Various products of representative series and some applications

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Abstract

Special functions such as polyzetas, multiple harmonic sums and polylogarithms are defined over $\mathcal{H}_r := \{(s_1, \ldots, s_r) \in \mathbb{N}^r_{\geq 1}, s_1 > 1\}$. Polyzetas values are given by the formula:

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}, \tag{1}$$

polylogarithms (denoted $(Li_{s_1,\ldots,s_r})$ with $s_j \geq 1, r \geq 1$) and multiple harmonic sums (denoted $(H_{s_1,\ldots,s_r})$ with $s_j \geq 1, r \geq 1$). They are defined as follows (with $n \in \mathbb{N}_{\geq 1}$):

$$Li_{s_1,\ldots,s_r}(z) = \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r} z^{n_1} \tag{2}$$

and

$$H_{s_1,\ldots,s_r}(n) = \sum_{n \geq n_1 > \cdots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}. \tag{3}$$

They are compatible with algebraic structures of quasi-shuffle products, in some different cases of the parameter q:

$$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u, \quad y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + q y_{i+j}(u \uplus v), \tag{4}$$

where ϵ is the empty word, y_i, y_j, y_{i+j} are letters of the alphabet $Y = \{y_k\}_{k \in \mathbb{N}_{\geq 1}}$, and u, v are words in the monoid Y^*.

For a commutative ring A containing the field of rational numbers $\mathbb{Q}$, we examine the set of noncommutative formal series, denoted by $A\langle\langle \mathcal{X} \rangle\rangle$. Within this set, representative series, that are closed under various products, form a module. This is a central focus of our research.

In this presentation, we will delve into how to factorize and decompose these noncommutative rational series and explore their relevance to theoretical computer science.

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