One-parameter groups and combinatorial physics

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Abstract.

1. Introduction

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In the preceding articles the cases $a^r(a^+)^s$ were studied.

2. Bonson string matrices

Let $w \in \{a, a^+\}^*$ be a word in the letters $\{a, a^+\}$, and define $e = |w|_{a^+} - |w|_a$ (the excess), then the normal form of w^n reads

$$\mathcal{NF}(w^n) = (a^+)^{ne} \left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k \right) \tag{1}$$

when e is positive (i.e. there is more creations than annihilations).
In the opposite case (i.e. there is more annihilations than creations) the normal form of w^n is

$$\mathcal{NF}(w^n) = \left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k \right) (a)^{n|e|} \tag{2}$$

in each case, the coefficients S_w are well defined by the corresponding equation (1 and 2).
Now, for any boson string u one has

$$\mathcal{NF}(u) = (a^+)^{|u|_{a^+}} a^{|u|_a} + \sum_{|v|<|u|} \lambda_v v. \tag{3}$$

It has been observed that the numbers λ_v which can be easily computed from the string u‡.

Let us give, as examples, the upper-left corner of these (doubly infinite) matrices.

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

‡ This remark is due to Mark van Leeuwen

For $w = a^+aa^+$, we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

For $w = a^+aaa^+a^+$, one gets

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6}$$

Remark that, in each case, the matrix S_w has a staircase form and the "step" depends of the number of a's in the word w. More precisely, due to equation (3) one can prove that the ones ending each row are have (n, nr) as addresses (where $r = |w|_a$). Thus all the matrices are row finite and unitriangular iff $r = 1$, which case will be of special interest in the following.

3. Sequence transformations

In this section, we deal with sequence transformations performed by means of *row-finite* matrices with complex coefficients. These matrices will be indexed by $\mathbf{N} \times \mathbf{N}$ and hence such a matrix is a double sequence $(M(n,k))_{n,k\geq 0}$ such that, for every $n \in \mathbf{N}$ the sequence $(M(n,k))_{k\geq 0}$ has finite support. Their set $\mathcal{RF}_{\mathbf{C}}$ is exactly the algebra of continuous endomorphisms of the vector space of complex sequences $\mathbf{C}^{\mathbf{N}}$ endowed with the Frechet product topology. For a sequence $A = (a_n)_{n\geq 0}$, the transformed sequence $B = MA$ is given by $B = (b_n)_{n\geq 0}$ with

$$b_n = \sum_{k\geq 0} M(n,k) a_k \tag{7}$$

The combinatorial coefficients S_w defined above are indeed row-finite matrices.

3.1. The group of regular triangular transformations

3.2. Substitutions with prefunctions

4. Unipotent transformations

4.1. Definition and Lie group structure

4.2. Examples

4.2.1. Provided by the exponential formula

4.2.2. Provided by linearization of powers of boson strings

4.3. A criterium

5. One-parameter subgroups of $UT_\infty(\mathbf{C})$

5.1. General form of the exponential functions

5.2. Link with local Lie groups : Straightening vector fields on the line

5.3. Another conjugacy trick and a shocking formula

5.4. Characteristic series $\leftrightarrow$ one parameter group correspondence

6. Conclusion and remaining problems

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