

Free Quasi-Symmetric Functions, Product Actions and Quantum Field Theory of Partitions

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Introduction

In a relatively recent paper, Bender, Brody and Meister introduce a special Field Theory described by

$$G(z) = \left(e^{\left(\sum_{n \geq 1} L_n \frac{z^n}{n!} \frac{\partial}{\partial x} \right)} \left(e^{\left(\sum_{m \geq 1} V_m \frac{x^m}{m!} \right)} \right) \right) \Big|_{x=0}$$

in order to prove that any sequence of numbers $\{a_n\}$ can be generated by a suitable set of rules applied to some type of Feynman diagrams . These diagrams actually are bicoloured multigraphs with no isolated vertex.

Actions of a direct product of permutation groups

Direct product actions

Two pairs (G_1, X_1) and (G_2, X_2) , each G_i is a permutation group acting on X_i .

Intransitive action of $G_1 \times G_2$ on $X_1 \sqcup X_2$:

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1 \\ \sigma_2 x & \text{if } x \in X_2 \end{cases}$$

$(G_1, X_1) \rightarrow (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$.

Cartesian action of $G_1 \times G_2$ on $X_1 \times X_2$:

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2).$$

$(G_1, X_1) \times (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$.

Explicit realization

Denote

- by \circ_N the natural action of \mathfrak{S}_n on $\{0, \dots, n-1\}$,
- by \circ_I the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \dots, n+m-1\}$
- by \circ_C the cartesian action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \dots, nm-1\}$.

More precisely,

$$(\sigma_1, \sigma_2) \circ_I i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \leq i \leq n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \leq i \leq n+m-1 \end{cases}$$

and

$$(\sigma_1, \sigma_2) \circ_C (j+nk) = (\sigma_1 \circ_N j) + n(\sigma_2 \circ_N k)$$

for $0 \leq i \leq n+m-1$, $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$.

Let the map $\rightarrow : \mathfrak{S}_n \times \mathfrak{S}_m \rightarrow \mathfrak{S}_{n+m}$ defined by

$$\sigma_1 \rightarrow \sigma_2 = \sigma_1 \sigma_2 [n]$$

$$\sigma_1 = 1320 \in \mathfrak{S}_4, \sigma_2 = 534120 \in \mathfrak{S}_6.$$

$$\sigma_1 \rightarrow \sigma_2 = 1320978564, \sigma_2 \rightarrow \sigma_1 = 5341207986$$

Proposition

$$(\sigma_1 \rightarrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_I i.$$

Let the map $\times : \mathfrak{S}_n \times \mathfrak{S}_m \rightarrow \mathfrak{S}_{nm}$ defined by

$$\sigma_1 \times \sigma_2 = \prod_{i,j} c_i \times c'_j$$

where $\sigma_1 = c_1 \cdots c_k$ and $\sigma_2 = c'_1 \cdots c'_{k'}$ are the decompositions of σ_1 and σ_2 in a product of cycles and

$$c \times c' = \prod_{s=0}^{l \wedge l' - 1} (\phi(s, 0), \phi(s+1, 1) \cdots, \phi(s+l \vee l' - 1, l \vee l' - 1)),$$

($\wedge := \text{gcd}$, $\vee := \text{lcm}$, $c = (i_0, \dots, i_{l-1})$, $c' = (j_0, \dots, j_{l'-1})$) are two cycles and $\phi(k, k') = i_k \bmod l + n j_{k'} \bmod l'$

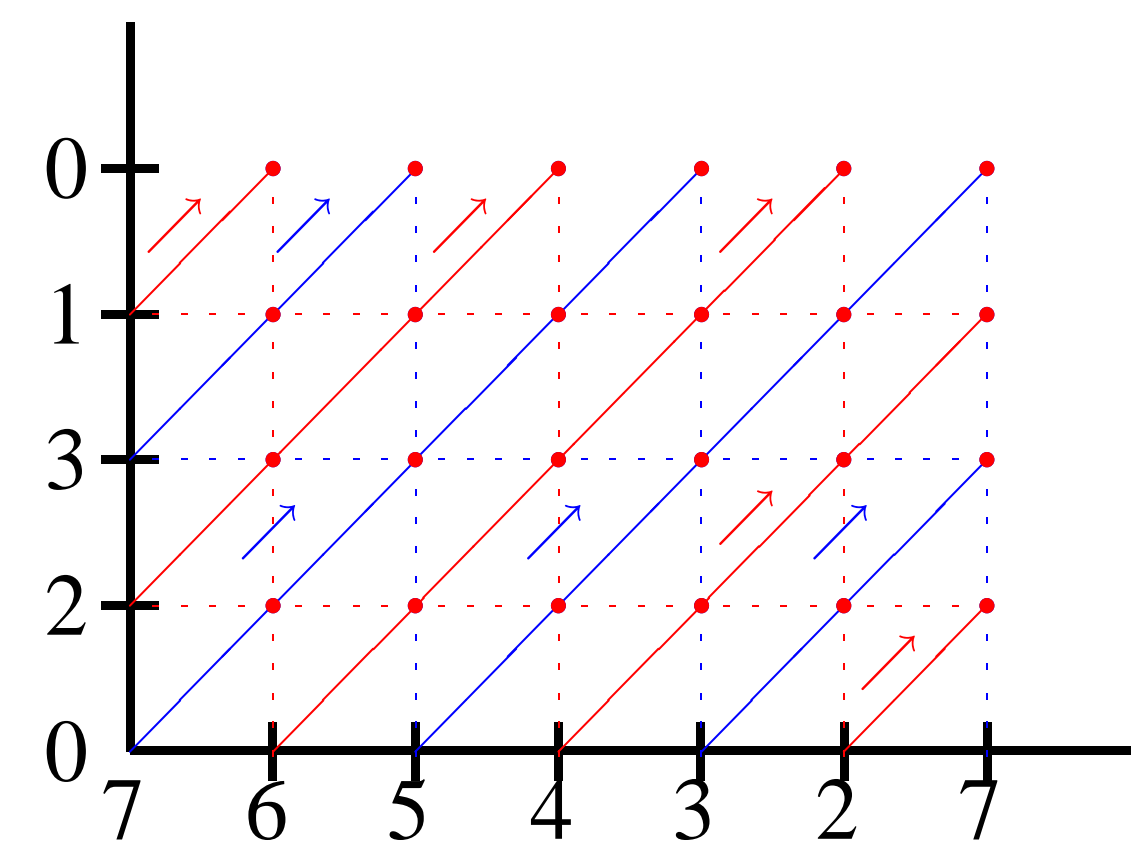
The cartesian action is compatible with the natural action.

Proposition

$$(\sigma_1 \times \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_C i.$$

$$c_1 = (0, 2, 3, 1), c_2 = (7, 6, 5, 4, 3, 2).$$

$$c_1 \times c_2 :$$



$$(28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9) \\ (30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8).$$

Algebraic structure

Proposition Associativity

Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

1. $\sigma_1 \rightarrow (\sigma_2 \rightarrow \sigma_3) = (\sigma_1 \rightarrow \sigma_2) \rightarrow \sigma_3$
2. $\sigma_1 \times (\sigma_2 \times \sigma_3) = (\sigma_1 \times \sigma_2) \times \sigma_3$

Proposition Semi-distributivity

$\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$

$$\sigma_1 \times (\sigma_2 \rightarrow \sigma_3) = (\sigma_1 \times \sigma_2) \rightarrow (\sigma_1 \times \sigma_3)$$

Cycle index algebra

Cartesian product in Sym

Let $c_j(\sigma)$ be the number of cycles of σ of length j

$$z(\sigma) = \prod_{j=0}^{\infty} \psi_j^{c_j(\sigma)} \quad (1)$$

where ψ_i is a power sum symmetric function.

One defines

$$\prod_{1 \leq i \leq \infty} \psi_i^{\alpha_i} \star \prod_{1 \leq j \leq \infty} \psi_j^{\beta_j} = \prod_{1 \leq i, j \leq \infty} \psi_{i \vee j}^{\alpha_i \beta_j (i \wedge j)}$$

Proposition i) $z : \bigoplus_{n \geq 0} \mathbb{Q}[\mathfrak{S}_n] \mapsto Sym$ is morphism sending the two laws $\rightarrow \times$ respectively to \star ;

More precisely, for $\sigma, \tau \in \sqcup_{n \geq 0} \mathfrak{S}_n$ one has

$$z(\sigma \rightarrow \tau) = z(\sigma)z(\tau); \quad z(\sigma \times \tau) = z(\sigma) \star z(\tau) \quad (2)$$

ii) \star is commutative and distributive over \times .

Proposition $P := \{\prod_{i=1}^{\infty} \psi_i^{\alpha_i}\}_{(\alpha_i)_{i \geq 1} \in \mathbb{N}^{\mathbb{N}^*}}$ is closed by \times and \star .

More precisely: (P, \times, \star) is isomorphic to a subsemiring of the semiring $\mathbb{N}[(\mathbb{N}^{\mathbb{P}}), sup]$ (\mathbb{P} := the set of prime numbers).

Cycle index polynomial

Polyà cycle index polynomial :

$$Z(G) = z \left(\frac{1}{|G|} \sum_{\sigma \in G} \sigma \right).$$

1. Symmetric group $\mathfrak{S}_n : Z(\mathfrak{S}_n) = h_n$.
2. Alternating group $A_n : Z(A_n) = h_n + e_n$.

As z is a morphism of 2-associative algebra, one recovers the classical relations

$$Z(G_1 \rightarrow G_2) = Z(G_1)Z(G_2), \quad Z(G_1 \times G_2) = Z(G_1) \star Z(G_2)$$

1. Intransitive product of \mathfrak{S}_n and \mathfrak{S}_m :

$$Z(\mathfrak{S}_n \rightarrow \mathfrak{S}_m) = h_n h_m.$$

2. Cartesian product of \mathfrak{S}_n and \mathfrak{S}_m :

$$Z(\mathfrak{S}_n \times \mathfrak{S}_m) = h_n \star h_m = \sum_{\substack{|\lambda|=n, \\ |\rho|=m}} \frac{1}{z_\lambda z_\rho} \prod_{i,j} \psi_{i \vee j}^{\lambda_i \wedge \rho_j}$$

where $z_\lambda = \prod i^{n_i} n_i!$ if n_i is the number of parts of λ equal to i .

Enumeration of a type of Feynman diagrams related to quantum field theory of partitions

The *type* $t(f)$ of $f : X \rightarrow L = \{l_0 < l_1 \cdots < l_k \cdots\}$ is the vector (i_0, \dots, i_p, \dots) where i_k is the number of elements of X whose image by f is l_k .

The *shape* $s(f)$ of f is the partition obtained by sorting in the decreasing order $t(f)$ and erasing the zeroes.

The number $d_\lambda^s(G, L)$ of G -classes on L^X with the shape λ is the coefficient of m_λ in the expansion of $Z(G)$ in the basis of monomial symmetric functions:

$$Z(G) = \sum_{\lambda} d_\lambda^s(G, L) m_\lambda.$$

Let us define the generating series of the type of our Feynman diagrams

$$F(n, m) := \sum_{I=(i_0, \dots, i_p, \dots)} f_I^t(n, m) \prod_{k=0}^{\infty} y_k^{i_k},$$

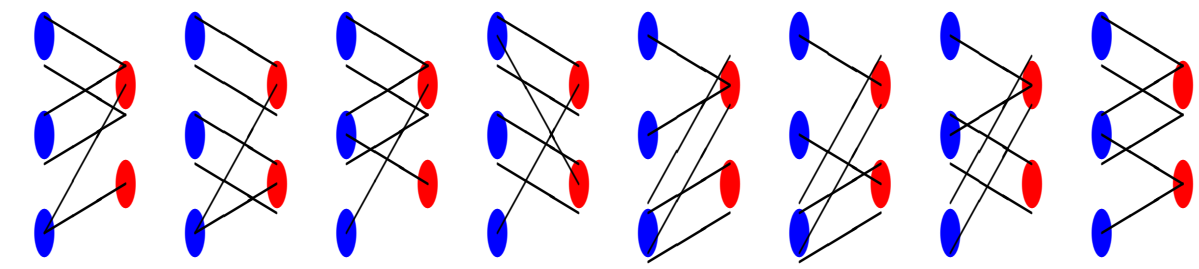
where $f_I^t(n, m)$ denotes the number of Feynman diagrams of type I .

Theorem One has the following decomposition of the cycle index polynomial.

$$Z(\mathfrak{S}_n \times \mathfrak{S}_m) = \sum_{(1,1) \leq_{lex} (k,p) \leq_{lex} (n,m)} F(k, p) y_0^{nm+kp-np-mk} + y_0^{nm}.$$

$$\begin{aligned} F(2, 3) &= Z(\mathfrak{S}_3 \times \mathfrak{S}_2) - F(1, 3)y_0^3 - F(2, 2)y_0^2 \\ &\quad - F(2, 1)y_0^4 - F(1, 2)y_0^4 - F(1, 1)y_0^5 - y_0^6 \\ &= y_0^6 + y_2^5 y_1 + 3y_2^4 y_1 + 3y_2^4 y_1 y_0 + 2y_2^4 y_0^2 \\ &\quad + 3y_2^3 y_1^3 + 6y_2^3 y_1^2 y_0 + 5y_2^3 y_1 y_0^2 + y_2^3 y_0^3 + 3y_2^2 y_1^4 \\ &\quad + 3y_2^2 y_1^3 y_0 + 8y_2^2 y_1^2 y_0^2 + 3y_2^2 y_1 y_0^3 + y_2 y_1^5 + 3y_2 y_1^4 y_0 \\ &\quad + 5y_2 y_1^3 y_0^2 + 3y_2 y_1^2 y_0^3 + y_1^6 + y_1^5 y_0 + y_1^4 y_0^2 + 2y_1^3 y_0^3. \end{aligned}$$

For example, there is 8 (2, 2, 2)- Feynman diagrams:



Non commutative realizations

Free quasi-symmetric cycle index algebra

Recall that the algebra \mathbf{FQSym} is defined by one of its bases, indexed to \mathfrak{S} and defined as follows

$$\mathbf{F}_\sigma = \sum_{Std(w)=\sigma^{-1}} w \in \mathbb{Z}\langle\langle A \rangle\rangle$$

One has

$$F^\sigma F^\tau = F^{\sigma \rightarrow \tau}.$$

This induces naturally a morphism of algebra

$$\begin{aligned} \underline{z} : (\bigoplus \mathbb{Q}[\mathfrak{S}_n], \rightarrow, +) &\rightarrow (\mathbf{FQSym}, \cdot, +) \\ \sigma &\rightarrow \mathbf{F}^\sigma. \end{aligned}$$

One defines the product \star on \mathbf{FQSym} by $F^\sigma \star F^\tau := F^\sigma \times \tau$. By this way, \underline{z} becomes a morphism of 2-associative algebra. Furthermore, \star is associative, distributive over the sum and semi-distributive over the shifted concatenation.

Free quasi-symmetric Polyà cycle index polynomial

$$\underline{Z}(G) := \underline{z} \left(\frac{1}{|G|} \sum_{\sigma \in G} \sigma \right).$$

Proposition

Let $G_1, G_2 \in \mathfrak{S}_{sg}$ be two permutation groups, one has

1. $\underline{Z}(G_1 \rightarrow G_2) = \underline{Z}(G_1)\underline{Z}(G_2)$.
2. $\underline{Z}(G_1 \times G_2) = \underline{Z}(G_1) \star \underline{Z}(G_2)$.

$$\begin{array}{ccc} \mathfrak{S}_{sg} & \xrightarrow{\underline{z}} & \mathbf{FQSym} \\ Z \downarrow z \swarrow \uparrow \underline{z} & & \\ Sym & \xleftarrow{\underline{z}} & \bigoplus \mathbb{Q}[\mathfrak{S}_n] \end{array}$$

1. Free quasi-symmetric cycle index of \mathfrak{S}_n :

$$\mathbf{H}_n := \underline{Z}(\mathfrak{S}_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} F^\sigma.$$

Analogous of $z(\mathbf{H}_n) = Z(\mathfrak{S}_n) = h_n$.

2. Free cycle index polynomial of the alternative groups:

$$\mathbf{E}_n := \underline{Z}(A_n) - \underline{Z}(\mathfrak{S}_n).$$

analogous of elementary symmetric functions :

$$z(\mathbf{E}_n) = Z(A_n) - Z(\mathfrak{S}_n) = e_n.$$