

Equivariant functions in topological quantum mechanics.

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Abstract. We deal in the paper with topological quantum mechanics. We show on the example of anyon system that the space of quantum states depends on a Hamiltonian.

1. Introduction

The aim of the paper is to show that the space of quantum states in topological quantum mechanics (TQM) depends on the Hamiltonian of the system. In quantum mechanics (QM) wave functions are complex square-integrable functions on the classical configuration space. For different Hamiltonians the systems of eigenstates are different but the Hilbert space spanned on any system does not depend on a Hamiltonian, what is a consequence of the spectral theorem. States in QM form the Hilbert space of complex square-integrable functions on the classical configuration space, and Hamiltonians are Hermitian operators.

In TQM the situation is totally different. Wave functions can be defined on the universal covering of the classical configuration space, or as multivalued functions defined on this space. The wave functions are equivariant with respect to the standard action of the fundamental group of the classical configuration space. Hamiltonians are not Hermitian operators and we cannot use the spectral theorem. Hamiltonians in TQM are Hermitian operators on their domains, but the domain depends on the Hamiltonian in contrary to QM.

The paper is organised as follows. In Section 2 we give some general information about TQM. In Section 3 we consider a special case of TQM – anyons on a plane. We show that the space of quantum states of a system of free anyons is the whole Hilbert space of square-integrable equivariant functions. Further we show that the space of quantum states for the system of free anyons and for the system of anyons in a homogeneous potential field has only the zero element as the intersection. At last we show that the space of quantum states for the anyon system in anisotropic harmonic oscillator potential is the zero space.

2. Topological quantum mechanics

Let M be a non-simply connected Riemann manifold of classical configurations of a physical system, with the fundamental group $\pi_1(M)$. Let us consider the universal covering

$$p : \widetilde{M} \longrightarrow M \tag{1}$$

consistent with the Riemann structure on M . It means that \widetilde{M} is also a Riemann manifold, and p is a local isometry.

Let

$$L : \pi_1(M) \times \widetilde{M} \longrightarrow \widetilde{M} \tag{2}$$

be the standard left free action of the fundamental group on the total space \widetilde{M} and

$$\chi : \pi_1(M) \longrightarrow U(1) \quad (3)$$

be a unitary one-dimensional representation. A function $f : \widetilde{M} \longrightarrow C$ is called χ -equivariant when

$$f(L([\gamma], \widetilde{m})) = \chi([\gamma])f(\widetilde{m}) \quad (4)$$

for $[\gamma] \in \pi_1(M)$, $\widetilde{m} \in \widetilde{M}$. For the case of χ being the trivial representation a χ -equivariant function is $\pi_1(M)$ -invariant and it is the lifting of a function defined on the base space M .

We can define the scalar product of equivariant functions by the formula

$$\langle f, g \rangle = \int_M f g^* d\mu \quad (5)$$

because the integrand is $\pi_1(M)$ -invariant ($d\mu$ denotes the volume form on M). The space of quantum space in TQM is a subspace of the Hilbert space of square-integrable χ -equivariant functions [1], [2]

$$\mathcal{H} = L^2_\chi(M, C, d\mu) \quad (6)$$

satisfying some boundary conditions.

The kinetic energy operator is defined as

$$\hat{T} = -\frac{1}{2m} \widetilde{\Delta}, \quad (7)$$

where $\widetilde{\Delta}$ is the Laplace operator on \widetilde{M} (it is the lifting of the Laplace operator Δ on M) and we put $\hbar = 1$. Although Δ is a Hermitean operator in $L^2(M, C, d\mu)$ the \hat{T} is in general not an operator in $L^2_\chi(M, C, \widetilde{d\mu})$, as we will see on an example of anyons. \hat{T} is defined only on a subspace of $L^2_\chi(M, C, \widetilde{d\mu})$ Moreover for Hamiltonians of the form

$$\hat{H} = \hat{T} + V \quad (8)$$

the domain of the operator \hat{H} depends on the potential V .

3. Anyons on a plane.

Anyons are hard core particles, so two or more particles can not be placed at the same point. Let us consider a system of several kinds of anyons on a plane: N_1 anyons of the first kind, N_2 anyons of the second, \dots , N_K anyons of the K -th. Anyons of the same kind are indistinguishable [3], [4]. We identify the plane with the complex plane C . The classical configuration space of the system is the quotient space

$$Q_{N_1, \dots, N_K} = \frac{C^N \setminus D_N}{S_{N_1} \times \dots \times S_{N_K}}, \quad N = N_1 + \dots + N_K, \quad (9)$$

where the direct product of symmetric groups $S_{N_1} \times \dots \times S_{N_K}$ acts on C^N in the standard way and D_N denotes the fat diagonal of C^N . The symmetric group S_{N_K} permutes positions of the k -th kind particles. The stratification (9) result with that anyons of the same kind are indistinguishable even on the classical level.

The fundamental group of the configuration space is the generalized braid group

$$B_{N_1, \dots, N_K} = \pi_1(Q_{N_1, \dots, N_K}). \quad (10)$$

Let us specify two special kinds of loops in Q_{N_1, \dots, N_K} at the point $[z_{11}, \dots, z_{KN_K}]$. We put the square bracket to denote the $S_{N_1} \times \dots \times S_{N_K}$ -orbit of $(z_{11}, \dots, z_{KN_K})$. Let

$$\gamma : I \longrightarrow Q_{N_1, \dots, N_K}, [\gamma_{11}(t), \dots, \gamma_{KN_K}(t)] \quad (11)$$

be a loop with $I = [0, 1]$.

I. We say that the loop (11) is a proper transposition of two anyons of the k -th kind placed at z_{kp} and z_{kq} if

$$\begin{aligned} \gamma_{lr}(t) &= z_{lr} \text{ for } (l, r) \neq (k, p), (k, q) \\ \gamma_{kp}(0) &= \gamma_{kq}(1) = z_{kp}, \quad \gamma_{kp}(1) = \gamma_{kq}(0) = z_{kq} \end{aligned} \quad (12)$$

and the composition of the curves γ_{kp}, γ_{kq} is a positively oriented Jordan loop with no particles inside the area limited by the loop.

II. We say that the loop (11) is a proper encircling of a particle of the l -th kind by a particle of the k -th kind placed at z_{lq} and z_{kp} respectively, if

$$\gamma_{ir}(t) = z_{ir} \text{ for } (i, r) \neq (k, p) \quad (13)$$

and γ_{kp} is a positively directed Jordan loop with the only (l, q) -th particle inside.

Let $\chi_{\nu\mu}$ be a unitary one-dimensional representation of the fundamental group $\pi_1(Q_{N_1, \dots, N_K})$. The representation is labeled by statistical parameters $\nu = (\nu_1, \dots, \nu_K)$ and encircling parameters $\mu = (\mu_{k,l}), 1 \leq k < l \leq K$, and is uniquely determined by two conditions:

$$\chi_{\nu\mu}(\alpha_k) = e^{\pi i \nu_k}, \quad (14)$$

where α_k is a proper transposition of anyons of the k -th kind, and

$$\chi_{\nu\mu}(\alpha_{k,l}) = e^{2\pi i \mu_{k,l}} \quad (15)$$

is a proper encircling of a particle of the l -th kind by a particle of the k -th kind.

Let $L^2_{\chi_{\nu\mu}0}(Q_{N_1, \dots, N_K}, C, \widetilde{d\mu})$ denote the subspace of all equivariant square-integrable functions satisfying the boundary conditions:

$$\lim_{z_{k,p} \rightarrow z_{l,q}} \Psi(z_{1,1}, \dots, z_{K,N_K}) = 0. \quad (16)$$

The space of quantum states of the anyon system is a subspace of $L^2_{\chi_{\nu\mu}0}(Q_{N_1, \dots, N_K}, C, \widetilde{d\mu})$.

We consider hamiltonians of the form $\hat{H} = \hat{T} + V$. The kinetic energy operator is given by

$$\hat{H}_0 = \hat{T} = - \sum_{k,p} \alpha_k \frac{\partial^2}{\partial z_{kp} \partial z_{kp}^*}, \quad \alpha_k = \frac{2}{m_k}, \quad (17)$$

where m_k denotes the mass of an anyon of the k -th kind. We will analyse free anyon system, anyons in homogeneous potential field and anyons in anisotropic harmonic oscillator potential. For these hamiltonians solutions of the Schrödinger equation can be decomposed into two classes of equivariant functions:

$$\begin{aligned} \Phi &= \left[\prod_{k=1}^K \prod_{1 \leq p < q \leq N_k} (z_{kq} - z_{kp})^{\nu_k} \right] \times \\ &\left[\prod_{1 \leq k < l \leq K} \prod_{p=1}^{N_k} \prod_{q=1}^{N_l} (z_{lq} - z_{kp})^{\nu_{kl}} \right] \times \end{aligned} \quad (18)$$

$$G(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots)$$

and

$$\begin{aligned} \Phi = & \left[\prod_{k=1}^K \prod_{1 \leq p < q \leq N_k} (z_{kq}^* - z_{kp}^*)^{2-\nu_k} \right] \times \\ & \left[\prod_{1 \leq k < l \leq K} \prod_{p=1}^{N_k} \prod_{q=1}^{N_l} (z_{lq}^* - z_{kp}^*)^{1-\nu_{kl}} \right] \times \\ & G(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots), \end{aligned} \quad (19)$$

where G denotes a power series of holomorphic z_{kp} and antiholomorphic z_{kp}^* variables with nonnegative integer exponents. The topological factors

$$\left[\prod_{k=1}^K \prod_{1 \leq p < q \leq N_k} (z_{kq} - z_{kp})^{\nu_k} \right] \left[\prod_{1 \leq k < l \leq K} \prod_{p=1}^{N_k} \prod_{q=1}^{N_l} (z_{lq} - z_{kp})^{\nu_{kl}} \right]$$

and

$$\left[\prod_{k=1}^K \prod_{1 \leq p < q \leq N_k} (z_{kq}^* - z_{kp}^*)^{2-\nu_k} \right] \times \left[\prod_{1 \leq k < l \leq K} \prod_{p=1}^{N_k} \prod_{q=1}^{N_l} (z_{lq}^* - z_{kp}^*)^{1-\nu_{kl}} \right]$$

are harmonic functions (the first is a holomorphic and the second is an antiholomorphic function). Let

$$\Psi(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots; t) \quad (20)$$

be an equivariant solution of the Schrödinger equation

$$\hat{T}\Psi = i \frac{\partial \Psi}{\partial t} \quad (21)$$

for the system of free anyons with an equivariant initial condition:

$$\begin{aligned} \Psi(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots; t = 0) = \\ \Phi(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots). \end{aligned} \quad (22)$$

Now let us modify the initial condition

$$\Psi_a(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots; t = 0) = \quad (23)$$

$$\begin{aligned} \sum_{\sigma \in S_{N_1} \times \dots \times S_{N_K}} \exp \left[i \sum_{k=1}^K \sum_{p=1}^{N_k} (a_{\sigma(kp)}^* z_{kp} + a_{\sigma(kp)} z_{kp}^*) \right] \\ \Phi(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots). \end{aligned}$$

At [5] we have analysed a similar modification but with an unbounded factor at the all plane. Although the modified initial condition is also equivariant, the solution

$$\begin{aligned} \Psi_a(\dots, z_{kp}, \dots; \dots, z_{kp}^*, \dots; t) = \\ \sum_{\sigma} \exp \left[i \sum_{k,p} (-\alpha_k |a_{\sigma(kp)}|^2 t + a_{\sigma(kp)}^* z_{kp} + a_{\sigma(kp)} z_{kp}^*) \right] \end{aligned} \quad (24)$$

$$\Psi(\dots, z_{kp} - \frac{a_{\sigma(kp)}}{\alpha_k} t, \dots; \dots, z_{kp}^* - \frac{a_{\sigma(kp)}^*}{\alpha_k} t, \dots)$$

is not equivariant. Equivariancy for transpositions of the same kind of particles is broken and equivariancy for encircling of one kind of particle by other kind is broken as well because the argument at the last factor is translated by a vector quantity proportional to time t . So the all space of square-integrable equivariant functions spanned by (18, 19) is not the space of quantum states of the free anyon system. Let us denote the space of quantum states by \mathcal{H}_0 . One can see that the space \mathcal{H}_0 is infinity dimensional

$$\dim \mathcal{H}_0 = \infty .$$

Any quantum state from the space \mathcal{H}_0 can be put as an initial condition (22) for the Schrödinger equation (21). The function (23) is also square-integrable equivariant function but (24) shows that (23) is not a quantum state from \mathcal{H}_0 .

Now let us consider the anyon system in homogeneous potential field. At [6] we have considered an analogical problem for a particle on a plane with one point removed. The Hamiltonian of the system is

$$\hat{H}_f = \hat{T} - \frac{1}{2} \sum_{k,p} (f_k^* z_{k,p} + f_k z_{k,p}^*) ,$$

where f_k is the force acting on the k -th kind particle. The evolution operator is given by the formula

$$\hat{U}_f(t) = \prod_{k,p} \exp \frac{it^3}{3m_k} \hat{\text{Tr}}_{\Pi_{kp}} \left(\frac{-t^2}{2m_k} f \right) \exp i \langle f_k, z_{kp} \rangle U_0(t) , \quad (25)$$

where $\hat{U}_0(t)$ is the evolution operator of the free anyon system and $\hat{\text{Tr}}_{\Pi}(v)$ is the translation operator in the plane Π by a vector v . Let the initial condition for the Schrödinger equation

$$\hat{H}_f \Psi = i \frac{\partial \Psi}{\partial t} \quad (26)$$

be any wave function Φ from the space \mathcal{H}_0 . The solution

$$\hat{U}_f(t) \Phi$$

is not equivariant in general. Equivariancy for transpositions of the same kind of particles is conserved, but equivariancy for encircling one particle by other kind is broken. As we see from the formula (eq. 25) the singularities move with uniform accelerations

$$a_k = \frac{f_k}{m_k}$$

for the k -kind particle, what is identical formula with the second Newton's law of motion. The solution is equivariant only for the case $a_k = a_l$ for any k, l . Although

$$\dim \mathcal{H}_f = \infty , \quad (27)$$

for $a_k \neq a_l$ we get

$$\mathcal{H}_0 \cap \mathcal{H}_f = \{0\}. \quad (28)$$

Finally let us consider the anyon system in an anisotropic harmonic oscillator potential. Let the Hamiltonian of the system be given by

$$\hat{H}_{\omega_1, \omega_2} = \hat{T} + \frac{1}{2} \sum_{kp} m_k (\omega_1^2 \hat{x}_{kp}^2 + \omega_2^2 \hat{y}_{kp}^2), \quad (29)$$

where $z_{kp} = x_{kp} + iy_{kp}$. The evolution operator is

$$\begin{aligned} \hat{U}_{\omega_1, \omega_2}(t) &= \frac{1}{\sqrt{\cos \omega_1 t \cos \omega_2 t}} \\ &\prod [\exp[-i \frac{m_k}{2} (\omega_1^2 \hat{x}_{kp}^2 \tan \omega_1 t + \omega_2^2 \hat{y}_{kp}^2 \tan \omega_2 t)] \\ &\hat{d}_{x_{kp}} \left(\sqrt{\frac{\tan \omega_1 t}{\omega_1}} \right) \hat{d}_{y_{kp}} \left(\sqrt{\frac{\tan \omega_2 t}{\omega_2}} \right)] \\ &\hat{U}_0(1) \prod_{kp} \hat{d}_{x_{kp}} \left(\sqrt{\frac{2\omega_1}{\sin 2\omega_1 t}} \right) \hat{d}_{y_{kp}} \left(\sqrt{\frac{2\omega_2}{\sin 2\omega_2 t}} \right), \end{aligned}$$

where $\hat{d}_x(a)$ is the dilatation operator at the direction x in the scale a . The composition $\hat{d}_{x_{kp}} \left(\sqrt{\frac{2\omega_1}{\sin 2\omega_1 t}} \right) \hat{d}_{y_{kp}} \left(\sqrt{\frac{2\omega_2}{\sin 2\omega_2 t}} \right)$ of dilatations deforms the holomorphic variable z_{kp} . So deformed topological factors at the equations (18, 19) are also equivariant functions but they are not harmonic functions for $\omega_1 \neq \omega_2$. Consequently the deformed functions do not belong to the domain of the operator $\hat{U}_0(1)$. So the space of quantum states for anisotropic oscillator is the zero space

$$\mathcal{H}_{\omega_1, \omega_2} = \{0\}. \quad (30)$$

4. Conclusion remarks.

I. We see that if the function (22) is a quantum state for free anyons system, then the function (23) (also equivariant) is not a quantum state for the system. So the quantum states of states for the system is not the all Hilbert space of equivariant square-integrable functions.

II. The spaces of quantum states for the free anyon system and for the free anyon system and for the system of anyons in homogeneous potential field are infinitely dimensional Hilbert subspaces of the space of square-integrable equivariant functional. The equation (28) shows that these quantum states of states have only the zero common vector.

III. The space of quantum states for the anyon system in anisotropic harmonic oscillator potential is the zero space.

So we see that the space of quantum states in quantum mechanics of anyons depends on the Hamiltonian.

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