Multiple Divided Bernoulli Polynomials and Numbers

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Abstract

This work defines multiple divided Bernoulli polynomials by solving a system of difference equations that generalizes the classical Bernoulli case. These polynomials are required to span an algebra whose product matches the M basis of *QSym*. Although not unique, an explicit and notable solution is constructed using the reflection equation for Bernoulli polynomials.

1 Introduction

The aim of this paper is to define a family of polynomials $B_{n_1,\cdots,n_r}$, for $r \in \mathbb{N}$, depending on non-negative integers $n_1, \cdots, n_r$, generalizing the classical family $\left(\dfrac{B_{n+1}}{n+1}\right)_{n\geq 0}$. We will call them divided multiple Bernoulli polynomials and define the associated multiple divided Bernoulli numbers $b_{n_1,\cdots,n_r}$, $r \in \mathbb{N}$, as their constant terms, generalizing the divided Bernoulli numbers $\left(\dfrac{B_{n+1}(0)}{n+1}\right)_{n\geq 0}$.

The Riemann and Hurwitz zeta functions extend to multiple cases, respectively as multiple zeta values (MZV) and the Hurwitz multiple zeta function, (HMZF) , for $s_1, \cdots, s_r \in \mathbb{C}$ such that $\Re(s_1+\cdots+s_k) > k$, $k \in [\![1 \,;\, r]\!]$ (See [6] for the MZV; see [1] for the Hurwitz multiple zeta functions):

$$\mathcal{Z}e^{s_1,\cdots,s_r} = \sum_{0<n_r<\cdots<n_1} \frac{1}{{n_1}^{s_1}\cdots{n_r}^{s_r}} \quad , \quad \mathcal{H}e^{s_1,\cdots,s_r} : z \longmapsto \sum_{0<n_r<\cdots<n_1} \frac{1}{(n_1+z)^{s_1}\cdots(n_r+z)^{s_r}} \,. \quad (1)$$

It is well known that the Riemann zeta function (resp. the Hurwitz zeta function) has a meromorphic continuation to $\mathbb{C}$ with a unique pole at 1, whose values at negative integers are related to the divided Bernoulli numbers (resp. the divided Bernoulli polynomials). Similarly, MZVs and HMZFs also have meromorphic continuations (see *e.g.*, [3], [5]), suggesting that their values at non-positive integers define respectively multiple divided Bernoulli numbers and polynomials.

2 Required conditions on multiple divided Bernoulli polynomials

On the one hand, Hurwitz multiple zeta functions specialize the monomial quasi-symmetric functions basis M of *QSym*. Therefore, MZVs and HMZFs multiply accordint to the product of the M's, namely the stuffle product $\ast$ (see [4]).

On the other hand, the Hurwitz multiple zeta functions satisfy a difference equation:

Lemma 1 *For all non-negative integers r and all positive integers $s_1, \cdots, s_r$ with $s_1 \geq 2$, we have:*

$$\mathcal{H}e^{s_1,\cdots,s_r}(z-1) - \mathcal{H}e^{s_1,\cdots,s_r}(z) = \begin{cases} \dfrac{1}{z^{s_r}} & \text{if } r = 1 \ . \\ \mathcal{H}e^{s_1,\cdots,s_{r-1}}(z) \cdot \dfrac{1}{z^{s_r}} & \text{if } r \geq 2 \ . \end{cases} \tag{2}$$

Reinterpreted at negative integers, this suggests basing the study of multiple divided Bernoulli polynomials on an analogue of Equation (2) and on multiplication via the stuffle product, i.e. defining polynomials that satisfy:

$$\begin{cases} Be^{\emptyset}(z) = 1 \text{ and } Be^{n}(z) = \dfrac{B_{n+1}(z)}{n+1} \text{ , where } n \geq 0 \ , \\ Be^{n_1,\cdots,n_r}(z+1) - Be^{n_1,\cdots,n_r}(z) = Be^{n_1,\cdots,n_{r-1}}(z) z^{n_r} \text{ , where } r \geq 1 \text{ and } \ n_1,\cdots,n_r \geq 0 \ , \\ \text{the } Be^{n_1,\cdots,n_r} \text{ multiply by the stuffle product.} \end{cases} \tag{3}$$

3 Algebraic reformulation of System (3)

Let us consider an infinite (commutative) alphabet of indeterminates $\mathsf{X} = \{X_1, X_2, X_3, \cdots\}$.

To any family of polynomials $Be^{n_1,\cdots,n_r}$ satisfying System (3), we associate a family of exponential generating functions $(\mathcal{B}eeg^{Y_1,\cdots,Y_r})$, with $r \in \mathbb{N}$ and $Y_1, \cdots, Y_r \in \mathsf{X}$:

$$\mathcal{B}eeg^{X_{i_1},\cdots,X_{i_r}}(z) \quad = \sum_{n_1,\cdots,n_r \geq 0} Be^{n_1,\cdots,n_r}(z) \frac{X_{i_1}^{n_1}}{n_1!} \cdots \frac{X_{i_r}^{n_r}}{n_r!}, \text{ for all } r \in \mathbb{N}^* \ , \ X_{i_1}, \cdots, X_{i_r} \in \mathsf{X} \ . \tag{4}$$

Then, extending the commutative alphabet X to

$$\mathsf{Y} = \mathbb{N}\mathsf{X} - \{0\} = \left\{ \sum_{X \in \mathsf{X}} \lambda_X \cdot X \ ; \ (\lambda_X)_{X \in \mathsf{X}} \in \mathbb{N}^{\mathsf{X}} \text{ has finitely nonzero terms} \right\} - \{0\} \ , \tag{5}$$

we define the noncommutative series $\mathcal{B}eeg$ over an infinite alphabet $\mathsf{A} = \{a_Y \ ; \ Y \in \mathsf{Y}\}$, *i.e.* a formal mould (see [?]) by:

$$\mathcal{B}eeg(z) = 1 + \sum_{r>0} \sum_{Y_1,\cdots,Y_r \in \mathsf{Y}} \mathcal{B}eeg^{Y_1,\cdots,Y_k}(z) \ a_{Y_1} \cdots a_{Y_r} \in \mathbb{C}[z][\![\mathsf{Y}]\!]\langle\langle \mathsf{A} \rangle\rangle \ . \tag{6}$$

The algebra $\mathbb{C}[z][\![\mathsf{Y}]\!]\langle\langle \mathsf{A} \rangle\rangle$ turns out to be a Hopf algebra, with its coproduct defined by:

$$\Delta_{\uplus}(a_Y) = 1 \otimes a_Y + \sum_{\substack{U,V \in \mathsf{Y} \\ U+V=Y}} a_U \otimes a_V + a_Y \otimes 1 \text{ , where } Y \in \mathsf{Y} \ . \tag{7}$$

Therefore, System (3) is equivalent to

$$\begin{cases} \mathfrak{B}(z+1) = \mathfrak{B}(z) \cdot \mathfrak{E}(z) \text{ , where } \mathfrak{E}(z) = 1 + \displaystyle\sum_{Y \in \mathsf{Y}} e^{zY} \ a_Y \ , \\ \mathfrak{B} \text{ is a group-like element of } \mathbb{C}[z][\![\mathsf{Y}]\!]\langle\langle \mathsf{A} \rangle\rangle \ , \\ \langle \mathfrak{B}(z) | a_Y \rangle = \dfrac{e^{zY}}{e^Y - 1} - \dfrac{1}{Y} \text{ for all } Y \in \mathsf{Y} \ . \end{cases} \tag{8}$$

4 A particular solution of Systems (3) and (8)

System (3) admits a unique solution $(Be_0^{n_1,\cdots,n_r}(z))$ of polynomials with zero constant term. Using these polynomials and their associated series $\mathcal{B}eeg_0 \in \mathbb{C}[z][\![\mathsf{Y}]\!]\langle\!\langle \mathsf{A} \rangle\!\rangle$, we fully characterize the solutions of System (8).

Theorem 1 *A family of polynomials* $(Be^{n_1,\cdots,n_r})_{\substack{r>0 \\ n_1,\cdots,n_r\in\mathbb{N}}}$ *is a solution of System* (3) *if, and only if,* $\mathcal{B}eeg(z) = \mathcal{X}\cdot\mathcal{B}eeg_0(z)$ *where* $\mathcal{X} \in \mathbb{C}[\![\mathsf{Y}]\!]\langle\!\langle \mathsf{A} \rangle\!\rangle$ *is a group-like element satisfying* $\langle \mathcal{X} | a_Y \rangle = \dfrac{1}{e^Y - 1} - \dfrac{1}{Y}$ *for all* $Y \in \mathsf{Y}$.

From this, we obtain a notable solution, called the **multiple divided Bernoulli polynomials**:

Definition 1 *Define* $\sqrt{sg} \in \mathbb{C}[\![\mathsf{Y}]\!]\langle\!\langle \mathsf{A} \rangle\!\rangle$ *by* $\sqrt{sg} = 1 + \displaystyle\sum_{r>0} \sum_{Y_1,\cdots,Y_r\in\mathsf{Y}} \frac{(-1)^r}{2^{2r}} \binom{2r}{r} a_{Y_1}\cdots a_{Y_r}$.
Let $\mathfrak{v}$ *be the primitive nonconmmutative series of* $\mathbb{C}[\![\mathsf{Y}]\!]\langle\!\langle \mathsf{A} \rangle\!\rangle$ *defined by:*

$$\langle \mathfrak{v} | a_{Y_1}\cdots a_{Y_r} \rangle = \frac{(-1)^{r-1}}{r} \left(\frac{1}{e^{S_r} - 1} - \frac{1}{S_r} + \frac{1}{2} \right) \text{ , where } S_r = Y_1 + \cdots + Y_r \tag{9}$$

We call multiple divided Bernoulli polynomials *(resp.* numbers*) the coefficients of the noncommutative series* $\mathfrak{B}eeg(z) = exp(\mathfrak{v}) \cdot \sqrt{sg} \cdot \mathcal{B}eeg_0(z)$ *(resp.* $\mathfrak{b} = exp(\mathfrak{v}) \cdot \sqrt{sg}$*).*

For a generic series $s \in \mathbb{C}[z][\![\mathsf{X}]\!]\langle\!\langle \mathsf{A} \rangle\!\rangle$ defined by $s(z) = 1 + \displaystyle\sum_{r>0} \sum_{k_1,\cdots,k_r>0} s^{X_{k_1},\cdots,X_{k_r}}(z)\, a_{k_1}\cdots a_{k_r}$, we denote respectively by $\overline{s}(z)$ and $\widetilde{s}(z)$ the reverse and retrograde series of $s(z)$:

$$\overline{s}(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} s^{Y_r,\cdots,Y_1}(z)\, a_{Y_1}\cdots a_{Y_r} \ , \quad \widetilde{s}(z) = 1 + \sum_{r>0} \sum_{k_1,\cdots,k_r>0} s^{-Y_1,\cdots,-Y_r}(z)\, a_{Y_1}\cdots a_{Y_r} \ . \tag{10}$$

Thus, the solution $\mathcal{B}eeg$ generalizes the reflection formula of Bernoulli polynomials.

Proposition 1 $\widetilde{\mathcal{B}eeg}(1-z) \cdot \overline{\mathcal{B}eeg}(z) = 1$.

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