb{B}_{\mathbf{i}}$,

Theorem (Kashiwara-N)

Let $w \in W$ *and* $i \in I$ *satisfy* $w' = ws_i < w$. *Define* $\mathbf{E}^*_{w',w} : \mathrm{Irr}(\widetilde{\mathscr{C}}_w) \longrightarrow \mathrm{Irr}(\widetilde{\mathscr{C}}_{w'})$ *by*

$$(\widetilde{C}_\Lambda)^{-1} \circ Q_w(M) \mapsto Q_{w'}(\widetilde{E}_i^{*\max} C_\Lambda)^{-1} \circ Q_{w'}(\widetilde{E}_i^{*\max} M).$$

Then we obtain a bijective map

$$\mathrm{Irr}(\widetilde{\mathscr{C}}_w) \longrightarrow \mathrm{Irr}(\widetilde{\mathscr{C}}_{w'}) \times \mathbb{Z} \quad (X \mapsto (\mathbf{E}^*_{w',w}(X), \varepsilon_i^*(X)).$$

And indeeed, for $X \in \mathrm{Irr}(\widetilde{\mathscr{C}}_w)$ *we also get*

$$K_{\underline{w}}(X) = K_{\underline{w'}}(\mathbf{E}^*_{w',w}(X)) \otimes \tilde{f}_i^{\varepsilon_i^*(X)}(0)_i$$

On the set of reduced words of $w \in W$, we define braid moves.

$$\cdots ij \cdots \to \cdots ji \cdots (a_{ij}a_{ji} = 0), \quad \cdots iji \cdots \to \cdots jij \cdots (a_{ij}a_{ji} = 1),$$
$$\cdots ijij \cdots \to \cdots jiji \cdots (a_{ij}a_{ji} = 2),$$

which are called 2-move, 3-move, 4-move, respectively.

In the sequel, let $\mathfrak{g}$ be of type $A_n,\ B_n,\ C_n,\ D_n$ and w_0 be the longest element of W. Then, for any $\underline{w_0}$, any $i \in I$ can be moved to the rightmost position of the reduced longest word by the braid moves, and each braid-move induces the braid type isomorphism $\phi_{ij}^{(k)}$. Thus, for $w_0 = w' s_i$ this procedure induces an isomorphism of the crystals

$$\phi : \mathbb{B}_{\underline{w_0}} \to \mathbb{B}_{\underline{w'}} \otimes B_i \quad (x \mapsto \phi(x) = b' \otimes (a)_i)$$

where ϕ is the composition of the braid type isom. $\phi_{ij}^{(k)}$'s, which is induced by $K_{\underline{w_0}}$. For $x \in \mathbb{B}_{\underline{w_0}}$, we then define $\varepsilon_i^*(x) = -a$.

Explicit formula for ε_i^* I

ε_i^* depends neither on the choice of the reduced word nor on the process of the braid moves.

Proposition (Matsuura-N)

Let $\mathbb{B}_{\underline{w_0}}$ be the cellular crystal associated to the reduced longest word $\underline{w_0}$.
Type A_n : Fix $\underline{w_0} = (1)(21)\ldots(n-1\ldots21)(n\ldots21)$ as a reduced longest word. For $x \in \mathbb{B}_{\underline{w_0}}$ expressed as

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{2,1})_1 \otimes \cdots \otimes (z_{1,n})_n \otimes \cdots \otimes (z_{n-1,2})_2 \otimes (z_{n,1})_1 \in \mathbb{B}_{\underline{w_0}},$$

$\varepsilon_i^*(x)$ is given by

$$\varepsilon_i^*(x) = \max_{1\le k\le i}(z_{n-i+2,k-1} - z_{n-i+1,k}) \quad (1 \le i \le n).$$

We understand $z_{j,k} = 0$, if $k = 0$.

Explicit formula for ε_i^* II

Proposition (Matsuura-N)

1. *Type B_n : Fix $\underline{w_0} = (12\dots n)^n$ as a reduced longest word. For $x \in \mathbb{B}_{\underline{w_0}}$ expressed as*

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes \cdots (z_{1,n})_n \otimes (z_{2,1})_1 \otimes \cdots \otimes (z_{n,n-1})_{n-1} \otimes (z_{n,n})_n \in \mathbb{B}_{\underline{w_0}},$$

$\varepsilon_i^*(x)$ is given by

$$\varepsilon_i^*(x) = \max_{1\le k\le n-i-1}(-\zeta_i, z_{i+k+1,n-k} - z_{i+k+1,n-k-1})\ (1 \le i \le n-2),\ \ \varepsilon_{n-1}^*(x) = -\zeta_{n-1},\ \ \varepsilon_n^*(x) = -z_{n,n},$$

$$\zeta_i = -\max(-z_{i+1,n-1} + z_{i+1,n}, -\eta_i, z_{i+1,n-1} - z_{i,n}),\ \ (1 \le i \le n-1)$$

$$\eta_i = -\max_{1\le k\le n-1}(z_{i+1,k-1} - z_{i,k}).\ (i \le i \le n-1)$$

2. *Type C_n : Fix $\underline{w_0} = (12\dots n)^n$ as a reduced longest word. For $x \in \mathbb{B}_{\underline{w_0}}$ with the same expression as in type B_n, $\varepsilon_i^*(x)$ and $\eta_i(x)$ are the same as those in type B_n. ζ_i is given by*

$$\zeta_i = -\max(-2z_{i,n} + z_{i+1,n-1}, -\eta_i, z_{i+1,n-1} - 2z_{i+1,n}).\ (1 \le i \le n-1)$$

Proposition (Matsuura-N)

1. *Type D_n : Fix $\underline{w_0} = (12\ldots n)^{n-1}$ as a reduced longest word. For*

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes \cdots (z_{1,n})_n \otimes (z_{2,1})_1 \otimes \cdots \otimes (z_{n-1,n-1})_{n-1} \otimes (z_{n-1,n})_n \in \mathbb{B}_{\underline{w_0}},$$

$\varepsilon_i^*(x)$ is given by

$$\begin{aligned}
\varepsilon_i^*(x) &= \max_{1\le k\le n-i-2}(-\kappa_i, z_{i+k+1,n-k-1} - z_{i+k+1,n-k-2}),\ (1 \le i \le n-3)\\
\kappa_i &= -\max(-\theta_i,\ z_{i+1,n-1} - z_{i,n},\ z_{i+1,n-2} - z_{i,n-1} - z_{i,n},\\
&\qquad z_{i+1,n} - z_{i,n-1},\ z_{i+1,n} + z_{i+1,n-1} - z_{i+1,n-2}),\ (1 \le i \le n-2)\\
\theta_i &= -\max_{1\le k\le n-2}(z_{i+1,k-1} - z_{i,k}),\ (1 \le i \le n-2)\\
\varepsilon_{n-2}^*(x) &= -\kappa_{n-2},\ \varepsilon_{n-1}^*(x) = -z_{n-1,n-1},\ \varepsilon_n^*(x) = -z_{n-1,n}.
\end{aligned}$$

Characterization of the unit object

Now, let us state the characterization of the unit object.

Theorem (Matsuura-N)

Let W be the Weyl group of classical finite type, $\underline{w_0} = i_1 \cdots i_l$ be a reduced longest word, and $\mathbb{B}_{\underline{w_0}}$ be the cellular crystal associated to $\underline{w_0}$. For $x \in \mathbb{B}_{\underline{w_0}}$, the following are equivalent.

1. $\mathrm{wt}(x) = 0$ *and* $\varepsilon_i^*(x) = 0$ $({}^{\forall} i \in I)$.
2. $x = (0)_{i_1} \otimes \cdots \otimes (0)_{i_l} \in \mathbb{B}_{\underline{w_0}}$.

The following follows immediately from the fact that $K_{\underline{w_0}}$ is an isom. of crystals.

Corollary (Matsuura-N)

For $X \in \mathrm{Irr}(\widetilde{\mathscr{C}_{w_0}}) = \mathrm{Irr}(\widetilde{R\text{-gmod}})$, the following are equivalent.

1. $\mathrm{wt}(X) = 0$ *and* $\varepsilon_i^*(X) = 0$ $({}^{\forall} i \in I)$.
2. $X = \mathbf{1}$ *(up to grading shifts).*

It is known that for $x \in B(\infty)$, $\varepsilon_i^*(x) = 0$ $({}^\forall i \in I)$ without $\mathrm{wt}(x) = 0$ implies $x = u_\infty (= 1 \bmod qL(\infty))$. But, this is not true in the localized case. For example, in case $\underline{w} = 121321$ (type A_3) we have for

$$x = (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{2,1})_1 \otimes (z_{1,3})_3 \otimes (z_{2,2})_2 \otimes (z_{3,1})_1,$$

$$\varepsilon_1^*(x) = -z_{3,1},\ \varepsilon_2^*(x) = \max(-z_{2,1}, z_{3,1} - z_{2,2}),\ \varepsilon_3^*(x) = \max(-z_{1,1}, z_{2,1} - z_{1,2}, z_{2,2} - z_{1,3}).$$

Now, take

$$x = (1)_1 \otimes (1)_2 \otimes (0)_1 \otimes (1)_3 \otimes (1)_2 \otimes (0)_1,$$

this satisfies $\varepsilon_i^*(x) = 0$ for any $i \in I$, while $x \neq (0)_1 \otimes (0)_2 \otimes (0)_1 \otimes (0)_3 \otimes (0)_2 \otimes (0)_1$. Thus, we need the condition $\mathrm{wt}(x) = 0$.

Sketch of proof for type A_n I

We exibit the procedure to move a given letter i to the rightmost position by the braid moves. (with an example of moving $i = 3$ for type A_4)

Step1 By applying 2-moves, move the i-th occurrence of the letter 1 (reading from **right to left**) as far to the right as possible. Then we obtain the subword 121 and apply 3-move : $121 \mapsto 212$.

$$1213214321 \to 1231214321 \to 1232124321$$

Step2 By applying 2-moves, move the right 2 appeared in (Step1) as far to the right as possible. Then we obtain the subword 232 and apply 3-move : $232 \mapsto 323$.

$$1232124321 \to 1232142321 \to 1232143231$$

Step3 Iterate (Step 2) successively for $3, 4, \ldots, i-1$ until the subword $i(i-1)i$ appears.

(Since the rank is small, Step 3 is not necessary in this case.)

Step4 Since there only exist letters strictly less than $i-1$ on the right side of the right i appeared in (Step3), one can move the letter i to the rightmost position by applying 2-moves.

Sketch of proof for type A_n II

Then, the procedure above induces the following composition of the **braid-type isomorphisms**.

$$\text{Step1}: (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{2,1})_1 \otimes (z_{1,3})_3 \otimes (z_{2,2})_2 \otimes (z_{3.1})_1 \otimes (z_{1,4})_4 \otimes (z_{2,3})_3 \otimes (z_{3,2})_2 \otimes (z_{4,1})_1$$
$$\longmapsto (z_{1,1})_1 \otimes (z_{1,2})_2 \otimes (z_{1,3})_3 \otimes (A)_2 \otimes (B)_1 \otimes (-\max(-z_{2,1}, z_{3,1} - z_{2,2})_2 \otimes (z_{1,4})_4 \otimes \cdots$$
$$\text{Step2}: \cdots \otimes (-\max(-z_{2,1}, z_{3,1} - z_{2,2})_2 \otimes (z_{1,4})_4 \otimes (z_{2,3})_3 \otimes (z_{3,2})_2 \otimes (z_{4,1})_1$$
$$\longmapsto \cdots \otimes (z_{1,4})_4 \otimes (C)_3 \otimes (D)_2 \otimes (-\max(\max(-z_{2,1}, z_{3,1} - z_{2,2}), z_{3,2} - z_{2,3}))_3 \otimes (z_{1,4})_1$$

Hence, We obtain

$$\varepsilon_3^*(x) = \max(-z_{2,1}, z_{3,1} - z_{2,2}, z_{3,2} - z_{2,3})$$

Sketch of proof for type A_n III

Caluculating in this vein, we obtain the follwing:

$$\varepsilon_1^*(x) = -z_{4,1} = 0,$$
$$\varepsilon_2^*(x) = \max(-z_{3,1}, z_{4,1} - z_{3,2}) = 0 \implies z_{3,1} \geq 0,\ z_{4,1} \leq z_{3,2},$$
$$\varepsilon_3^*(x) = \max(-z_{2,1}, z_{3,1} - z_{2,2}, z_{3,2} - z_{2,3}) = 0$$
$$\implies z_{2,1} \geq 0,\ z_{3,1} \leq z_{2,2},\ z_{3,2} \leq z_{2,3},$$
$$\varepsilon_4^*(x) = \max(-z_{1,1}, z_{2,1} - z_{1,2}, z_{2,2} - z_{1,3}, z_{2,3} - z_{1,4}) = 0$$
$$\implies z_{1,1} \geq 0,\ z_{2,1} \leq z_{1,2},\ z_{2,2} \leq z_{1,3},\ z_{2,3} \leq z_{1,4}$$
$$\therefore z_{1,1} z_{2,1}, z_{3,1} \geq 0,\ 0 = z_{4,1} \leq z_{3,2} \leq z_{2,3} \leq z_{1,4},\ z_{3,1} \leq z_{2,2} \leq z_{1,3},\ z_{1,2} \leq z_{2,2}.$$
$$wt(x) = 0 \implies z_{1,1} + z_{2,1} + z_{3,1} + z_{4,1} = z_{1,2} + z_{2,2} + z_{3,2} = z_{1,3} + z_{2,3} = z_{1,4} = 0.$$

Then, we obtain $z_{i,j} = 0\ {}^\forall i, j$.

Merci beaucoup pour votre attention