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Title (to be defined)

Résumé

In this paper, we study, by means of a statistical version of the exponential formula, various sequences transforms associated to partitions with constraints and idempotent functions statistics. The link with Bender's diagrams with arbitrary constraints is given and we completely solve the problem of monomial separability in free zero-dimensional QFT.

1 Sequence transformations

1.1 Framework

1.1.1 General setting

The main tool is here to consider the sequences as "measuring" polynomials (i.e. the space of formal power series is dual to polynomials [4]). The linear fonctional defined by $S = \sum_{k=0}^{\infty} a_k z^k$ acts on a polynomial $P = \sum_{k=0}^{n} c_k z^k$ by (we set $c_k = 0$ for $k > n$)

$$\langle S|P\rangle = \sum_{k\geq 0} a_k c_k \tag{1}$$

It is now possible to seek when some power series have a meaning for classical analysis whenever they do not represent a Taylor expansion.
Let's give an example.
It is possible to write the OGF of the Bell numbers

$$B(z) = \sum_{n} B_n z^n \tag{2}$$

and compute formally (give some identities, prove obstructions) [6], but one can also address the problem of the meaning of $B(z)$ as a functional even if it's convergence radius is 0.
The answer is that $B(z)$ *as a fonctional over the polynomials* coincide with (and therefore is) a distribution. The expression is given by Dobiǹski's formula

$$B_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

so, the functional ϕ such that $\phi(z^n) = B_n$ is $\frac{1}{e}\sum_{k=0}^{\infty} \frac{\epsilon_k}{k!}$ (where ϵ_k denote the Dirac measure with support $\{k\}$). And we have, as distributions [9]

$$B(z) \equiv \frac{1}{e} \sum_{k=0}^{\infty} \frac{\epsilon_k}{k!}. \tag{3}$$

More general expressions can be found in [1].

Now it is natural to consider sequences of such functionals. A functional as above can be written as a raw $(a_0,a_1,\cdots)$ and then, sequences of such functionals can be seen as infinite matrices.
We therefore will be concerned with "kernel-type" transformations of sequences such that, to every integer sequence $(a_k)_{k\geq 0}$, the transform is $(b_n)_{n\geq 0}$ such that

$$b_n = \sum_{k=0}^{\infty} K(n,k) a_k \tag{4}$$

where $K(n,k)$ is a complex-valued double sequence and which are able to transform *any* integer sequence. The following statement gives us the framework[1].

Proposition 1.1 *Let $K(n,k)_{n,k\geq 0}$ be a complex-valued double sequence. In order that, for every integer sequence $(a_k)_{k\geq 0}$, the series*

$$\sum_{k=0}^{\infty} K(n,k)a_k$$

be convergent, it is necessary and sufficient that every row $(K(n,k))_{k\geq 0}$ be with finite support.

Proof — The sufficiency is straightforward.
Then write $K = K_1 + iK_2$ (real and imaginary parts of the kernel). We know that $\sum_{k=0}^{\infty} K(n,k)a_k$ converges iff $\sum_{k=0}^{\infty} K_1(n,k)a_k$ and $\sum_{k=0}^{\infty} K_2(n,k)a_k$ do (since the "test" sequences are real). So the criterium can be proven for K real. Now suppose that there is a row (with index n_0) such that the support of the sequence $(K(n_0,k))_{k\geq 0}$ be infinite. Let P (resp. M) be the set of indices k such that $K(n_0,k) > 0$ (resp. $K(n_0,k) < 0$). One of these subsets (at least) must be infinite. Suppose it is P (the reasonning is similar in case it is M) and define

$$a_k = \begin{cases} 0 & \text{if } k \notin P \\ \lceil \frac{1}{K(n_0,k)} \rceil & \text{if } k \in P \end{cases}$$

a simple computation then gives $\sum_{k=0}^{\infty} K(n_0,k)a_k = +\infty$. □

In the sequel, we will be specially interested by sequence transformations induced by substitutions.
Let us give a first example of such a transformation, induced by the subtitution $z \to \frac{z}{1-z}$. If we attach to $(a_k)_{k\geq 0}$ the OGF $f(z) = \sum_{k\geq 0} a_k z^k$, then $f(\frac{z}{1-z}) = \sum_{n\geq 0} b_n z^n$ with $b_n = \sum_{k\geq 0} K(n,k)a_k$ for some kernel K. An easy computation gives

1. $K(0,0) = 1$, $K(0,k) = K(n,0) = 0$, for $k,n \neq 0$
2. for $0 < k \leq n$, $K(n,k) = (-1)^{n-k} \frac{(n-1)!}{(n-k)!(k-1)!} = (-1)^{n-k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$
3. if $k > n$, $K(n,k) = 0$.

Note 1.2 *i) Remark that (3) above means that the matrix $K(n,k)$ is lower triangular (this phenomenon will be seen as caracteristic of substitutions of type $z \to \phi(z) = \alpha z + \sum \text{higher terms}$ - see section (1.9) for more details).*
ii) In case $\alpha = 1$, $\phi(z) = z + \sum \text{higher terms})$ the corresponding matrix is unipotent, so we will all these substitutions unipotent (see (2.4)).

1.2 A statistical (categorical) version of the exponential formula

Throughout the paper, we will be interested to compute various examples of EGF for combinatorial objects having (a finite set of) nodes (their set-theoretical support) so we use as central concept the mapping σ which associates to every structure, its set of nodes.
We need to draw what could be called "square-free decomposable objects" (SFD). This version is suited to our needs for the "exponential formula". It is sufficiently general to contain, as a

1. An abundant litterature is devoted to such transformations, the interested reader is refered to [10].

particular case, the case of multivariate series. For other points of view, see [5, 7, 8]
Let $\mathcal{C}$ be a class of (combinatorial) objects and $\mathbf{FSt}$ the catgory of finite sets, $\mathcal{C}$ will be called (SFD) if it fulfills the two following conditions.

(DS) *Direct sum.* — There is a (partial) binary law $\oplus$ on $\mathcal{C}$, defined for couples of objects (ω_1,ω_2) such that $\sigma(\omega_1)\cap\sigma(\omega_2)=\emptyset$, which is associative, commutative and such that

$$\mathcal{C}_{F_1}\times\mathcal{C}_{F_2}\stackrel{\oplus}{\to}\mathcal{C}_{F_1\cup F_2} \tag{5}$$

is into.
Moreover, $\mathcal{C}_\emptyset$ consists in a single element $\{\epsilon\}$ which is neutral in the sense that, identically

$$\epsilon\oplus\omega=\omega\oplus\epsilon=\omega \tag{6}$$

(LP) *Levi's property.* — Let $\omega=\omega_1\oplus\omega_2=\omega^1\oplus\omega^2$ be two decompositions. Then it can be found a four terms decomposition $\omega=\oplus_{i,j=1,2}\omega_j^i$ refining the original data in the sense that the maginal sums give the factors of the decompositions i.e.

$$\omega_j=\omega_j^1\oplus\omega_j^2 \text{ and } \omega^i=\omega_1^i\oplus\omega_2^i;\ i,j=1,2 \tag{7}$$

Note 1.3 *Condition (5) implies that* $\sigma(\omega_1\oplus\omega_2)=\sigma(\omega_1)\sqcup\sigma(\omega_2)$.

Now, an *atom* is any object $\omega\neq\epsilon$ which cannot be split, formally

$$\omega=\omega_1\oplus\omega_2\Longrightarrow\epsilon\in\{\omega_1,\omega_2\} \tag{8}$$

Example 1.4 *As example of this setting we have:*

1. *the positive square-free integers $\sigma(n)$ being the set of primes which divide n, the atoms being the primes.*
2. *the positive integers $\sigma(n)$ being the set of primes which divide n, the atoms being the primes.*
3. *all graphs, hypergraphs and weighted graphs, $\sigma(G)$ being the set of nodes and $\oplus$ the juxtaposition, here the atoms are connected graphs.*
4. *the class of endofunctions f with $\sigma(f)=dom(f)$*
5. *the (multivariate) polynomials in $\mathbf{N}[X]$ with $\sigma=Alph$ and $\oplus=+$.*
6. *the square-free monic (for a given order on the variables) polynomials ; $\sigma(P)$ being the set of irreducible monic divisors of P and $\oplus$ being the multiplication.*
7. *complex algebraic curves ; $\sigma(V)$ being the set of monic irreducible bivariate polynomials vanishing on an infinite subset of V.*

The prescriptions (DS,LP) imply that decomposition of objects into atoms always exists and is unique.
Proposition 1.5 *Let $\omega\in\mathcal{C}$ then $\omega=\omega_1\oplus\omega_2\oplus\cdots\oplus\omega_l$ the ω_i being (dinstinct) atoms and the set $\{\omega_1,\omega_2\cdots\omega_l\}$ depends only on ω.*
In the class $\mathcal{C}$, objects are conceived to be "measured" by different parameters (data in statistical language). So, to get a general purpose tool, we suppose that the statistics takes it's values iin a ring K which contains $\mathbf{Q}$ (as, to write EGFs it is convenient to have no trouble with the fractions $\frac{1}{n!}$). Let then $c:\mathcal{C}\to K$ be the given statistic. In order to write generating series, we need

1. that the sum $c_F=\sum_{\omega\in\mathcal{C}_F}c(w)$ exists

2. that $F \to c_F$ should depend only of the cardinality of F.
3. $c(\omega_1 \oplus \omega_2) = c(\omega_1).c(\omega_2)$

We formalize it in

(LF) *Local finiteness.* — For each finite set F, the subclass

$$\mathcal{C}_F = \{\omega \in \mathcal{C} | \sigma(\omega) = F\} \tag{9}$$

is a finite set.
(Eq) *Equivariance.* —

$$|F_1| = |F_2| \Longrightarrow c_{F_1} = c_{F_2} \tag{10}$$

(Mu) *Multiplicativity.* —

$$c(\omega_1 \oplus \omega_2) = c(\omega_1).c(\omega_2) \tag{11}$$

Note 1.6 *a) In fact, (LF) is a property of the class $\mathcal{C}$, while (Eq) is a property of the statistics. In practice, we choose $\mathcal{C}$ which is locally finite and choose equivariant statistics for instance*

$$c(\omega) = x^{(\text{number of cycles})} y^{(\text{number of fixed points})}$$

for some variables x,y.

b) More generally, it is typical to take integer-valued partial (additive) statistics $c_1, \cdots c_i, \cdots, c_r$ (for every $\omega \in \mathcal{C}$, $c_i(\omega) \in \mathbf{N}$) and set $c(\omega) = x_1^{c_1(\omega)} x_2^{c_2(\omega)} \cdots x_r^{c_r(\omega)}$.

c) The class of examples 1.4.2 is not locally finite, but other examples satisfy (LF): 1.4.3 if one asks that the number of arrows and weight is finite, 1.4.1 and 1.4.5 to 1.4.7 in any case.

Now, we are in position to state the exponential formula as it will be used throughout the paper.

Proposition 1.7 *Let $\mathcal{C}$ be a locally finite (SFD) and $c : \mathcal{C} \to K$ an equivariant statistics on $\mathcal{C}$. For every subclass $\mathcal{F}$ one sets the following exponential generating series*

$$E(\mathcal{F}) = \sum_{n=0}^{\infty} c(\mathcal{F}_{[1..n]}) \frac{z^n}{n!} \tag{12}$$

Let $\mathcal{C}^a$ be the set of atoms of $\mathcal{C}$. Then, one has

$$E(\mathcal{C}) = e^{E(\mathcal{C}^a)} \tag{13}$$

Proof — (First Step). — We consider the subclasses of objects the atoms of which have a support of cardinality n i.e.

$$\mathcal{C}[n] = \{\omega \in \mathcal{C} | \omega = \omega_1 \oplus \omega_2 \oplus \cdots \oplus \omega_s = \omega \text{ with } \omega_i \in \mathcal{C}^a,\ i = 1..s \text{ then } |\sigma(\omega)| = n, i = 1..s\} \tag{14}$$

These subclasses are closed under compositions (i.e. under $\oplus$) and decompositions and their atoms $\mathcal{C}^a[n] = \{\omega \in \mathcal{C}[n] \cap \mathcal{C}^a\}$. Now, one has, thanks to the partitions of $[1..n]$

$$\mathcal{C}_{[1..n]} = \bigsqcup_{\substack{k \geq 0,\ 0 < n_1 < n_2 < \cdots < n_k \\ n_1 + n_2 + \cdots n_k = n}} \bigsqcup_{\substack{|P_j| = n_j \\ P_1 \sqcup P_2 \sqcup \cdots P_k = [1..n]}} \mathcal{C}_{P_1} \oplus \mathcal{C}_{P_2} \oplus \cdots \oplus \mathcal{C}_{P_k} \tag{15}$$

$$c(\mathcal{C}_{[1..n]}) = \sum_{k \geq 0} \sum_{\substack{0 < n_1 < n_2 < \cdots < n_k \\ n_1 + n_2 + \cdots n_k = n}} \sum_{\substack{|P_j| = n_j \\ P_1 \sqcup P_2 \sqcup \cdots P_k = [1..n]}} c(\mathcal{C}_{P_1}) c(\mathcal{C}_{P_2}) \cdots c(\mathcal{C}_{P_k}) \tag{16}$$

as, for disjoint sets, it is easy to check that $c(\mathcal{C}_X \oplus \mathcal{C}_Y) = c(\mathcal{C}_X)c(\mathcal{C}_Y)$. Now, due to the equivariance of c and to the fact that partitions $(P_1,P_2,\cdots,P_k)$ such that $P_j = n_j$ and $P_1 \sqcup P_2 \sqcup \cdots P_k = [1..n]$ are in number

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

we get

$$c(\mathcal{C}_{[1..n]}) = \sum_{k\geq 0} \sum_{\substack{0<n_1<n_2<\cdots<n_k \\ n_1+n_2+\cdots n_k=n}} \frac{n!}{n_1!n_2!\cdots n_k!} c(\mathcal{C}_{[1..n_1]})c(\mathcal{C}_{[1..n_2]})\cdots c(\mathcal{C}_{[1..n_k]}) \tag{17}$$

and then

$$E(\mathcal{C}) = \prod_{n>0} E(\mathcal{C}[n]) \tag{18}$$

We then compute the factors.

$$E(\mathcal{C}[n]) = \sum_{k\geq 0} c(\mathcal{C}[n]_{[1..nk]})\frac{z^{nk}}{(nk)!} \tag{19}$$

but

$$E(\mathcal{C}^a[n]) = c(\mathcal{C}^a_{[1..n]})\frac{z^n}{n!} \tag{20}$$

(one monomial) and

$$e^{E(\mathcal{C}^a[n])} = \sum_{k\geq 0} c(\mathcal{C}^a_{[1..n]})^k \frac{z^{nk}}{(n!)^k k!} = \sum_{k\geq 0} c(\mathcal{C}^a_{[1..n]})^k \frac{z^{nk}}{(nk!)}\frac{(nk)!}{(n!)^k k!} = \sum_{k\geq 0} c(\mathcal{C}[n]_{[1..nk]})\frac{z^{nk}}{(nk)!} = E(\mathcal{C}[n]) \tag{21}$$

due to the fact that the number of (unordered) partitions of $[1..nk]$ into k blocs of cardinality n is $\frac{(nk)!}{(n!)^k k!}$. To end the proof, it suffices to remark that $\mathcal{C}^a = \sqcap_{n>0}\mathcal{C}^a[n]$ and then

$$E(\mathcal{C}) = \prod_{n>0} E(\mathcal{C}[n]) = \prod_{n>0} e^{E(\mathcal{C}^a[n])} = e^{\sum_{n>0} E(\mathcal{C}^a[n])} = e^{E(\mathcal{C}^a)} \tag{22}$$

□

Note 1.8 *The proof suggests us that it is fruitful to factor a class $\mathcal{C}$ into (full) subclasses i.e. that are generated by a partition of the atoms (see below section (2.6)).*

1.3 Transformations induced by substitutions

We consider substitutions of the form $f(z) \to f(\phi(z))$ where

$$\phi(z) = c_1 z + \sum_{n\geq 2} c_k z^k. \tag{23}$$

Then, setting $f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ and $f(g(z)) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!}$ one can see that the transformation $(a_k)_k \to (b_n)_n$ is of the type (4).
Then, one can address the question: when a transformation of type (4)?

First, remark that in the case of substitutions of type (23), the matrix of coefficients $K(l,c)$ is lower triangular.

One considers conversely a transformation defined by $(K(l,c))_{l,c\geq 0}$ which is lower triangular. The transformed sequence is

$$b_l = \sum_c K(l,c)a_c \tag{24}$$

Now, if the transformation is the substitution by a function ϕ (as in (23)), the transform of $\frac{z^c}{c!}$ reads

$$\frac{(\phi(z))^c}{c!} = \sum_{l=0}^{\infty} K(l,c)\frac{z^l}{l!} \tag{25}$$

giving the answer.

Proposition 1.9 *Let $K(l,c)$ be a lower-triangular kernel.*
i) The following conditions are equivalent

1. *the transformation defined by K on exponential sequences is induced by a substitution by a function ϕ without constant term ($\phi(z) = c_1 z + \sum_{k\geq 2} c_k z^k$))*
2. $K(0,0) = 1$, $K(0,k) = K(n,0) = 0$, for $k,n \neq 0$ *and if one sets* $\phi(z) = \sum_l K(l,1)\frac{z^l}{l!}$, *one has, for every other c (i.e. $c \geq 2$)*

$$\frac{(\phi(z))^c}{c!} = \sum_{l=0}^{\infty} K(l,c)\frac{z^l}{l!} \tag{26}$$

3. $\sum_{n,k\geq 0} K(n,k)x^k \frac{z^n}{n!} = e^{x\phi(z)}$

ii) In the preceding conditions, $K(n,n) = c_1^n$. This implies that the matrix K is unitriangular (unipotent, i.e. with every diagonal coefficient equal to 1) iff $c_1 = 1$.

2 Applications

2.1 Generalized Stirling Numbers

In their papers, [1, 2, 3], P. Błasiak, K. Penson, A. I. Solomon, defined generalized Stirling numbers by

$$(u^r d^s)^n = u^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)u^k d^k \tag{27}$$

Note 2.1 *These numbers have a clear interpretation in terms of coefficients of rook polynomials of certain Ferrers boards (see (**??**)).*

From one of their formula ([3] *form. 16*), one can prove that the double sequence $S_{r,1}(n,k)$ defines a substitution (for $s > 1$, the $S_{r,s}(n,k)$ do not define a lower triangular matrix). This formula is the following

$$\prod_{j=1}^{n}(x + (j-1)(r-s))^{\underline{s}} = \sum_{k=s}^{ns} S_{rs}(n,k)x^{\underline{k}} \tag{28}$$

where $X^{\underline{m}} = X(X-1)\cdots(X-m+1)$ is the falling factorial.
For $s = 1$, $r > 1$ one has

$$\sum_{k=1}^{n} S_{r1}(n,k)x^{\underline{k}} = \prod_{j=1}^{n}(x + (j-1)(r-1)) = (r-1)^n \prod_{j=1}^{n}\left(\frac{x}{r-1} + j - 1\right) =$$

$$(r-1)^n\left(\frac{x}{r-1}\right)^{\overline{n}} = (r-1)^n\sum_{l=1}^{n}|St1(n,l)|\left(\frac{x^l}{(r-1)^l}\right) =$$
$$(r-1)^n\sum_{l=1}^{n}|St1(n,l)|\frac{1}{(r-1)^l}\sum_{k=1}^{l}St2(l,k)x^{\underline{k}} \qquad (29)$$

with $X^{\underline{m}} = X(X+1)\cdots(X+m-1)$ the ascending factorial.
Thus, since $(x^{\underline{k}})_{k\geq 0}$ is a basis of the space of univariate polynomials, one has

$$S_{r1}(n,k) = (r-1)^n\sum_{l=1}^{n}|St1(n,l)|\frac{1}{(r-1)^l}\sum_{k=1}^{l}St2(l,k) \qquad (30)$$

where Sti are the Stirling numbers of the $ieth$ kind.
In view of proposition (1.9) let us compute the double series $\sum_{n,k\geq 0} S_{r1}(n,k)\frac{x^n}{n!}y^k$.

$$\sum_{n,k\geq 0} S_{r1}(n,k)\frac{x^n}{n!}y^k = 1 + \sum_{n,k\geq 1} S_{r1}(n,k)\frac{x^n}{n!}y^k =$$
$$1 + \sum_{n,k\geq 1}(r-1)^n\sum_{l=1}^{n}|St1(n,l)|\frac{1}{(r-1)^l}\sum_{k=1}^{l}St2(l,k)\frac{x^n}{n!}y^k \qquad (31)$$

which is, due to the boundary properties of the Stirling numbers,

$$\sum_{n,l,k\geq 0}(r-1)^n|St1(n,l)|\frac{1}{(r-1)^l}St2(l,k)\frac{x^n}{n!}y^k =$$
$$\sum_{n,l,k\geq 0}\frac{1}{(r-1)^l}St2(l,k)y^k(r-1)^n|St1(n,l)|\frac{x^n}{n!} =$$
$$\sum_{l,k\geq 0}\frac{1}{(r-1)^l}St2(l,k)y^k\sum_{n\geq 0}|St1(n,l)|\frac{(r-1)^n x^n}{n!} =$$
$$\sum_{l,k\geq 0}\frac{1}{(r-1)^l}St2(l,k)y^k\frac{(-log(1-(r-1)x))^l}{l!} = \sum_{l,k\geq 0}St2(l,k)y^k\frac{\left(\frac{-log(1-(r-1)x)}{r-1}\right)^l}{l!} =$$
$$e^{\left(y(e^{\frac{-log(1-(r-1)x)}{r-1}}-1)\right)} = e^{\left(y(\frac{1}{1-(r-1)x})^{\frac{1}{r-1}}-1)\right)} \qquad (32)$$

which proves that the matrix $S_{r1}(n,k)$ defines the substitution $\{z \to (-1 + (\frac{1}{1-(r-1)z})^{\frac{1}{r-1}})\}$ which can be checked to be unipotent. For $r = 2$, we have the substitution
$\{z \to (-1 + \frac{1}{1-z}) = \frac{z}{1-z}\}$. This will be checked in (**??**) by other means.

2.2 Idempotent transformations and the like

2.2.1 Idempotent transformations

Let $I(n,k)$ be the number of idempotent endofunctions (i.e. such that $f^2 = f$) on a (fixed) set with n elements and k fixed points. In the setting (1.2), the class $\mathcal{C}$ is the class of idempotent endofunctions, the composition is just the juxtaposition of endofunctions (that we will denote by $\mathcal{E}$ in the next section) defined on disjoint domains, the atoms being the (idempotent) endofunctions having only one fixed point and the statistic is $c(f) = x^{(\text{number of fixed points})}$. It is clear then that $c(f \oplus g) = c(f)c(g)$. One has, by definition

$$E(\mathcal{C}) = \sum_{n,k\geq 0} I(n,k)x^k\frac{z^n}{n!} = e^{E(\mathcal{C}^a)} = e^{\sum_{n\geq 0}\frac{nxz^n}{n!}} = e^{xze^z} \qquad (33)$$

using the criterium (1.9), one gets that $I(n,k)$ defines the substitution $z \to ze^z$.

2.3 Higher order idempotent transformations

The idempotent endofunctions are exactly the endofunctions satisfying $f^2 = f$.
Can we pursue the investigation of such equations?
The answer is positive, but we have to analyse carefully arithmetic phenomena that are not apparent in the case of idempotent endofunctions.
Call $\mathcal{E}_m$ the class of endofunctions satisfying the equation $f^m = f^{m+1}$. It is a full subclass of $\mathcal{E}$. We first remark that, if $k|m$, one has $\mathcal{E}_k \subset \mathcal{E}_m$. So, we will have a finer classification if we consider the *strict* classes.
We denote $\mathcal{ES}_m$ the class of endofunctions that have only m-cycles and satisfy $f^m = f^{m+1}$. Then it is not difficult to check that

1. the $\mathcal{ES}_m$ are full subclasses of $\mathcal{E}$
2. $(\mathcal{E}_m)^a = \sqcup_{k|m}(\mathcal{ES}_m)^a$
3. $((\mathcal{ES}_m)^a)$ consists in the endofunctions having only a m-cycle (and satisfying $f^m = f^{m+1}$, which is equivalent to say that the image of every point is cyclic)
4. $E((\mathcal{ES}_m)^a) = \sum_{n\geq m} \begin{pmatrix} n \\ m \end{pmatrix} (m-1)!m^{n-m}\frac{z^n}{n!} = \frac{z^m}{m}e^{mz}$

we deduce from (2) and (4) above that

$$E(\mathcal{E}_m) = e^{\sum_{\substack{k|m \\ k\geq 1}} \frac{z^k}{k}e^{kz}} \tag{34}$$

and if one counts the number of cycles so that $I_m(n,c)$ is the number of endofunctions in $[1..n]$ satisfying $f^m = f^{m+1}$ and having c cycles, one has the double statistics

$$\sum_{n,c\geq 0} I_m(n,c)(x^c)\frac{z^n}{n!} = e^{x(\sum_{\substack{k|m \\ k\geq 1}} \frac{z^k}{k}e^{kz})} \tag{35}$$

which implies that $I_m(n,c)$ is the matrix associated to the substitution $z \to \sum_{\substack{k|m \\ k\geq 1}} \frac{z^k}{k}e^{kz}$.

Let $\mathcal{E}_\infty$ be the class of all endofunctions satisfying some equation $f^m = f^{m+1}$ (Burnside condition with $k = 1$,see below), and denote $I(n,c)$ the number of such endofunctions on $[1..n]$ with c cycles. We have (directly or by limiting process)

$$\sum_{n,c\geq 0} I(n,c)(x^c)\frac{z^n}{n!} = e^{x(\sum_{k\geq 1} \frac{z^k}{k}e^{kz})} = e^{xlog(\frac{1}{1-ze^z})} \tag{36}$$

so, again, $I(n,c)$ is the matrix associated to the substitution $z \to log(\frac{1}{1-ze^z})$. In particular

$$E(\mathcal{E}_\infty) = e^{log(\frac{1}{1-ze^z})} = \frac{1}{1-ze^z} \tag{37}$$

this rather simple equation, which seems to belong to the realm of OGFs and is, for us, a mystery.

2.4 The Lie group of unipotent substitutions

The substitutions $z \to \phi(z) = z + \sum \text{higher terms}$ form a group under composition. in the next section, we see that, in a certain sense, this group is an (infinite dimensional) Lie group. In particular it is possible to compute one parameter subgroups of substitutions.

The aim of his section is then to prove that for any subsitution $z \to \phi(z) = z + \sum$ higher terms it is possible to compute a continuous family $\phi_t;\ t \in \mathbf{R}$ such that

$$\phi_0(z) = z;\ \phi_1 = \phi;\ \phi_r(\phi_s(z)) = \phi_{r+s}(z) \tag{38}$$

Example 2.2 *For example, with $\phi(z) = \frac{z}{1-z}$, one has $\phi_t(z) = \frac{z}{1-tz}$.*

Hence the analytic substitutions form a Lie group $\mathbf{Subs}$.

Example 2.3 *We consider the generalized Stirling numbers of [3] $S_{21}(n,k)$. One has*

$$\sum_{n,k\geq 0} S_{21}(n,k)\frac{x^n}{n!}y^k = e^{y\frac{x}{1-x}} \tag{39}$$

Before claiming that $\mathbf{Subs}$ is a Lie group, we need to put some topology on it and study an algebra in which it is embedded.

The (topological) algebra $\mathcal{T}$. —
Let $\mathcal{T}$ be the ($\mathbf{C}$-) algebra of infinite $\mathbf{N} \times \mathbf{N}$ lower triangular matrices with complex coefficients

$$\mathcal{T} = \{K \in \mathbf{C}^{\mathbf{N}\times\mathbf{N}} |\ c > l \Longrightarrow K(l,c) = 0\} \tag{40}$$

the following properties are not difficult to check

1. $\mathcal{T}$ is a $\mathbf{C}$-algebra of infinite matrices
2. let $\tau_n :\ \mathcal{T} \to \mathcal{T}_n$ be the *truncation* mapping such that $\tau_n(K)$ is the $[0..n]\times[0..n]$ submatrix of K (we denote $\mathcal{T}_n$ the algebra of $[0..n] \times [0..n]$ triangular matrices).
3. For $n \leq m$ let $\tau_{nm} :\ \mathcal{T}_m \to \mathcal{T}_n$ be the truncation mapping. One has $\tau_{nm}\tau_m = \tau_n$

We put on $\mathcal{T}$ the topology of local convergence i.e. determined by the morphisms τ_n (more precisely, it is the coarsest topology for which the τ_n are continuous, the algebras $\mathcal{T}_n$ being endowed with the standard topology namely a filter, a sequence converges iff all it's tuncations do.
In view of proposition (1.9), to be the matrix of a substitution is a local condition as the equality $\sum_{n,k\geq 0} K(n,k)x^k\frac{z^n}{n!} = e^{x\phi(z)}$ is equivalent to the fact that, for all k

$$\sum_{n\geq 0} K(n,k)\frac{z^n}{n!} = \frac{1}{k!}(\sum_{n\geq 0} K(n,1)\frac{z^n}{n!})^k \tag{41}$$

which, in turn expresses that, for all $k,N \geq 0$

$$\sum_{0\leq n\leq N} K(n,k)\frac{z^n}{n!} = \frac{1}{k!}\left((\sum_{0\leq n\leq N} K(n,1)\frac{z^n}{n!})^k\right)_{\leq N} \tag{42}$$

where $(T(z))_{\leq N}$ denotes the truncation of the series after at order N.

2.5 (Internal) More on Burnside matrices

For $m,k \geq 1$ and $n,c \geq 0$, let $B_{m,k}(n,c)$ be the number of endofunctions of $[1..n]$ such that $f^m = f^{m+k}$ and having k cycles. It is not difficult to prove, as there is one and only one cycle in each connected component, that

$$\sum_{n,c\geq 0} B_{m,k}(n,c)(x^c)\frac{z^n}{n!} = e^{x(\sum_n B_{m,k}(n,1)\frac{z^n}{n!})} \tag{43}$$

but the computation of $B_{m,k}(n,1)$ seems difficult.

2.6 Multisections and full subclasses

Bi- and multisection is the art of selecting terms in a given power series. The first procedures are:

1. take one term over 2 (i.e. the even and odd part of a given function)
2. generalization: the first, the second ... the m-ieth in each m terms
3. take the first m terms, withdraw the first m terms

Applying one of these precedures to the exponent of an exponential formula amounts to select some among the atoms of the class under consideration. Then the formula counts the objects having only the selected atoms in their maximal decomposition.

3 Separability in zero-dimensional QFT

3.1 Hadamard product of Taylor series

In this section, the central rôle is played by an interesting expression of the *Hadamard product* of two sequences. Recall that the *Hadamard product* [4, **?**] or *pointwise product* of two sequences $A = (a_k)_{k\geq 0}$, $B = (b_k)_{k\geq 0}$ is

$$A \odot B = (a_k b_k)_{k\geq 0} \tag{44}$$

this can, of course be transposed to the corresponding Taylor series $E_A(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!}$, $E_B(z) = \sum_{k\geq 0} b_k \frac{z^k}{k!}$ by

$$E_{A\odot B}(z) = \sum_{k\geq 0} a_k b_k \frac{z^k}{k!} \tag{45}$$

Many interesting (and non-trivial) features are known about Hadamard product such as the fact that the Hadamard product of the Taylor expansions of two rational functions is of the same type (the computation of the coefficients of the result is however non-trivial).
A formula, classical in QFT, says that the Hadamard product of $A(z)$, $B(z)$ can be obtained by

$$E_{A\odot B}(y) = E_A(y\frac{d}{dz})E_B(z)|_{z=0} \tag{46}$$

[motivation, explanation, interest]. ... as I could understand the following formula is of utmost importance in QFT ??

$$e^{\sum_{m\geq 1} L_m \frac{(y\frac{d}{dz})^m}{m!}} e^{\sum_{m\geq 1} V_m \frac{z^m}{m!}}|_{z=0} = \sum_{n\geq 0} F_n((L_m)_{m\geq 1},(V_m)_{m\geq 1})\frac{y^n}{n!} \tag{47}$$

it is not difficult to prove that the functions F_n are, in fact, polynomials in the double set of variables $(L_m)_{m\geq 1}$,$(V_m)_{m\geq 1}$. So, the separation problem by these quantity (potential, field) as they vary reads

What are the coefficients and the monomials of the polynomials F_n?

3.2 Separation theorem

As we know that $\sum_{n\geq 0} F_n((L_m)_{m\geq 1},(V_m)_{m\geq 1})\frac{z^n}{n!}$ is the Hadamard product of $e^{\sum_{m\geq 1} L_m \frac{z^m}{m!}}$ and $e^{\sum_{m\geq 1} V_m \frac{z^m}{m!}}$ we must begin in computing the coefficients of the expansion

$$e^{\sum_{m\geq 1} L_m \frac{z^m}{m!}} = \sum_{n\geq 0} P_n((L_m)_{m\geq 1})\frac{z^n}{n!} \tag{48}$$

We have

$$e^{\sum_{m\geq 1} L_m \frac{z^m}{m!}} = \sum_{k\geq 0} \frac{(\sum_{m\geq 1} L_m \frac{z^m}{m!})^k}{k!} \tag{49}$$

with the following conventions

1. $\mathbf{N}^{(\mathbf{N}^*)}$ is the set of (integer) sequences $(\alpha(1),\alpha(2),\cdots\alpha(j)\cdots)$ with finite support
2. for $\alpha \in \mathbf{N}^{(\mathbf{N}^*)}$, $|\alpha(i)| = \sum_{i\geq 1}\alpha(i)$ and $||\alpha|| = \sum_{i\geq 1} i\alpha(i)$
3. for $\alpha \in \mathbf{N}^{(\mathbf{N}^*)}$, $\alpha! = \prod_{i\geq 1}\alpha(i)!$ and $\mathbf{L}^\alpha = \prod_{i\geq 1} L_i^{\alpha(i)}$

one has

$$\frac{1}{k!}(\sum_{m\geq 1} L_m \frac{z^m}{m!})^k = \frac{1}{k!} \sum_{\substack{\alpha\in\mathbf{N}^{(\mathbf{N}^*)}\\ |\alpha|=k}} \frac{k!}{\alpha!}\mathbf{L}^\alpha \frac{z^{||\alpha||}}{\prod_{i\geq 1}(i!)^{\alpha(i)}} \tag{50}$$

from what we derive

$$P_n((L_m)_{m\geq 1}) = \sum_{\substack{\alpha\in\mathbf{N}^{(\mathbf{N}^*)}\\ ||\alpha||=n}} \frac{n!}{\alpha!\prod_{i\geq 1}(i!)^{\alpha(i)}}\mathbf{L}^\alpha \tag{51}$$

but now the coefficient of $\mathbf{L}^\alpha$ has a clear combinatorial interpretation. It is the number of (unordered) partitions of $[1..n]$ into (nonempty) subsets: $\alpha(1)$ with one element (singletons), $\alpha(2)$ with two elements (unordered pairs), $\alpha(3)$ with three elements,$\cdots$ $\alpha(j)$ with j elements, $\cdots$ hence the condition $||\alpha|| = n$. Such partitions will be called *of type* α and their number will be denoted by $numpart(\alpha)$. Then

$$numpart(\alpha) = \frac{n!}{\alpha!\prod_{i\geq 1}(i!)^{\alpha(i)}}. \tag{52}$$

From (47), we see that

$$F_n((L_m)_{m\geq 1},(V_m)_{m\geq 1}) = \sum_{||\alpha||=||\beta||=n} numpart(\alpha)numpart(\beta)\mathbf{L}^\alpha\mathbf{V}^\beta \tag{53}$$

We then, have proved the following separation property which solves the question of the preceding section.

Theorem 3.1 *Let* $\mathbf{L} = (L_m)_{m\geq 1}$; $\mathbf{V} = (V_m)_{m\geq 1}$) *two (disjoint) alphabets (sets of free variables), and* F_n*, the functions defined by*

$$\sum_{n\geq 0} F_n((L_m)_{m\geq 1},(V_m)_{m\geq 1})\frac{y^n}{n!} = e^{\sum_{m\geq 1} L_m \frac{(y\frac{d}{dz})^m}{m!}} e^{\sum_{m\geq 1} V_m \frac{z^m}{m!}}|_{z=0} \tag{54}$$

then

1. *the functions* F_n *are polynomials*
2. *we have, with the usual mlultiindex notations*

$$F_n(\mathbf{L},\mathbf{V}) = \sum_{||\alpha||=||\beta||=n} numpart(\alpha)numpart(\beta)\mathbf{L}^\alpha\mathbf{V}^\beta \tag{55}$$

3.3 Drawn diagrams, diagrams and their separation power

In this section, we deal with the combinatorial interpretation of the products $numpart(\alpha)numpart(\beta)$ linked with some special bipartite graphs.

Remark that $numpart(\alpha)numpart(\beta)$ is the number of couples of partitions with bitype (α,β).

First, let us review some basic knowledge about partitions and types.

Definition 3.2 *Let X be a set and $P \subset \mathfrak{P}(X) - \{\emptyset\}$.*
i) The collection P is called a partition (unordered) of X iff

1. $\cup_{Y \in P} Y = X$
2. $Y,Z \in P \Longrightarrow (Y = Z)$ or $(Y \cap Z = \emptyset)$

in another denotation $X = \sqcup_{Y \in P} Y$.
ii) If X is finite, the type of X is $\alpha \in \mathbf{N}^{\mathbf{N}^*}$

$$\alpha(i) = \#\{Y \in P | \#Y = i\} \tag{56}$$

Note 3.3 *i) The condition of definition (3.2) i)2) implies that, if the set P is empty then so is X. The converse is obvious.*
ii) The number of partitions of a set of n elements is called the Bell number of order n, $B(n)$. Below the first Bell numbers.

n	0	1	2	3	4	5	6	7	8	9
$B(n)$	1	1	2	5	15	52	203	877	4140	21147

iii) The "partition type" of a partition of $[1..n]$ is a partition in the Diophantine sense. Below the partitons of $[1..4]$ and their types (if there are several choices, the multiplicity is indicated between brackets).

types	[1111]	[112]	[2,2]	[13]	[4]
partitions	$\{\{1\},\{2\},\{3\},\{4\}\}$	$\{\{a,b\},\{c\},\{d\}\}$ *(6)*	$\{\{a,b\},\{c,d\}\}$ *(3)*	$\{\{a\},\{b,c,d\}\}$ *(4)*	$\{1,2,3,4\}$

From what has been said it is clear that, for two (Diophantine) partitions α,β of n, the number $numpart(\alpha)numpart(\beta)$ (which is the coefficient of the monomial $\mathbf{L}^\alpha\mathbf{V}\beta$ in the function F_n) is the number of (ordered) pairs P_1,P_2, the $P_i;\ i = 1,2$ being (set-theoretical) partitions of $[1..n]$.

Bi-partitions. —
These couples are representable by bipartite (multi-)graphs between the sets P_1, P_2, the node $Y_1 \in P_1$ and $Y_2 \in P_2$ being joined by as many arrows as they have elements in common. So, for every couple of partitions of the same set, we get an incidence matrix $\mathcal{M}(P_1,P_2)$ (the incidence matrix of the bipartite graph described previously. The table below shows the incidence matrices for $P_1 = \{\{1\},\{2\},\{3,4\}\}$ and P_2 of type $[2,2]$

$P_1 \backslash P_2$	$\{\{1,2\},\{3,4\}\}$	$\{\{1,3\},\{2,4\}\}$	$\{\{1,4\},\{2,3\}\}$
$\{\{1\},\{2\},\{3,4\}\}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

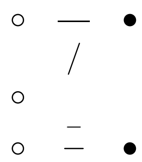
for example, the graph associated to the first matrix is

$$\begin{array}{ccc} \{1\} & — & \{1,2\} \\ & / & \\ \{2\} & & \\ \{3,4\} & = & \{3,4\} \end{array}$$

Note 3.4 *It is equivalent also to "unlabel" the vertives and label the edges from 1 to n, since every vertex contains the set of labels which are connected to it.*

First oblivion: drawn diagrams. —

We can forget some of the information in "unlabelling" the parts of the partitions involved, replacing the subsets of the left column by white spots (∘) and the others (right column) by black spots (•)



of course, the vertices remain labelled in some way because as far a a choice is made to draw the spots, we can consider that they are labelled from top to bottom. This is what we call "drawn diagrams" their data is equivalent to the (integer) matrices with no zero raw and no zero column. The information contained in such a diagram is not sufficient to recover the bi-partition (see above the second and third matrix) they come from but it is to recover the bi-type as the first type is given by the sums of the raws and the second type is given by the sums of the columns. In this sense, we can say that the coefficient $numpart(\alpha)numpart(\beta)$ "contains" all the drawn diagrams of the given bi-type and enumerate them with their multiplicities (which is the number of bi-partitions giving the drawn diagram under consideration).

Second oblivion: diagrams. —

When manipulating the diagrams, one says that two diagrams are "the same" if they differ by permutations of the white spots and black spots (between themselves). This amounts to consider the orbit of the matrix representing the diagram under permutations of the raws and th columns (between themselves). Again these items contain the information of the bi-type, so the coefficients of the generic formula (47) can be seen to be enumerations of diagrams with their multiplicities.

3.4 Restricted Bender diagrams

4 Concluding remarks

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