Invitation to Random Tensor Models: from random geometry, enumeration of tensor invariants, to characteristic polynomials

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Introducing random tensor models...

Random tensor models and tensor field theories

- Consider a field theory defined by a "field" $\phi : G^d \to \mathbb{C}, \mathbb{R}$, etc, where G is a compact Lie group admitting Peter-Weyl decomposition.
- The "Fourier" transform of ϕ yields an order-d tensor [1] $\phi_{\mathbf{P}}$, with $\mathbf{P} = (p_1, p_2, \ldots, p_d)$ a multi-index, where $p_1, p_2, \ldots, p_d \in I$, where I is a discrete set.
- e.g., take $\phi : U(1)^d \to \mathbb{C}$, $\phi_{\mathbf{P}}$ an order-d complex tensor, and $\bar{\phi}_{\mathbf{P}}$ its complex conjugate, where $p_1, p_2, \ldots, p_d \in \mathbb{Z}$.

The partition function is $\quad \mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi}\; e^{-(S^{\text{kinetic}}[\bar{\phi},\phi] + S^{\text{interaction}}[\bar{\phi},\phi])}$,

where the action $S^*[\bar{\phi}, \phi]$ is given by convolutions of tensors, e.g.,

$$S^{\text{kinetic}}[\bar{\phi}, \phi] \;=\; \sum_{\mathbf{P},\mathbf{P}'} \bar{\phi}_{\mathbf{P}}\, K(\mathbf{P};\mathbf{P}')\, \phi_{\mathbf{P}'} =: \mathrm{Tr}_2(\bar{\phi} \cdot K \cdot \phi)\,,$$

where Tr_{2n_B} represents sums over all indices p_s of $\mathbf{P}$ on n_B tensors ϕ and $\bar{\phi}$.

> We are studying the space of tensors $\phi = \phi_{p_1,\ldots,p_d}$, equipped with the measure
>
> $$d\mu(\bar{\phi}, \phi) = d\nu_K(\bar{\phi}, \phi)\, e^{-S^{\text{interaction}}[\bar{\phi},\phi]} \qquad \text{and} \qquad \mathcal{Z} = \int d\mu(\bar{\phi}, \phi)\,,$$
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> where $d\nu_K(\bar{\phi}, \phi)$ is a Gaussian measure with covariance K^{-1}.

[1] Considering $\phi_{\mathbf{P}}$ as a tensor is a slight abuse because the modes p_i range up to infinity. We cut off at N, then $\phi_{\mathbf{P}}$ is a tensor.

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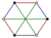
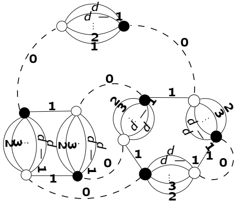
$$S^{\text{kinetic}}[\phi,\bar\phi] = \mathrm{Tr}_2(\bar\phi\cdot K\cdot\phi)$$

$$S^{\text{interaction}}[\phi,\bar\phi] = \sum_{\substack{\mathcal{B}\text{ bubbles}\\ \text{(tensor invariants)}}} \lambda_{\mathcal{B}}\,\mathrm{Tr}_{2n_{\mathcal{B}}}(\bar\phi^{n_{\mathcal{B}}}\cdot\mathcal{V}_{\mathcal{B}}\cdot\phi^{n_{\mathcal{B}}})\,,\quad n_{\mathcal{B}}\in\mathbb{Z}_+$$

$\overset{d=3}{=}$ $\lambda_2^{(3)}$  $+\,\lambda_4^{(3)}$  $+\,\lambda_{6,1}^{(3)}$  $+\,\lambda_{6,2}^{(3)}$  $+\,\lambda_{6,3}^{(3)}$  $+\cdots$

$\overset{d=4}{=}$ $\lambda_2^{(4)}$  $+\,\lambda_{4,1}^{(4)}$  $+\,\lambda_{4,2}^{(4)}$  $+\,\lambda_{6,1}^{(4)}$  $+\,\lambda_{6,2}^{(4)}$  $+\,\lambda_{6,3}^{(4)}$  $+\cdots$

After Wick contraction, it generates $(d+1)$-edge-colored Feynman graphs, e.g.,



Remark

- *If $K(\mathbf{P};\mathbf{P}')=\delta_{\mathbf{P};\mathbf{P}'}$ (trivial delta function), then this model is a statistical model i.e., a random tensor model.*
- *Otherwise, if the propagator is nontrivial, e.g., $K(\mathbf{P};\mathbf{P}')=\delta_{\mathbf{P};\mathbf{P}'}\mathbf{P}^{2b}$, this is a QFT, i.e., tensor field theories (generalisation of Grosse-Wulkenhaar model).*

Random tensor models and tensor field theories

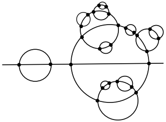
- After Wick contraction, random tensor models (with d indices) generates $(d+1)$-edge-colored Feynman graphs.
- $(d+1)$-edge-colored graphs (also, called graph encoding manifolds (GEM)) are dual to simplicial triangulations of piecewise linear (PL) d-dimensional pseudo-manifolds.
  [Pezzana 1974; Bandieri, Gagliardi 1982; Ferri, Gagliardi, Grasselli 1986]
- In other words, tensor models generate discrete (pseudo-)manifolds, and the path integral formulation provides us a way to sum over all of them.

Relevant for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$.

- Encouragingly, the lower dimensional counterpart ($d = 2$), matrix models yield the Brownian sphere at criticality and are rigorously proven to be equivalent to 2-dimensional Liouville quantum gravity.
  [Le Gall, Miermont 2011; Miller, Sheffield 2015]

Promising for random geometric (path integral) approach to quantum gravity in dimensions $d \geq 3$!

Random tensor models



Melons (graphs $\mathcal{G}$ with $\omega(\mathcal{G}) = 0$, i.e., a subclass of the sphere) dominate in the large N (dimension of tensors) limit. [Gurau Rivasseau 2011]

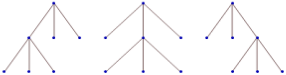
$$\mathcal{Z} = \sum_{\omega \geq 0} N^{d - \frac{2}{(d-1)!}\omega} Z_\omega , \quad \text{where Gurau degree } \omega(\mathcal{G}) = \sum_{\substack{\text{jackets, } \mathcal{J}(\mathcal{G}) \\ \text{(regular embeddings)}}} g_{\mathcal{J}(\mathcal{G})} \geq 0 \,.$$

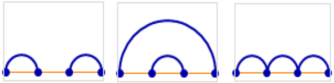
The melonic 2-point function admits the following expansion

$$G_{\mathrm{melonic}}(t) = \sum_{n=0}^{\infty} t^n \, FC_n^{(d+1)} , \qquad FC_n^{(d+1)} = \frac{1}{(d+1)n+1} \binom{(d+1)n+1}{n} .$$

Fuss-Catalan numbers $FC_n^{(d+1)}$ ($d = 1$ is Catalan) correspond to

- the numbers of planar $(d+1)$-ary trees with n vertices and with $dn + 1$ leaves.
- the numbers of non-crossing partitions of the set $\{1, 2, \cdots, dn\}$ that contain only subsets of size d.
- etc.





$(d = 2, \ n = 2)$

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Enumeration of $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

(with Rémi Cocou Avohou, Joseph Ben Geloun)
Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404 [hep-th]]

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

Consider

- A tensor T transforms under the action of the fundamental representation of the Lie group $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$.

$$T_{a_1,a_2,\dots,a_r,b_1,b_2,\dots,b_q} \to U^{(1)}_{a_1 c_1} U^{(2)}_{a_2 c_2} \dots U^{(r)}_{a_r c_r} O^{(1)}_{b_1 d_1} O^{(2)}_{b_2 d_2} \dots O^{(q)}_{b_q d_q} T_{c_1,c_2,\dots,c_r,d_1,d_2,\dots,d_q} .$$

- A $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$ invariant (UO-invariant) is constructed by contractions of complex tensors of order $r+q$ (of a given number, n, of tensors T and the same number of complex conjugate $\bar{T}$.)

  $\to$ Therefore, UO invariants are tensor model invariants/bubbles .

- An UO-invariant is algebraically denoted

$$\mathrm{Tr}_{K_n}(T,\bar{T}) = \sum_{a^i_k, b^i_k, a'^i_k, b'^i_k} K_n(\{a^i_k, b^i_k\}; \{a'^i_k, b'^i_k\}) \prod_{i=1}^{n} T_{a^i_1,a^i_2,\dots,a^i_r,b^i_1,b^i_2,\dots,b^i_q} \bar{T}_{a'^i_1,a'^i_2,\dots,a'^i_r,b'^i_1,b'^i_2,\dots,b'^i_q} .$$

  K_n is a kernel composed of a product of Kronecker delta functions that match the indices of n copies of T's and those of n copies of $\bar{T}$'s. A given tensor contraction dictates the pattern of an edge-colored graph, which can, in turn, be used to label the invariant.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

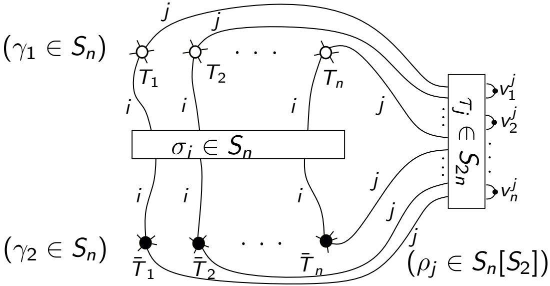


Diagram of contraction of n tensors T and n tensors $\bar{T}$. For a given color $i = 1, 2, \ldots, r$, σ_i represents the contraction in the unitary sector and, for any color $j = 1, 2, \ldots, q$, τ_j represents the contraction in the orthogonal sector.

Consider $(r, q) = (3, 3)$. An UO-invariant is defined by a $(3+3)$-tuple of permutations $(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$ from the product space $(S_n)^{\times 3} \times (S_{2n})^{\times 3}$.

We will remove the vertex labeling (two configurations are equivalent if their resulting unlabeled graphs coincide), which introduces more permutations $\gamma_1, \gamma_2 \in S_n$, and $\varrho_1, \varrho_2, \varrho_3 \in S_n[S_2]$ the so-called wreath product subgroup of S_{2n}.

The equivalence relation is

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \quad \sim \quad (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3)$$

Counting UO tensor invariants

Idea:

We work with the equivalence relation to count the graphs, i.e., tensor invariants

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \sim (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3).$$

- $G \times X \to X$.
- Recall: an orbit of an element x in X: the set of elements in X to which x can be moved by an element of G. $G \cdot x = \{g \cdot x : g \in G\}$.
- a point ($\in X$) on an orbit $\to$ another point on the orbit.
- number of equivalence classes of graphs = number of orbits
- Burnside's lemma
  $\sharp\mathrm{orb} = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$, where $\mathrm{Fix}(g) = \{x \in X : gx = x\}$.

Therefore the counting of UO invariants of order (r, q) is

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q} \sum_{\gamma_1,\gamma_2\in S_n} \sum_{\varrho_1,\dots,\varrho_q\in S_n[S_2]} \sum_{\substack{\sigma_1,\dots,\sigma_r\in S_n \\ \tau_1,\dots,\tau_q\in S_{2n}}} \left[\prod_{i=1}^{r}\delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\right]\left[\prod_{i=1}^{q}\delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\right].$$

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

example: $\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

$\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants are enumerated in the increasing number of tensors: $1, 108, 20385, 27911497, 101270263373, 808737763302769, \ldots$

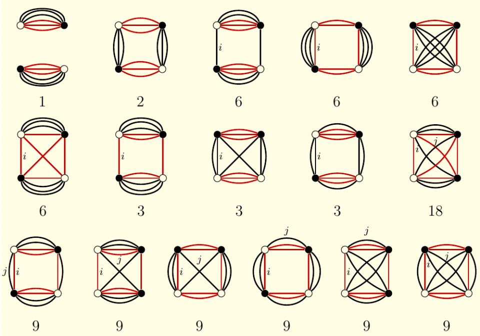


Figure: UO-invariant graphs at order $(r, q) = (3, 3)$ with 4 tensors ($n = 2$). The integer below each graph enumerates various possibilities based on index colors, summing to 108 for all configurations. Black edges are in the U-sector, and red are in the O-sector.

TQFT (lattice gauge theories)

- On a **cellular complex of a manifold** X, we can define a partition function for a finite **group** G by assigning a group element g_e to each edge (1-cell) and to each **plaquette** (2-cell) P a weight $\delta\Big(\prod_{e\in P} g_e\Big)$. The partition function of this **lattice gauge theory** is

$$Z[X;G] = \frac{1}{|G|^V} \sum_{g_e} \prod_{P} \delta\Big(\prod_{e\in P} g_e\Big),$$

  with V the number of vertices (0-cell) in the cell decomposition.
- Moreover [Dijkgraaf, Witten]

$$Z[X;G] = \frac{1}{|G|} |\mathrm{Hom}(\pi_1(X), G)|.$$

- The theory is **topological** because it is invariant under refinement of the cellular decomposition.
- The partition function for a topological space X counts homomorphisms from $\pi_1(X)$ to $G = S_n$ (permutation group), i.e., counts covering spaces of X of degree n counted with a certain weight.

permutation TQFT for UO tensor invariants

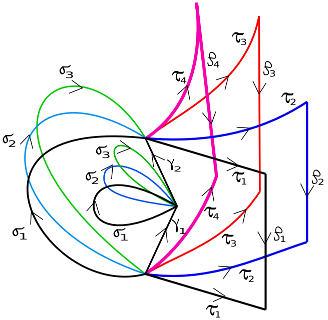
[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

Recall the counting of UO invariants of order (r, q)

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q}$$

$$\sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\varrho_1,\ldots,\varrho_q \in S_n[S_2]} \sum_{\substack{\sigma_1,\ldots,\sigma_r \in S_n \\ \tau_1,\ldots,\tau_q \in S_{2n}}} \Big[\prod_{i=1}^{r} \delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\Big]\Big[\prod_{i=1}^{q} \delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\Big].$$

TQFT reformulates our enumeration as a lattice gauge theory.



2-cellular complex associated with the TQFT$_2$ of $Z_{(3,4)}$ made of 3+4 cylinders sharing boundaries.

permutation TQFT for UO tensor invariants

The counting of UO invariants of order $(r \geq 2, q)$ can be massaged:

$$Z_{(r\geq 2,q)}(n) = \frac{1}{n!}\sum_{\gamma\in S_n} Z^q_{n;\gamma} \sum_{\sigma_0,\sigma_2,\sigma_3\dots,\sigma_r\in S_n} \Big[\prod_{i=2}^{r}\delta(\gamma^{-1}\sigma_i\gamma\sigma_i^{-1})\Big]\delta(\gamma^{-1}\sigma_0\gamma\sigma_0^{-1})\delta(\sigma_0\prod_{i=2}^{r}\sigma_i)\,,$$

$$\text{with} \quad Z^q_{n;\gamma} = \frac{1}{(n!)[n!(2!)^n]^q} \sum_{\substack{\sigma_1\in S_n \\ \tau_1,\dots,\tau_q\in S_{2n} \\ \tau_i^{-1}\sigma_1\gamma^{-1}\sigma_1^{-1}\gamma\tau_i\in S_n[S_2]}} 1\,.$$

We are counting equivalence classes of r permutations σ_i under the conjugation $\sigma_i \sim \gamma\sigma_i\gamma^{-1}$, and the group generated by r generators subject to one relation by the last constraint $\sigma_0\prod_{i=2}^{r}\sigma_i = \text{id}$, i.e., the fundamental group of the 2-sphere with r-punctures.

Therefore, $Z_{(r\geq 2,q)}(n)$ enumerates $Z^q_{n;\gamma}$-weighted equivalence classes of branched covers of the sphere with r branched points.

On the other hand, $Z_{(r=1,q)}(n)$ counts the number of covers of the sphere with $(q+3)$-punctures.

Summary of results

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

- The sequences of numbers corresponding to our enumerations [6] [7] are new and unknown before in OEIS (Online Encyclopedia of Integer Sequences).
- So far, regardless of whether the tensor invariants are unitary [Ben Geloun, Ramgoolam 2013], orthogonal [Avohou, Ben Geloun, Dub 2019], or UO symmetric, we consistently find a correspondence with covers of various cellular complexes via permutation TQFT, but also with (branched) covers of the sphere (possibly with punctures).

The counting of tensor invariants, in addition to their essential role in the analysis of tensor models in theoretical physics, reveals connections between combinatorics, algebra, and topology.

What is intriguing is the connection between tensor models and branched covers of the 2-sphere suggests that 2-dimensional holomorphic maps know about higher dimensional ($d \geq 3$) combinatorial topology.

[6] except purely U case ($r, q = 0$) was reported before [Ben Geloun, Ramgoolam 2013] and also ($r = 2, q = 1$) case was reported in [Bulycheva, Klebanov, Milekhin Tarnopolsky 2017].

[7] Remark that our formulation cannot be reduced to purely O case which was studied before [Avohou, Ben Geloun,Dub 2022].

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Characteristic polynomials of tensors via Grassmann integrals and distributions of roots for random Gaussian tensors

(with Nicolas Delporte, Giacomo La Scala, Naoki Sasakura)
[arXiv:2510.04068 [math-ph]]

Gurau's "resolvent" of tensors

Gurau [arXiv:2004.02660[math-ph]] defined a resolvent of tensors, $T \in \otimes^p \mathbb{R}^N$ a real symmetric tensor of order p, via the 2-point function of a field theory,

$$\begin{aligned} \Omega(w;T) &:= \frac{1}{w\,\mathcal{Z}(w;T)} \int_{\mathbb{R}^N} \mathcal{D}\phi \frac{\phi^2}{N} \exp\Big(-\frac{\phi^2}{2} + \frac{1}{w}\frac{T\cdot\phi^p}{p} \Big) \\ &= \frac{1}{N} \sum_{n\geq 0} \frac{1}{w^{n+1}} \sum_{b\in\mathcal{B}_n} \mathrm{Tr}_b(T) \\ &= \text{``}\frac{1}{N}\mathrm{Tr}\Big(\frac{1}{w-T}\Big)\text{''} \qquad \text{(resolvent)} \end{aligned}$$

where ϕ is a (bosonic) vector in $\mathbb{R}^N$, $\mathcal{D}\phi = (2\pi)^{-N/2}\prod_{i=1}^N d\phi_i$, w is a complex variable, $T\cdot\phi^p \equiv \sum_{a_1,\dots,a_p}^N T_{a_1,\dots,a_p}\phi_{a_1}\dots\phi_{a_p}$, and the partition function is Gaussian p-spin model, $\mathcal{Z}(w;T) = \int_{\mathbb{R}^N} D\phi\, e^{-S(\phi)}$, $\quad S(\phi) = \frac{\phi^2}{2} - \frac{1}{w}\frac{T\cdot\phi^p}{p}$.

At saddle points,

$$\frac{\partial S}{\partial \phi} = 0 \quad \leftrightarrow \quad T\cdot\phi^{p-1} = w\,\phi\,.$$

The number of eigenpairs of a tensor is proportional to the exponential of its dimension (N). [Cartwright, Sturmfels. (2013)] [Auffinger, Ben Arous, Cerny, (2013)]

Resolvent for random matrices

If $p = 2$ (a random Gaussian matrix M),

$$\begin{aligned} \langle \Omega(w; M) \rangle_M &:= \lim_{N\to\infty} \frac{1}{N} \Big\langle \mathrm{Tr}\Big(\frac{1}{w - M} \Big) \Big\rangle_M \\ &= \lim_{N\to\infty} \frac{1}{N} \Big\langle \frac{1}{w} \sum_{n\geq 0} \frac{1}{w^n} \mathrm{Tr}(M^n) \Big\rangle_M \\ &= \int_{-\infty}^{\infty} d\lambda \frac{\rho_{Wigner}(\lambda)}{w - \lambda}, \end{aligned}$$

where $\rho_{Wigner}(\lambda)$ is the asymptotic spectral density, and λ are eigenvalues of matrices.

$\rightarrow$ Wigner's semicircle law.

- Illustrates a clear relation between the resolvent and eigenvalues of matrices.
- Suggests that the tensor resolvent (in the way we have defined earlier) should have a relation to eigenvalues of tensors.

Understanding the resolvent of tensors may help develop eigenvalue decomposition techniques for tensor models. (Eigenvalue decomposition techniques are widely used and developed in matrix models.)

Wigner's semicircle law for matrices

Theorem (Wigner 1955)

Consider $N \times N$ real symmetric, Hermitian, or Hermitian quaternionic random matrices with independent identically distributed entries, where the probability distribution of each matrix element gives zero mean. Each spectrum of the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE), and the Gaussian Symplectic Ensemble (GSE) for a matrix $M/\sqrt{N}$ converges at large N to the Wigner's semicircle law: $\rho_{Wigner}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}\,, \lambda \in [-2, 2]$.

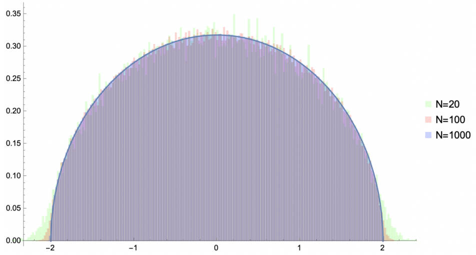

Figure: Histograms showing the empirical eigenvalue distribution of $N \times N$ real symmetric Gaussian random matrices of different sizes. (credit: Nicolas Delporte)

Resolvent for random matrices

If $p = 2$ (so that tensor is a matrix M),

$$\begin{aligned} \left\langle \Omega(w; M) \right\rangle_M &:= \lim_{N\to\infty} \frac{1}{N} \left\langle \mathrm{Tr}\left(\frac{1}{w - M} \right) \right\rangle_M \\ &= \lim_{N\to\infty} \frac{1}{N} \left\langle \frac{1}{w} \sum_{n\geq 0} \frac{1}{w^n} \mathrm{Tr}(M^n) \right\rangle_M \\ &= \int_{-\infty}^{\infty} d\lambda \frac{\rho_{Wigner}(\lambda)}{w - \lambda}, \end{aligned}$$

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Gurau's generalised Wigner's semicircle law for tensors

[Gurau, arXiv:2004.02660[math-ph]]

Further, taking the Gaussian average over random tensors

$$d\nu(T) = \mathcal{N} \left(\prod_{a_1 \leq \cdots \leq a_p} dT_{a_1 \cdots a_p} \right) e^{-\frac{N^{p-1}}{2p} \sum_{a_1, \cdots, a_p = 1}^{N} \left(T_{a_1 \cdots a_p} \right)^2}$$

and saddle point approximation, Gurau derived the resolvent of tensors to be

$$\text{``} \lim_{N \to \infty} \frac{1}{N} \left\langle \mathrm{Tr}\left(\frac{1}{w - T} \right) \right\rangle_T \text{''} := \left\langle \Omega(w; T) \right\rangle_T = \frac{1}{w} \sum_{n \geq 0} \left(\frac{1}{w^2} \right)^n FC_p(n) ,$$

where the **Fuss-Catalan numbers** $FC_p(n)$ appear.

Then, obtained the spectral density $\rho_{\mathrm{Gurau}}(w) = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \Big(\Omega(w + i\epsilon) - \Omega(w - i\epsilon) \Big)$.

$\to$ **generalises the Wigner's semicircular law**.

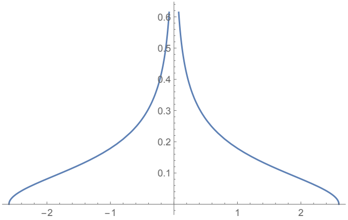


Figure: $\rho_{\mathrm{Gurau}}(w)$ for $p = 3$

Our proposition

Characteristic polynomial for a matrix

Recall that the eigenvalues λ of a given matrix M can be obtained from solving for the zeros of the characteristic polynomial,

$$\det(\lambda \mathbb{1} - M) = 0\,.$$

Characteristic polynomial can be computed via Grassmann integral of two species of Grassmann variables ψ and $\bar{\psi}$,

$$\det(\lambda \mathbb{1} - M) = \int \mathcal{D}(\psi, \bar{\psi}) \exp\{ \sum_{a,b=1}^{N} \bar{\psi}_a (\lambda \delta_{ab} - M_{ab}) \psi_b \}\,,$$

$$\mathcal{D}(\psi, \bar{\psi}) = \prod_{a=1}^{N} d\psi_a d\bar{\psi}_a\,,$$

where we assume the standard anti-commutation relations and the normalisation,

$$\{\psi_a, \psi_b\} = \{\bar{\psi}_a, \psi_b\} = \{\bar{\psi}_a, \bar{\psi}_b\} = 0\,, \quad \int \mathcal{D}(\psi, \bar{\psi}) \prod_{a=1}^{N} \bar{\psi}_a \psi_a = 1\,.$$

Catalan numbers for random matrix eigenvalues

[Kornyik, Michaletzky, J. of Approx. Theory, 211:29-41, 2016]

Observe

$$\left\langle \det(\lambda\mathbb{1} - M)\right\rangle_M = \sigma^N \mathrm{He}_N(\lambda/\sigma)\,,$$

where $N \times N$ real symmetric matrix M is Gaussian distributed with a variance σ^2 for its off-diagonal elements and where He_N are the (probabilistic) Hermite polynomials.

Besides, the generating function $\mathcal{X}_N$ of the sum of the k-th powers of the (normalised) zeroes of the Hermite polynomials converges weakly to the generating function of the Catalan numbers

$$\frac{1}{N}\mathcal{X}_N(z/\sqrt{N}) \to \sum_{k\geq 0} C_k z^{2k} \quad (0 \leq z \leq 1/3)\,, \quad C_k = \frac{1}{k+1}\binom{2k}{k}$$

$$\mathcal{X}_N(z) = \sum_{k\geq 0} \Xi_N(k) z^k\,, \quad \Xi_N(k) = \sum_{j=1}^{N}\left(\xi_j^{(N)}\right)^k,$$

where $\{\xi_j^{(N)}\}_{1\leq j\leq N}$ are the N roots of the N-th Hermite polynomial He_N, and they are related to the N eigenvalues of $M_{N\times N}$, $\{\lambda_j^{(N)}\}_{1\leq j\leq N}$, by the rescaling $\xi_j^{(N)} = 2\sqrt{N}\lambda_j^{(N)}$.

Sketch of our proposal

In analogy with matrices,

- We define a characteristic polynomial of a tensor via a Grassmann integral (as a partition function).
- Its zeros/roots remind us of “eigenvalues" of a tensor.
- Consider an average over tensors and see the distribution of these roots/zeros. ($\rightarrow$ obtain a “spectral density" of tensors)

A new definition of characteristic polynomials of tensors

Consider a certain partition function of two species of Grassmann variables ψ and $\bar{\psi}$, order-p tensor T, and λ, $\{g\}$, and $\{\tilde{g}\}$ are complex parameters. (for a tensor with odd p, or for a complex tensor, we introduce $\bar{T}$ respectively as another species of Grassmann odd tensor, or the complex conjugate)

$$Z(\lambda, T, \bar{T}, \{g\}, \{\bar{g}\}) = \int \mathcal{D}(\psi, \bar{\psi}) e^{S[\lambda, T, \bar{T}, \{\psi\}, \{\bar{\psi}\}, \{g\}, \{\tilde{g}\}]} ,$$

where the action [8] with $\psi_a^{(0)} := \psi_a$, $\psi_a^{(1)} := \bar{\psi}_a$,

$$S[\cdots] = \lambda \sum_{a=1}^{N} \bar{\psi}_a \psi_a + \sum_{a_1, \ldots, a_p = 1}^{N} \sum_{b_1, \ldots, b_p = 0}^{1} \left(g^{(b_1 \cdots b_p)} T_{a_1 \cdots a_p} + \tilde{g}^{(b_1 \cdots b_p)} \bar{T}_{a_1 \cdots a_p} \right) \prod_{i=1}^{p} \psi_{a_i}^{(b_i)} , \text{ with}$$

$$\begin{aligned} \sum_{b_1, \ldots, b_p = 0}^{1} \prod_{i=1}^{p} \psi_{a_i}^{(b_i)} = & (\psi_{a_1} \cdots \psi_{a_p}) \\ & + (\bar{\psi}_{a_1} \psi_{a_2} \psi_{a_3} \cdots \psi_{a_p}) + (\psi_{a_1} \bar{\psi}_{a_2} \psi_{a_3} \cdots \psi_{a_p}) + \cdots + (\psi_{a_1} \psi_{a_2} \cdots \psi_{a_{p-1}} \bar{\psi}_{a_p}) \\ & + (\bar{\psi}_{a_1} \bar{\psi}_{a_2} \psi_{a_3} \cdots \psi_{a_p}) + (\bar{\psi}_{a_1} \psi_{a_2} \bar{\psi}_{a_3} \cdots \psi_{a_p}) + \cdots + (\bar{\psi}_{a_1} \psi_{a_2} \cdots \psi_{a_{p-1}} \bar{\psi}_{a_p}) \\ & + \quad \ldots \quad + (\bar{\psi}_{a_1} \cdots \bar{\psi}_{a_p}) . \end{aligned}$$

Because of the Grassmann properties of the variables $\psi, \bar{\psi}$, the partition function $Z(\lambda, T, \{g\})$ is a polynomial of degree N in λ. There are N roots in contrast with the common exponential in N number of eigenvalues!

[8] To keep the action Grassmann even, for p odd, the tensor is Grassmann odd.

A new definition of spectrum of an ensemble of tensors

Let's take an Gaussian average over real tensors T, with the measure

$$d\nu(T) = \mathcal{N}\left[\prod_{a_1,\cdots,a_p} dT_{a_1\ldots a_p}\right]\exp\{-N^{p-1}\sum_{a_1,\cdots,a_p=1}^{N}(T_{a_1\ldots a_p})^2\},$$

or, over complex or Grassmann tensors T, $\bar{T}$ with the measure

$$d\nu(T,\bar{T}) = \mathcal{N}\left[\prod_{a_1,\cdots,a_p} dT_{a_1\ldots a_p} d\bar{T}_{a_1\ldots a_p}\right]\exp\{-N^{p-1}\sum_{a_1,\cdots,a_p=1}^{N}\bar{T}_{a_1\ldots a_p}T_{a_1\ldots a_p}\}.$$

The tensor averaged partition function

$$Z(\lambda,\mu) := \langle Z(\lambda,T,\bar{T},\{g\},\{\bar{g}\})\rangle_{T,\bar{T}} = \int \mathcal{D}(\psi,\bar{\psi})\ \exp\{\lambda\sum_{a=1}^{N}\bar{\psi}_a\psi_a - \mu\left(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right)^p\}$$

$$= \lambda^N \sum_{n=0}^{\lfloor N/p\rfloor}\frac{1}{n!}\left(-\frac{\mu}{\lambda^p}\right)^n \frac{N!}{(N-pn)!},$$

where $\mu = \tilde{\mu}/N^{p-1}$ with $\tilde{\mu} = \mathcal{O}(1)$. [9] We identify the tensor averaged partition function above to be a degree-N polynomial in λ. We solve for its roots.

[9] μ depends on the original action is a combination of the 2-point function of the tensor and combinatorial factors coming from the contraction of the Grassmann variables in a way that results into the term $\left(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right)^p$.

Zeros of the tensor averaged partition function

$$Z(\lambda,\mu) := \langle Z(\lambda, T, \bar{T}, \{g\}, \{\bar{g}\})\rangle_{T,\bar{T}} = \int \mathcal{D}(\psi,\bar{\psi}) \exp(\lambda \sum_{a=1}^{N} \bar{\psi}_a \psi_a - \mu \left(\sum_{a=1}^{N} \bar{\psi}_a \psi_a\right)^p)$$

$$= \lambda^N \sum_{n=0}^{\lfloor N/p \rfloor} \frac{1}{n!} \left(-\frac{\mu}{\lambda^p}\right)^n \frac{N!}{(N-pn)!} \,.$$

- Its zeros are located in p-fold symmetry on the complex λ plane: (for $N = 50$ and $\tilde{\mu} = 1$)

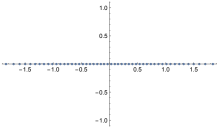


Figure: $p = 2$.

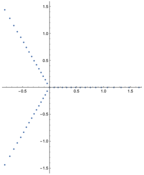


Figure: $p = 3$.

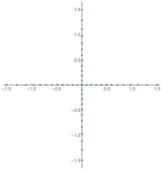


Figure: $p = 4$.

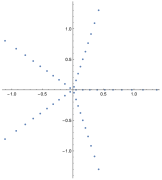


Figure: $p = 5$.

- In analogy with the matrices, whose characteristic polynomial is given by a similar integral and its ensemble average of matrices gives a polynomial

$$\left\langle \det(\lambda \mathbb{1} - M) \right\rangle_M = \left\langle \int \mathcal{D}(\psi,\bar{\psi}) \exp\{\sum_{a,b} \bar{\psi}_a (\lambda \delta_{ab} - M_{ab}) \psi_b\} \right\rangle_M = \sigma^N \mathrm{He}_N(\lambda/\sigma)\,,$$

one can see those zeros as the average location of the "eigenvalues" of the tensors.

Large N analysis

To obtain the large N limit we first have the equality at any N using a "radial" coordinate Q,

$$Z(\lambda,\mu) := \langle Z(\lambda,T,\{g\})\rangle_T = \int \mathcal{D}(\psi,\bar{\psi})\ \exp\{\lambda\Big(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\Big) - \mu\Big(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\Big)^p\}$$

$$= \frac{N!}{N^N}\frac{1}{2\pi i}\oint_{\mathcal{C}} dQ\,\frac{1}{Q^{N+1}}\exp(N\lambda Q - N\tilde{\mu}Q^p)\,, \quad \mu = \frac{\tilde{\mu}}{N^{p-1}}.$$

$$\text{At large } N, \quad Z \sim \oint_{\mathcal{C}} dQ\,\exp(NS[Q])\,, \quad \text{where} \quad S[Q] = \lambda Q - \tilde{\mu}Q^p - \log Q\,.$$

Obtain the saddle point equation $\frac{\partial S}{\partial Q}\Big|_{Q=Q_*} = 0$ in the new variables $q_* = \lambda Q_*$ and $z = \frac{p\tilde{\mu}}{\lambda^p}$,

$$q_*(z) = 1 + z q_*(z)^p$$

which is the Fuss-Catalan equation.

Fuss-Catalan numbers for random tensors

Consider the generating function of the sum of the k-th powers of the roots $\{\lambda_j^{(N)}\}_{1\leq j\leq N}$ of our partition function $Z(\lambda) = \sum_{k=0}^{N} b_k \lambda^k$,

$$\mathcal{X}_N(\lambda) = \sum_{k\geq 0} \Xi_N(k)\lambda^k \,, \quad \Xi_N(k) = \sum_{j=1}^{N} \left(\lambda_j^{(N)}\right)^k .$$

- We can show that $\mathcal{X}_N(1/\lambda) = \frac{\lambda Z'(\lambda)}{Z(\lambda)}$.
- The fermionic 2-point function is given by $\Omega(\lambda) := \frac{\lambda}{N}\frac{d}{d\lambda}\log Z(\lambda) = \frac{\lambda}{N}\frac{Z'(\lambda)}{Z(\lambda)}$.

Using the analysis at the leading order in N when one saddle contributes $\frac{1}{N}\log Z(\lambda) \sim S[Q_*]$, $\quad S[Q_*] = \lambda\, Q_* - \tilde{\mu} Q_*^p - \log Q_*$, $\quad \left.\frac{\partial S[Q]}{\partial Q}\right|_{Q=Q_*} = 0$,
we compute the fermionic 2-point function

$$\Omega(\lambda) \sim \lambda \frac{dS[Q_*]}{d\lambda} = \lambda \frac{\partial S[Q_*]}{\partial \lambda} + \lambda \left.\cancelto{0}{\frac{\partial S[Q]}{\partial Q}}\right|_{Q=Q_*} \frac{dQ_*}{d\lambda} = \lambda Q_* = q_* \,, \quad z \leq z_c = \frac{(p-1)^{p-1}}{p^p},$$

obeying the Fuss-Catalan equation $q_*(z) = 1 + z q_*(z)^p$.

(generalises Catalan numbers for Gaussian real symmetric matrices)

Existence of the zeros of the partition function

$$Z \sim \sum_{Q_* \text{ saddles}} \exp(NS[Q_*])\,.$$

- We observe that the partition function Z has zeros when two saddles Q_*'s have the same real part and opposite imaginary parts.

$$Z \sim e^{N\,\mathrm{Re}\left[S[Q_*]\right]}\,\cos\!\left(N\,\mathrm{Im}\left[S[Q_*]\right]\right),$$

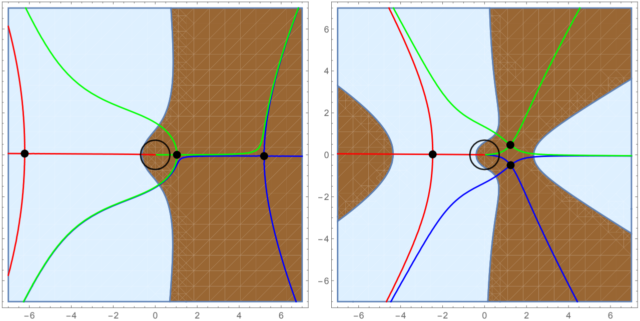
$$Z = 0 \quad \Leftrightarrow \quad \cos\!\left(N\,\mathrm{Im}\left[S[Q_*]\right]\right) = 0 \quad \Leftrightarrow \quad \mathrm{Im}S[Q_*] = \left(\frac{1}{2} + k\right)\frac{\pi}{N}\,, \quad k \in \mathbb{Z}\,.$$

- Focus on the radial component, r, of the zeros s.t. $\lambda = r\,e^{i2\pi k/p}$, $0 \le k \le p-1$.

- The distance between neighboring zeros is given by $\Delta r\,\frac{d\,\mathrm{Im}S[Q_*]}{dr} = \frac{\pi}{N}$, and their normalised[10] spectral density in large N limit is obtained

$$\begin{aligned}\rho(r) \quad &:= \quad \frac{p}{N}\frac{1}{|\Delta r|} = \frac{p}{\pi}\left|\frac{d\,\mathrm{Im}\left[S[Q_*]\right]}{dr}\right| = \frac{p}{\pi}\left|\frac{\partial\,\mathrm{Im}\left[S[Q_*]\right]}{\partial r} + \underbrace{\frac{\partial \mathrm{Im}S[Q]}{\partial Q}\Big|_{Q=Q_*}}_{\to\,0}\frac{dQ_*}{dr}\right| \\ &= \quad \frac{p}{N}\left|\mathrm{Im}\left[Q_*(r)\right]\right|.\end{aligned}$$

[10]recall that we have N zeros, and also p-fold symmetry.



(a) $z_0 = 0.03 < z_c$. $p = 3$. (b) $z_0 = 0.23 > z_c$. $p = 3$.

Figure: We represent in the complex q-plane, the Lefschetz thimbles ending in the light blue regions and their duals ending in the brown regions, for each saddle point of the Fuss-Catalan equation. The light blue (respectively brown) regions indicate where the real part of the large N action appearing in $Z \sim \oint_{\mathcal{C}} dq \exp(N\, S[q])$ with $S[q] = q - z\, q^p/p - \log(q/z^{1/p})$ is negative (respectively positive). The black circle around the origin is the original curve $\mathcal{C}$. The black points correspond to the p saddle points given by the p solutions of the Fuss-Catalan equation. For (a) $z < z_c$, the saddle point of the green thimble and dual thimble, contributes at leading order. For (b) $z > z_c$, the two saddles of the right (green and blue thimbles and dual thimbles) contribute at leading order in N. We have taken $z = z_0 e^{i\theta_0}$, $\theta_0 = 0.02$, with $z_c = (p-1)^{p-1}/p^p = 2^2/3^3 \approx 0.15$ for $p = 3$. $q = \lambda Q$ and $z = \frac{p\tilde{\mu}}{\lambda^p}$.

Our generalised Wigner's semicircle law for random tensors

We have

- the equivalence of two probability distributions $\frac{1}{2}\rho(r)dr = \rho_{\text{Gurau}}(w)\Theta(w)dw$,
- the scaling between Gurau's and ours by comparing the respective 2-point functions which are both the generating function of the Fuss-Catalan numbers $w^2 = \frac{1}{p\,\tilde{\mu}}r^p$,

and hence

$$\rho(r) = \left(\sqrt{\frac{p}{\tilde{\mu}}}\,r^{p/2-1}\right)\rho_{\text{Gurau}}\left(\frac{r^{p/2}}{\sqrt{p\tilde{\mu}}}\right).$$

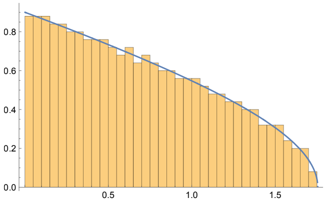


Figure: $\rho(r)$ vs. r. The distribution of the radius r of λ superimposed with the histogram absolute value of the roots of the polynomial $Z(\lambda) = \lambda^N \sum_{n=0}^{\lfloor N/p \rfloor} \frac{1}{n!}\left(-\frac{\mu}{\lambda^p}\right)^n \frac{N!}{(N-pn)!}$ for $N = 2000$ and $p = 4$ and $\mu = \tilde{\mu}/N^{p-1} = 1/(pN^{p-1})$.

Conclusions

- We proposed a new notion of characteristic polynomials of tensors (of size N) via Grassmann integrals.

  (Tensors are general (complex, real, Grassmann) including totally antisymmetric, but cannot be totally symmetric.)

  There are N zeros/roots (as opposed to $\sim e^{\text{constant } N}$ number for the standard existing definitions of eigenvalues of tensors.

- We provide an associated new notion of “spectral density", i.e., distributions of zeros of characteristic polynomials of random Gaussian tensors.

  This is again Fuss-Catalan!

  Actually, the eigenvalues of a product of complex Ginibre random matrices ($Y_{p-1}^* Y_{p-1}$ with $Y_{p-1} = X_1 \cdots X_{p-1}$ where $\{X_i\}_{1 \le i \le p-1}$ are complex Ginibre matrices) follow Fuss-Catalan distributions [K. A. Penson, K. Zyczkowski, Physical Review E - Statistical, Nonlinear, and Soft Matter Physics, 83(6):061118, 2011]. The polynomials associated with the zeros of characteristic polynomials of the product of Ginibre random matrices also is shown to be Fuss-Catalan. [T. Neuschel, Random Matrices: Theory and Applications Vol. 03, No. 01, 1450003 (2014)] the end