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Normal ordering and generalized combinatorial numbers versus Lie groups

Résumé

We consider the normal ordering problem of powers of strings of boson creation (a^+) and annihilation (a) operators satisfying $[a,a^+] = 1$, which (?? correspond to unipotent trranssformations). These strings are monomials in the form $(a^+)^{r-p}a(a^+)^p$. We give the solution of the normal ordering problem for every set of parameters $\{r,p\}$, $r \geq p$. In particular it can be expressed through the sets of generalized Stirling numbers of the second kind and generalized Bell numbers for which we give exact expressions, generating functions as well as combinatorial interpretations.

We demonstrate that the above is equivalent to a problem of an action of exponentials of certain differential operators on Taylor expandable functions. We formulate conditions that such an action be a substitution of variables. In the general case these operators form a group which, in turn, is equivalent to a Frechet Lie group structure of infinite dimension. We show that, such a formalism can be expressed in terms of so called Sheffer-type polynomials, thereby establishing a link between quantum statistics, combinatorics and Lie groups of infinite dimension. Many concrete and detailed examples of such structures are explicitly worked out in detail. In particular, we show that the one-parameter subgroups induced by these operators are conjugate of groups of homographic substitutions which are here explicitely given.

1 Introduction

2 .../...

3 First order boson strings as differential operators

3.0.1 General definitions

In this section, we deal with the *word Stirling numbers*. In the following a^+,a are the generators of the Heisenberg-Weyl algebra i.e. $[a,a^+] = 1$.

$$|w|_u = r;\ |w|_d = s;\ r - s = e;\ then,$$
$$if\ e \geq 0,\ one\ has\ \mathcal{N}(w^n) = u^{ne}\sum_{k=0}^{ns} S_w(n,k)u^k d^k$$
$$if\ e < 0,\ one\ has$$
$$\mathcal{N}(w^n) = (\sum_{k=0}^{nr} S_w(n,k)u^k d^k)d^{-ne} \tag{1}$$

Definition 3.1 *For any word $w \in \{a^+,a\}^*$ with $|w|_{a^+} = r \geq |w|_a = s$, denoting $e = r - s$, one has*

$$\mathcal{N}(w^n) = (a^+)^{ne}\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k \tag{2}$$

Remark that, due to the reordering relation $aa^+ = a^+a + 1$, the sum above is finite. In fact, the "last" coefficient is $S_w(n,ns) = 1$ and $S_w(n,k) = 0$ for $k > ns$. Then, the number S_w form a unipotent matrix iff $s = 1$. This case is very rich and will be treated below.

3.1 The numbers $S_{(a^+)^{r-p}a(a^+)^p}(n,k)$

Proposition 3.2 *Let* $w_p=(a^+)^{r-p}a(a^+)^p$ *then*

$$S_{w_p}(n,k)=\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l) \tag{3}$$

Proof — The formula is of course true for $p=0$. We first remark that $a^+w_{p+1}^n=w_p^na^+$, then, supposing $n>0$, one has on the one hand

$$\begin{aligned}a^+(a^+)^{ne}\sum_{k=0}^{\infty}S_{w_{p+1}}(n,k)(a^+)^ka^k=\mathcal{N}(a^+w_{p+1}^n)=\mathcal{N}(w_p^na^+)=(a^+)^{ne}\sum_{k=0}^{\infty}S_{w_p}(n,k)(a^+)^ka^ka^+=\\ (a^+)^{ne}\sum_{k=1}^{\infty}S_{w_p}(n,k)(a^+)^k(a^ka^+)=(a^+)^{ne}\sum_{k=1}^{\infty}S_{w_p}(n,k)(a^+)^k(a^+a^k+ka^{k-1})=\\ (a^+)(a^+)^{ne}\left(\sum_{k=1}^{\infty}S_{w_p}(n,k)(a^+)^ka^k+\sum_{k=1}^{\infty}S_{w_p}(n,k)(k)(a^+)^{k-1}a^{k-1}\right)=\\ (a^+)(a^+)^{ne}\left(\sum_{k=1}^{\infty}S_{w_p}(n,k)(a^+)^ka^k+\sum_{k=0}^{\infty}S_{w_p}(n,k+1)(k+1)(a^+)^ka^k\right)=\\ (a^+)(a^+)^{ne}\left(\sum_{k=0}^{\infty}\underbrace{S_{w_p}(n,k)+S_{w_p}(n,k+1)(k+1)}_{F(n,k)}\right)(a^+)^ka^k)=A\end{aligned} \tag{4}$$

Let us compute separately the factor $F(n,k)$, then, by induction hypothesis,

$$\begin{aligned}F(n,k)=\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l)+(k+1)\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+2)^{\bar{l}}S_{r1}(n,k+l+1)=\\ \sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l)+\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\overline{l+1}}S_{r1}(n,k+l+1)=\\ S_{r1}(n,k)+\sum_{l=1}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l)+\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\overline{l+1}}S_{r1}(n,k+l+1)=\\ S_{r1}(n,k)+\sum_{l=0}^{p-1}\begin{pmatrix} p \\ l+1 \end{pmatrix}(k+1)^{\overline{l+1}}S_{r1}(n,k+l+1)+\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\overline{l+1}}S_{r1}(n,k+l+1)=\\ S_{r1}(n,k)+\Big(\sum_{l=0}^{p-1}\begin{pmatrix} p+1 \\ l+1 \end{pmatrix}(k+1)^{\overline{l+1}}S_{r1}(n,k+l+1)\Big)+(k+1)^{\overline{p+1}}S_{r1}(n,k+p+1)=\\ \sum_{l=0}^{p+1}\begin{pmatrix} p+1 \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l)\end{aligned} \tag{5}$$

Hence

$$A=(a^+)(a^+)^{ne}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{p+1}\begin{pmatrix} p+1 \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l)\right)(a^+)^ka^k) \tag{6}$$

which proves the equality (3). □

The characteristic series of the numbers $S_{w_p}(n,k)$ is then given by

$$\sum_{n,k\geq 0}S_{w_p}(n,k)\frac{x^n}{n!}y^k=\sum_{n,k\geq 0}\Big(\sum_{l=0}^{p}\begin{pmatrix} p \\ l \end{pmatrix}(k+1)^{\bar{l}}S_{r1}(n,k+l)\Big)\frac{x^n}{n!}y^k=$$

$$\sum_{l=0}^{p}\sum_{k\geq 0}\left(\left(\begin{array}{c}p\\ l\end{array}\right)(k+1)^{\bar{l}}y^k\right)\sum_{n\geq 0}S_{r1}(n,k+l)\frac{x^n}{n!}=\sum_{l=0}^{p}\sum_{k\geq 0}\left(\left(\begin{array}{c}p\\ l\end{array}\right)(k+1)^{\bar{l}}y^k\right)\frac{\phi(x)^{k+l}}{(k+l)!} \quad (7)$$

where

$$\phi(x)=\sum_{n\geq 0}S_{r1}(n,1)\frac{x^n}{n!}=\left(y(e^{\frac{-log(1-(r-1)x)}{r-1}}-1)\right)\Big|_{y=1}=\left(\left(\frac{1}{1-(r-1)x}\right)^{\frac{1}{r-1}}-1\right) \quad (8)$$

Thus

$$\sum_{n,k\geq 0}S_{w_p}(n,k)\frac{x^n}{n!}y^k=\sum_{l=0}^{p}\left(\begin{array}{c}p\\ l\end{array}\right)\frac{\partial^l}{\partial y^l}\sum_{k\geq 0}\left(y^{k+l}\frac{\phi(x)^{k+l}}{(k+l)!}\right)=\sum_{l=0}^{p}\left(\begin{array}{c}p\\ l\end{array}\right)\frac{\partial^l}{\partial y^l}\sum_{m\geq l}\left(y^m\frac{\phi(x)^m}{(m)!}\right)=$$
$$\sum_{l=0}^{p}\left(\begin{array}{c}p\\ l\end{array}\right)\frac{\partial^l}{\partial y^l}\sum_{m\geq 0}\left(y^m\frac{\phi(x)^m}{(m)!}\right)=\sum_{l=0}^{p}\left(\begin{array}{c}p\\ l\end{array}\right)\frac{\partial^l}{\partial y^l}e^{y\phi(x)}=(1+\frac{\partial}{\partial y})^pe^{y\phi(x)}=(1+\phi(x))^pe^{y\phi(x)} \quad (9)$$

Note 3.3 *The same proof as above for (3) shows that*

$$S_{w(a^+)^p}(n,k)=\sum_{l=0}^{p}\left(\begin{array}{c}p\\ l\end{array}\right)(k+1)^{\bar{l}}S_{(a^+)^pw}(n,k+l) \quad (10)$$

3.2 The strings $(a^+)^{r-p}a(a^+)^p$ as infinitesimal operators

We here consider the string $(a^+)^{r-p}a(a^+)^p$ in the representation $a^+\to x$, $a\to\frac{d}{dx}$ acting on analytic functions. The one-parameter transformation group is then defined by

$$U_\lambda(f)=e^{\lambda\left((a^+)^{r-p}a(a^+)^p\right)}[f]=\sum_{k=0}^{\infty}\frac{\lambda^n}{n!}(a^+)^{r-p}a(a^+)^p[f] \quad (11)$$

where U_λ is a one parameter group of some fonction space. One has

$$U_\lambda(f)=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(a^+)^{r-p}a(a^+)^p[f]=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(a^+)^{n(r-1)}\sum_{k\geq 0}S_w(n,k)(a^+)^ka^k[f]=$$
$$\sum_{k\geq 0}\left(\sum_{n=0}^{\infty}\frac{\lambda^nx^{n(r-1)}}{n!}S_w(n,k)\right)x^k\frac{d^k}{dx^k}[f]=\sum_{k\geq 0}\left(1+\phi(\lambda x^{r-1})\right)^p\frac{\phi(\lambda x^{r-1})^k}{k!}x^k\frac{d^k}{dx^k}[f] \quad (12)$$

as

$$\sum_{n,k\geq 0}S_w(n,k)\frac{X^n}{n!}Y^k=(1+\phi(x))^p\,e^{Y\phi(X)}=(1+\phi(x))^p\sum_{k\geq 0}\frac{\phi(X)^k}{k!}Y^k \quad (13)$$

and then, if $f=\sum_{m\geq 0}c_m\frac{x^m}{m!}$ one has

$$U_\lambda[f]=\left(1+\phi(\lambda x^{r-1})\right)^p\sum_{k\geq 0}\sum_{m\geq k}\frac{\phi(\lambda x^{r-1})^kx^k}{k!}c_m\frac{x^{m-k}}{(m-k)!}=$$
$$\left(1+\phi(\lambda x^{r-1})\right)^p\sum_{k\geq 0}\sum_{m\geq k}\frac{c_mx^m}{m!}\left(\begin{array}{c}m\\ k\end{array}\right)\phi(\lambda x^{r-1})^k=$$
$$\left(1+\phi(\lambda x^{r-1})\right)^p\sum_{m\geq 0}\sum_{k\geq 0}\frac{c_mx^m}{m!}\left(\begin{array}{c}m\\ k\end{array}\right)\phi(\lambda x^{r-1})^k=$$
$$\left(1+\phi(\lambda x^{r-1})\right)^p\sum_{m\geq 0}\frac{c_mx^m}{m!}\sum_{0\leq k\leq m}\left(\begin{array}{c}m\\ k\end{array}\right)\phi(\lambda x^{r-1})^k=$$
$$\left(1+\phi(\lambda x^{r-1})\right)^p\sum_{m\geq 0}\frac{c_mx^m}{m!}\left(1+\phi(\lambda x^{r-1})\right)^m=\left(1+\phi(\lambda x^{r-1})\right)^pf\left[x\left(1+\phi(\lambda x^{r-1})\right)\right] \quad (14)$$

3.3 One parameter groups

3.3.1 Groups of substitutions

Considering an operator Ω on the sequences, the one-parameter group corresponding to Ω is, generally speaking, $\{e^{\lambda\Omega}\}$. It may be full (i.e. defined for every $\lambda \in \mathbb{R}$) or local $\lambda \in]-\epsilon,\epsilon[$. It always fulfills an additive rule, i.e., when it has sense, the relation $g_\lambda \star g_\mu = g_{\lambda+\mu}$, where $\star$ is some law (in the following, composition or pointwise multiplication). The simplest one-parameter group one can imagine is the *shift* i.e. the transformation

$$f(x) \to f(x+\lambda)$$

On can congugate this group by an invertible (at least locally) variable change $x \to u(x)$ with inverse u^{-1} (this means that, on appropriate domains $u(u^{-1}(y)) = y$ and $u^{-1}(u(x)) = x$). Thus, we get

$$f(x) \to f(u^{-1}(u(x)+\lambda)) \tag{15}$$

One can prove that, if two differentiable one-parameter groups have the same initial derivative, then they coincide. Unsing the chain rule, one gets the initial derivative of the group (15) which reads

$$\frac{d}{d\lambda}\Big(f(u^{-1}(u(x)+\lambda))\Big)\Big|_{\lambda=0} = \frac{d}{dx}(f)(u^{-1}(u(x)+\lambda)) \times \frac{d}{dx}(u^{-1})(u(x)+\lambda)\Big|_{\lambda=0} = \frac{d}{dx}(f)(x) \times \frac{d}{dx}(u^{-1})(u(x)) = \frac{1}{\frac{d}{dx}(u)(x)} \times \frac{d}{dx}(f)(x) \tag{16}$$

For example, with the vector fields on the half line of type $x^\alpha \frac{d}{dx}$, the method consists in resolving the equation

$$\frac{d}{dx}(u)(x) = x^{-\alpha} \tag{17}$$

in the case ($\alpha \neq 1$) one gets $u(x) = \frac{x^{1-\alpha}}{1-\alpha}$. The inverse of u is $u^{-1}(y) = ((1-\alpha)y)^{\frac{1}{1-\alpha}}$ and the one-parameter group (a substitution group) is given by

$$f(u^{-1}(u(x)+\lambda)) = f\left(\Big((1-\alpha)(\frac{x^{1-\alpha}}{1-\alpha}+\lambda)\Big)^{\frac{1}{1-\alpha}}\right) = f\left(x\Big((1+(1-\alpha)\lambda x^{\alpha-1})\Big)^{\frac{1}{1-\alpha}}\right) \tag{18}$$

which reproves the result of
??`précédent avec` r.
For the case $\alpha = 1$, an analog computation proves that the one parameter group is a group of dilatations $f(x) \to f(e^\lambda x)$.
With the field $(1+x^2)\frac{d}{dx}$ one gets $u = arctan;\ u^{-1} = tan$ the one-parameter group is then

$$s_\lambda(x) = u^{-1}(u(x)+\lambda) = tan(arctan(x)+\lambda) = \frac{tan(arctan(x)) + tan(\lambda)}{1 - tan(arctan(x))tan(\lambda)} = \frac{x + tan(\lambda)}{1 - xtan(\lambda)} = \frac{xcos(\lambda) + sin(\lambda)}{cos(\lambda) - xsin(\lambda)} \tag{19}$$

all the preceding results can be found in [**?**].
With $(1-x^2)\frac{d}{dx}$, one has

$$u(x) = arctanh(x) = ln(\sqrt{\frac{1+x}{1-x}});\ u^{-1}(x) = tanh(x);$$

$$s_\lambda(x) = tanh(arctanh(x)+\lambda) = \frac{xcosh(\lambda)+sinh(\lambda)}{cosh(\lambda)+xsinh(\lambda)} \tag{20}$$

With $\frac{\sqrt{(1+x^2)}}{x}\frac{d}{dx}$, one gets

$$\begin{aligned} u(x) = \sqrt{1+x^2};\ u^{-1}(x) = \sqrt{x^2-1}; \\ s_\lambda(x) = \sqrt{x^2+2\lambda\sqrt{1+x^2}+\lambda^2} \end{aligned} \tag{21}$$

3.3.2 Conjugates of one-parameter groups of substitutions and prefunctions

Let A be a continuous invertible operator over a certain function space (typically, in the following A will be the multiplication by a non-vanishing function. Then, one has $A^{-1}e^{\lambda\Omega}A = e^{\lambda A^{-1}\Omega A}$ (which can be proved using the fact that, if two differentiable one-parameter groups have the same initial derivatives then they coincide). Using this fact, we can act with two fonctionnal parameters instead of one, then

$$e^{\lambda(h_2\frac{h_1'}{h_1}+h_2\frac{d}{dx})} = e^{\lambda\frac{h_2}{h_1}(h_1'+h_1\frac{d}{dx})} = e^{\lambda\frac{h_2}{h_1}(\frac{d}{dx}h_1)} = e^{\lambda(\frac{1}{h_1}(h_2\frac{d}{dx})h_1)} = \frac{1}{h_1}e^{\lambda(h_2\frac{d}{dx})}h_1 \tag{22}$$

For example, we can treat the case $(a^+)^{r-p}a(a^+)^p$ remarking that $\Omega = x^{r-p}\frac{d}{dx}x^p = x^{-p}(x^r\frac{d}{dx})x^p$. Then

$$e^{\lambda\Omega} = x^{-p}e^{\lambda x^r\frac{d}{dx}}x^p \tag{23}$$

and denoting $s_\lambda(x) = x\Big((1+(1-r)\lambda x^{r-1})\Big)^{\frac{1}{1-r}}$, the one-parameter substitutionnal group corresponding to the vector field $x^r\frac{d}{dx}$, the transform of a function f reads

$$x^{-p}f(s_\lambda(x))(s_\lambda(x))^p = \left(\frac{s_\lambda(x)}{x}\right)^p f(s_\lambda(x)) \tag{24}$$

Remark 3.4 *Using h_1,h_2 as above, one can also treat by conjugacy the example of the differential operator of (Dattoli)* $\Omega = (q(x)\frac{d}{dx}+v(x))$ *with* $q = h_2;\ v = q\frac{h_1'}{h_1}$.