Rigged strings, Bethe Ansatz, and the geometry of the classical configuration space of the Heisenberg magnetic ring.

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1 Introduction

It is well known that the exact solutions of the one-dimensional Heisenberg magnetic ring, given by the famous Bethe Ansatz [1] (BA), are classified in terms of Takahashi [2] integers or half-integers, which combinatorially correspond to *rigged string configurations* [3], [4], [5], [6], [7]. Rigged string configurations are physically interpreted as a collection of strings, i.e. bound states of a number of spin deviations (Bethe pseudoparticles), each string equipped with a definite quasimomentum, expressed by the Takahashi number which varies over a definite range. In this way, each eigensolution of the Heisenberg Hamiltonian for the ring of N spins s=1/2 is fully characterized by the pair $(\nu, \mathcal{L})$, where, according to the terminology proposed by Kerov, Kirillov and Reshetikhin (KKR)[4], $\nu = (m_1, m_2, \dots)$ is the *string configuration*, with m_l being the number of strings of the length $l = 1, 2, \dots$, and $\mathcal{L}$ is the corresponding collection of integers which label quasimomenta of strings, referred to as the *rigging* of ν. The set of all BA eigensolutions is therefore decomposed into subsets of solutions with the same string configuration, which differ by the distribution of quasimomenta over strings.

We recall briefly that the space $\mathcal{H}$ of all quantum states of the Heisenberg magnet, with $\dim\mathcal{H} = 2^N$, decomposes as

$$\mathcal{H} = \sum_{r=0}^{N} \oplus \, \mathcal{H}^r, \tag{1}$$

into subspaces $\mathcal{H}^r$, $\dim\mathcal{H}^r = \binom{N}{r}$, with the fixed number r of Bethe pseudoparticles (spin deviations). Stated otherwise, $\mathcal{H}^r$ is the space of all states of the magnet with the projection

$$M = \frac{N}{2} - r \tag{2}$$

of the total spin, or of all states with the *weight*

$$\mu = \{N - r, r\}. \tag{3}$$

Clearly, $\mathcal{H}^r$ and $\mathcal{H}^{N-r}$ are particle-hole counterparts, so we restrict in the sequel to $r \leq N/2$. Then, each space $\mathcal{H}^r$ decomposes as

$$\mathcal{H}^r = \sum_{S=N/2-r}^{N/2} \oplus\, \mathcal{H}^{rS} \tag{4}$$

into subspaces $\mathcal{H}^{rS}$ with the total spin S (and $M = N/2 - r$). By the duality of Weyl [8] between the actions of the symmetric group Σ_N and the unitary group $U(n)$, $n = 2s+1 = 2$, in the space $\mathcal{H}$ (cf., e.g., [9], [10] for detail), it defines the partition λ of the integer N, of the *shape*

$$\lambda = \{N - r', r'\}, \tag{5}$$

such that λ labels the irrep Δ^λ of Σ_N,

$$\dim \Delta^\lambda = \dim \mathcal{H}^{rS}, \tag{6}$$

and

$$S = \frac{N}{2} - r', \;\; 0 \leq r' \leq r. \tag{7}$$

The extreme case

$$S_{h.w.} = \frac{N}{2} - r = M \tag{8}$$

is usually referred to as *the highest weight* subspace in $\mathcal{H}^r$. KKR [4] defined a bijection between the set of all rigged string configurations which span a space $\mathcal{H}^{rS}$ as exact solutions of BA, and the set $SYT(\lambda)$ of all standard Young tableaux of the shape λ which provides a basis in $\mathcal{H}^{rS}$ along the duality of Weyl, that is, the orthogonal Young basis in $\mathcal{H}^{rS}$, treated as the carrier space of the irrep Δ^λ of Σ_N (cf., e.g., [9]). Moreover, they pointed out for another bijection, provided by the Robinson [11] - Schensted [12] (RS) algorithm (cf. also [9], [10], [13], [14]), which relates the irreducible basis of the duality of Weyl with the basis of all magnetic configurations - the initial stage for all quantum calculations in the space $\mathcal{H}$.

The setting provided by KKR involves the general scheme of the duality of Weyl for an arbitrary $n = 2s + 1$, and is therefore applicable - after an appropriate generalization of the Heisenberg Hamiltonian - to an arbitrary spin s. In this paper we confine ourselves to the case $s = 1/2$, with an appropriate restriction of definitions and notation (in particular, we omit the notion of a *colour* of a string, which is redundand here).

In the present paper we aim to point out a somehow suprising conclusion, stemming from existence of the two bijections, RS and KKR. Solutions of highly non-linear BA equations involve a complete set of exact quantum numbers for each eigenstate. The combination of the two above bijections implies that these tiny results are already coded, in a combinatorially bijective way, in the set of all magnetic configurations. In other words, each *single* magnetic configuration anticipates a definite eigenstate, which is, as a rule, a highly correlated wavepacket of a number of magnetic configurations.

In other words, the results of algebraic calculations of BA equations are predicted and classified by purely combinatoric methods.

We also aim to examine properties of a composition, RSKKR = KKR $\circ$ RS, of these two bijections, under the action of the cyclic group C_N - the translational symmetry group of the problem, along the description of the basis of wavelets proposed in our earlier paper [15]. We have observed [15], [16] that the set $Q^{(r)}$ of all those magnetic configurations which span unitarily the space $\mathcal{H}^r$ has a natural interpretation of the classical configuration space of the system of r Bethe pseudoparticles - indistinguishable hard-core particles moving on the magnetic ring. In particular, we have shown that the geometry imposed on $Q^{(r)}$ by the dynamics resulting from the Heisenberg Hamiltonian, and by the action of the translation group C_N, yields a curvature of the manifold $\mathcal{M}^{(r)}$ into which the discrete set $Q^{(r)}$ can be regularly embedded. Such a curvature results simply from the observation that each C_N-orbit in $Q^{(r)}$ should be a loop. Here we investigate behaviour of these geometric properties under the RSKKR bijection, and thus study the origin of various configurations of strings in terms of "classical position" of the magnet in its configuration space $Q^{(r)}$. We put emphasis on the observation that the notion of a string configuration is not invariant under the translation group C_N. Namely, a single shift translates the string by one step from the left to right only when this string is *inside* the ring with *open* boundaries: the leftmost node j=1, and the rightmost j=N. The right boundary annihilates the rightmost box of a string whereas the left one creates one box of an arising string or hole.

As a result, the collection of exact BA solutions, expressed in terms of rigged strings, heritates some its structure from the geometry of the classical configuration space of the magnetic ring.

2 Geometry of the classical configuration space.

1. Preliminary definitions.

Let $\tilde{N} = \{j = 1, 2, \ldots, N\}$ be the set of nodes of a magnetic ring, such that consecutive nodes are nearest neighbours of each other, and also $j = 1$ is the nearest neighbour of $j = N$. Denote by $\tilde{2} = \{+, -\}$ the set of single-node spin projections. Then the set of all magnetic configurations is

$$\tilde{2}^{\tilde{N}} = \{f : \tilde{N} \longrightarrow \tilde{2}\}, \tag{9}$$

and the space $\mathcal{H}$ of all quantum states of the magnetic ring is the linear closure of this set over the field $\mathbb{C}$ of complex numbers, which we write as

$$\mathcal{H} = lc_{\mathbb{C}}\ \tilde{2}^{\tilde{N}}. \tag{10}$$

Moreover, the set $\tilde{2}^{\tilde{N}}$ provides an *orthonormal* basis in $\mathcal{H}$. The natural action $A : \Sigma_N \times \tilde{2}^{\tilde{N}} \to \tilde{2}^{\tilde{N}}$ of the symmetric group Σ_N of all permutations on the set $\tilde{N}$, lifted to the set $\tilde{2}^{\tilde{N}}$ of all magnetic configurations by the formula

$$A(\sigma) = \begin{pmatrix} f \\ f \circ \sigma^{-1} \end{pmatrix}, \;\; f \in \tilde{2}^{\tilde{N}}, \;\; \sigma \in \Sigma_N, \tag{11}$$

decomposes the latter set into orbits

$$\mathcal{O}_\mu = \{ f^\mu \circ \sigma^{-1} | \, \sigma \in \Sigma_N \}, \tag{12}$$

with $f^\mu \in \tilde{2}^{\tilde{N}}$ being the "initial" magnetic configuration, given by

$$f^\mu(j) = \begin{cases} + & \text{for} \quad 1 \le j \le N - r, \\ - & \text{for} \quad N - r + 1 \le j \le N, \end{cases} \tag{13}$$

and μ being the weight (3). Clearly, for $r \le N/2$, these orbits are classical configuration spaces $Q^{(r)}$ for the system of r spin deviations, such that

$$\mathcal{H}^r = lc_{\mathbb{C}} \mathcal{O}_\mu \equiv lc_{\mathbb{C}} Q^{(r)}, \tag{14}$$

and

$$\dim \mathcal{H} = |Q^{(r)}| = \begin{pmatrix} N \\ r \end{pmatrix}. \tag{15}$$

2. Structure of orbits of the translation group.

The classical configuration space $Q^{(r)}$ decomposes, as an Σ_N-orbit, into orbits of the subgroup $C_N \subset \Sigma_N$ - the translational symmetry group of the ring. We propose here, after [15], some handy conventons for a unique classification of these orbits, as well as magnetic configurations within such an orbit. Let

$$\mathbf{j} = (j_1, j_2, \ldots, j_r), \text{ for some set } 1 \le j_1 < j_2 < \cdots < j_r \le N, \tag{16}$$

be the distribution of Bethe pseudoparticles which specifies a magnetic configuration $f \in Q^{(r)}$, and let

$$\tau = (\tau_1, \tau_2, \ldots, \tau_r) \tag{17}$$

be defined by

$$\tau_\alpha = \begin{cases} j_{\alpha+1} - j_\alpha & \text{for} \quad 1 \le \alpha < r, \\ j_1 - j_r + N & \text{for} \quad \alpha = r, \end{cases} \tag{18}$$

so that

$$\sum_{\alpha \in \tilde{r}} \tau_\alpha = N. \tag{19}$$

Clearly, τ represents the *relative* distribution of Bethe pseudoparticles within an orbit of the translation group C_N. Yet, the vector τ does not classify such an orbit uniquely since all vectors of the form

$$C_r^{\beta-1} \tau = (\tau_\beta, \tau_{\beta+1}, \ldots, \tau_{\beta-1}), \;\; \beta \in \tilde{r}, \tag{20}$$

i. e. vectors which differ from τ by a cyclic permutation of their components, also yield the same C_N-orbit. We choose, by a convention, that of vectors (20), which is *lexically smallest* when looked at as a word of the length r in the alphabet $\tilde{N}$, and denote it by

$$\mathbf{t} = (t_1, t_2, \ldots, t_r), \ \sum_{\alpha \in \tilde{r}} t_\alpha = N. \tag{21}$$

Thus, in particular, t_1 is (possibly one of) the smallest component of τ, and $t_\alpha = 1$ implies that Bethe pseudoparticles α and $\alpha + 1$ are nearest neighbours in the chain $\tilde{N}$. The vector $\mathbf{t}$ provides a unique label for classification of C_N-orbits in $Q^{(r)}$, so that

$$Q^{(r)} = \bigcup_{\mathbf{t} \in T} \mathcal{O}_{\mathbf{t}}, \tag{22}$$

where $T = Q^{(r)}/C_N$ is the set of orbits, which we refer to, after [15], as a *skeleton* of the configuration space $Q^{(r)}$.

Let

$$f^{\mathbf{t}} = (\mathbf{t}, 1) = |\underbrace{- + + \cdots +}_{t_1}\underbrace{- + + \cdots +}_{t_2} - \ldots \underbrace{- + + \cdots +}_{t_r} > \tag{23}$$

be, by convention, the *initial* magnetic configuration in the orbit $\mathcal{O}_{\mathbf{t}}$. The distribution $\mathbf{j}^{\mathbf{t}}$ of Bethe pseudoparticles for the initial magnetic configuration $f^{\mathbf{t}}$ is given by

$$j_\alpha^{(\mathbf{t},1)} = 1 + \sum_{\alpha' < \alpha} t_{\alpha'}, \ \alpha \in \tilde{r}. \tag{24}$$

An arbitrary configuration in this orbit has the form of composition of mappings

$$(\mathbf{t}, j) = (\mathbf{t}, 1) \circ C_N^{1-j}, \ j \in \tilde{N}, \tag{25}$$

where the mapping $C_N : \tilde{N} \longrightarrow \tilde{N}$ is defined by

$$C_N = \begin{pmatrix} j \\ (j+1) \bmod N \end{pmatrix} \in \Sigma_N, \ j \in \tilde{N} \tag{26}$$

(we use the same symbol for the group $C_N \subset \Sigma_N$, and for its generating element C_N).

It is important to observe at this point that the magnetic configurations $(\mathbf{t}, j)$ of the form (25) are not necessarily all different. It is so when the initial configuration $(\mathbf{t}, 1)$ has no periodicity, and thus $\mathcal{O}_{\mathbf{t}}$ is a *regular* orbit of C_N. Otherwise, $(\mathbf{t}, 1)$ can be presented in a form

$$f^{\mathbf{t}} = \underbrace{ww \ldots w}_{\kappa \text{ times}}, \tag{27}$$

where κ is a divisor of N, and w is a word of the length $\overline{\kappa} = N/\kappa$ in the alphabet $\tilde{2}$, with no further periodicity. Then the orbit $\mathcal{O}_{\mathbf{t}}$ is κ-tuply *rarefied* [17]-[21], that is, it consists of $\overline{\kappa}$ elements. Moreover, the label $\mathbf{t}$ of this orbit has the form

$$\mathbf{t} = \underbrace{uu \ldots u}_{\kappa \, \text{times}}, \tag{28}$$

with

$$u = (t_1, t_2, \ldots, t_{r/\kappa}) \tag{29}$$

being a word of the length r/κ in the alphabet $\tilde{N}$, with no further periodicity, and

$$\sum_{\alpha=1}^{r/\kappa} t_\alpha = \overline{\kappa}. \tag{30}$$

A unique labeling of magnetic configurations within the orbit $\mathcal{O}_{\mathbf{t}}$, which is $\kappa(\mathbf{t})$-tuply rarefied, is given therefore by the formula

$$\mathcal{O}_{\mathbf{t}} = \{(\mathbf{t}, j) = (\mathbf{t}, 1) \circ C_N^{1-j} | j = 1, 2, \ldots, \overline{\kappa}(\mathbf{t})\}. \tag{31}$$

In general, $\kappa(\mathbf{t}) \in K(N)$, where $K(N)$ is the lattice of all divisors of N, and $\overline{\kappa}(\mathbf{t}) = N/\kappa(\mathbf{t})$ is the divisor complementary to $\kappa(\mathbf{t})$ in this lattice. In particular, $\kappa(\mathbf{t}) = 1$ when the orbit is regular, and then $|\mathcal{O}_{\mathbf{t}}| = N$. In the other extreme case, $\kappa(\mathbf{t}) = N$, the corresponding orbit $\mathcal{O}_{\mathbf{t}}$, $\mathbf{t} = (N)$, consists of a single magnetic configuration $|{+}{+} \cdots + >$, describing the ferromagnetic vacuum $r = 0$ for Bethe pseudoparticles.

The notation assumed here allows us to decompose each regular orbit $\mathcal{O}_{\mathbf{t}}$ into *sectors*

$$\mathcal{O}_{\mathbf{t}} = \bigcup_{\beta \in \tilde{r}} \mathcal{O}_{\mathbf{t}\beta}, \tag{32}$$

$$\mathcal{O}_{\mathbf{t}\beta} = \{(\mathbf{t}, j) | 1 + T_{\beta-1} \leq j \leq T_\beta\}, \tag{33}$$

where

$$T_\beta = \sum_{\beta' \leq \beta} t_{r-\beta'+1}, \ \beta \in \tilde{r}, \tag{34}$$

is the sum of the *last* β components of the vectorial label $\mathbf{t}$. Thus T_β is the number of consecutive translations C_N which put the Bethe pseudoparticle $\alpha = r - \beta + 1$ of the initial magnetic configuration $(\mathbf{t}, 1)$ (cf. Eq. (23)) to the rightmost node $j = N$ of the chain, that is

$$j_{r-\beta+1}^{(\mathbf{t},1)} + T_\beta - 1 = N, \ \beta \in \tilde{r}. \tag{35}$$

The next translation C_N puts this pseudoparticle into the leftmost node $j = 1$, and imposes therefore the relabelling of all pseudoparticles such that

$$\alpha \longmapsto (\alpha + 1)_{\bmod r}, \ \alpha \in \tilde{r}. \tag{36}$$

In our notation, all Bethe pseudoparticles conserve their labels α within each sector $\mathcal{O}_{\mathbf{t}\beta}$, $\beta \in \tilde{r}$, and undergo a single shift given by the cyclic group C_r (cf. Eq. (20)) when transgressing the boundary of a sector. The notion of sectors can be readily extended into rarefied orbits under appropriate modifications. Clearly, the number of sectors in a $\kappa(\mathbf{t})$-tuply rarefied orbit $\mathcal{O}_{\mathbf{t}}$ is equal to $r/\kappa(\mathbf{t})$ (cf. Eqs. (28)-(30)), and thus for regular orbits it coincides with r. We quote the formulas for the distribution $\mathbf{j}^{\mathbf{t}j}$ of Bethe pseudoparticles in an arbitrary magnetic configuration $\mathbf{t}j$ in the basis of C_N-orbits.

$j\backslash\alpha$	j_1	j_2	j_3	j_4
1	1	2	4	6
2	2	3	5	7
3	3	4	6	8
4	1	4	5	7
5	2	5	6	8
6	1	3	6	7
7	2	4	7	8
8	1	3	5	8

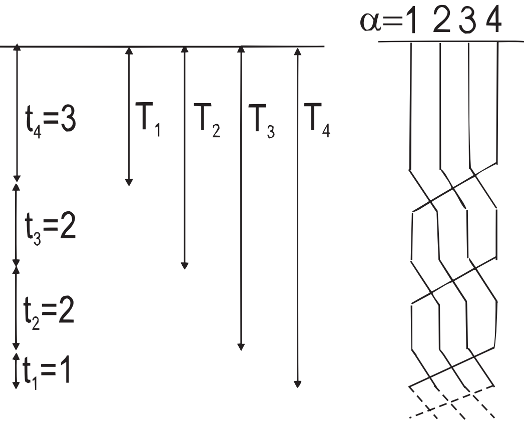

Figure 1: An example of a regular C_N-orbit $\mathcal{O}_{\mathbf{t}}$ for $N = 8$, $r = 4$. The orbit label is $\mathbf{t} = (1223)$ (cf. Eq. (21) and associated conventions). The first column of the Table lists the consecutive labels $j \in \tilde{N} = \tilde{8}$ of magnetic configurations. The corresponding magnetic configuration $(\mathbf{t}, j) = \mathbf{j} = (j_1, j_2, j_3, j_4)$ is listed in the second column. The component $j_\alpha, \alpha \in \tilde{r} = \tilde{4}$, point out the position of the α-th Bethe pseudoparticle in the configuration $\mathbf{j}$. The vertical arrows placed to the right of this column, of the length t_α of the vector $\mathbf{t}$, counted in reverse order from the top to the bottom ($t_4 = 3, t_3 = 2, t_2 = 2, t_1 = 1$), point out the range of the sector $\mathcal{O}_{\mathbf{t}\beta}$, $\beta = r - \alpha$. The next series T_β of vertical arrows, $\beta \in \tilde{4}$, denotes the distance between the Bethe pseudoparticle with the label $\alpha = r - \beta + 1$ in the initial magnetic configuration $(\mathbf{t}1)$ (the first row of the Table), and the terminal node $j = N$ of the magnetic chain. The lines in the right - hand side indicate the labelling of Bethe pseudoparticles: they have the same label within each sector $\mathcal{O}_{\mathbf{t}\alpha}$, and change by one step modulo r when transgressing a sector. Each of $r = 4$ lines is a loop with $N = 8$ nodes.

For the regular orbits, i. e. for $\kappa(\mathbf{t}) = 1$, the α-th Bethe pseudoparticle, counted from its leftmost node $j = 1$, reads

$$j_\alpha^{(\mathbf{t},j)} = j + N - T_{(r-\alpha+\beta) \bmod r}, \tag{37}$$

where β is the label of the sector, determined from the formula

$$1 + T_{\beta-1} \leq j \leq T_\beta, \tag{38}$$

which encloses this configuration ($(\mathbf{t}, j) \in \mathcal{O}_{\mathbf{t}\beta}$). In the case of a rarefied orbit, $\kappa(\mathbf{t}) \in K(N)$, the number of sectors is $r/\kappa(\mathbf{t})$, so that

$$1 \le \beta \le r/\kappa(\mathbf{t}), \tag{39}$$

and the length of the sector $\mathcal{O}_{\mathbf{t}\beta}$ is

$$|\mathcal{O}_{\mathbf{t}\beta})| = t_{r/\kappa(\mathbf{t})-\beta+1}, \; \beta = 1, 2, \ldots, r/\kappa(\mathbf{t}), \tag{40}$$

whereas the number of Bethe pseudoparticles remains r, and thus

$$j_\alpha^{(\mathbf{t},j)} = j + N + T_{(r/\kappa(\mathbf{t})-\alpha+\beta) \operatorname{mod} r/\kappa} + x\, T_{r/\kappa(\mathbf{t})}, \; \alpha \in \tilde{r}, \beta \in \{1, 2, \ldots, r/\kappa(\mathbf{t})\}, \tag{41}$$

the label β is determined from Eq. (38), and the integer x follows from the modular condition

$$\alpha = \alpha \operatorname{mod} r/\kappa + x \cdot r/\kappa, \tag{42}$$

with $1 \le \alpha \operatorname{mod} r/\kappa \le r/\kappa$. For the $\kappa(\mathbf{t})$-tuply rarefied orbit $\mathcal{O}_{\mathbf{t}}$, the integer $x \in \{1, 2, \ldots, \kappa\}$ labels all Bravais cells of a magnetic configuration (consecutive subword w in the word $(\mathbf{t}, 1)$ given by Eq. (27)).

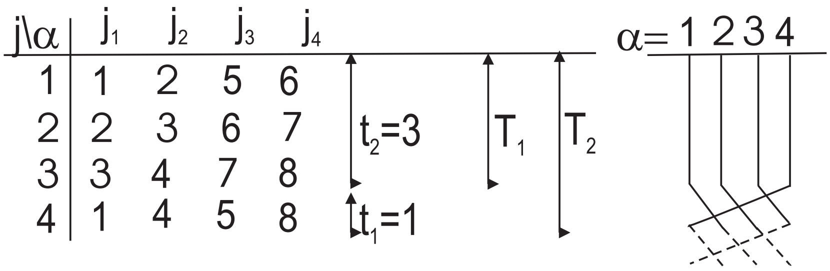

Figure 2: An example of a rarefied C_N-orbit for $N = 8$, $r = 4$, $\kappa = 2$. The label $\mathbf{t} = (1313)$ of the orbit exhibits $\kappa = 2$ - tuply periodicity: $\mathbf{t} = uu$, with $u = (t_1, t_2) = (13)$, so that $\overline{\kappa} = N/\kappa = 4$ is the number of magnetic configurations (i. e. the number of rows in the Table), and $r/\kappa = 2$ - the number of sectors in the orbit $\mathcal{O}_{\mathbf{t}}$ (cf. Eqs. (27)-(31)). The description of Fig. 1 applies correspondingly. Observe that again each line in the last column of the Figure is a loop, consisting of $N = 8$ nodes, but the total number of these loops is $r/\kappa = 4/2 = 2$, rather than $r = 4$.

We briefly recapitulate some properties of the basis (22) and (31) of orbits. The initial configuration (23) in an orbit $\mathcal{O}_{\mathbf{t}}$ corresponds to the (lexically) leftmost position of all Bethe pseudoparticles in the *open* chain $\tilde{N}$, that is in the chain with distinguished leftmost and rightmost node, $j = 1$ and $j = N$, respectively. Under consecutive translations C_N^{-j}, $j \in \tilde{N}$, all these pseudoparticles travel paralelly, step by step to the right, with mutual distances unchanged. The label $\alpha \in \tilde{r}$ of each pseudoparticle, counted by definition consecutively from the left to the right, remains unchanged within each sector

$\beta \in \{1, 2, .., r/\kappa(\mathbf{t})\}$, and increases by one modulo r when jumping between sectors, from β to $(\beta+1)\mathrm{mod}\, r/\kappa(\mathbf{t})$ (cf. Figs. 1 and 2).

3. Structure of islands of spin deviations.

We have described jumps between magnetic configurations, imposed by a single translation C_N. These jumps occur between the nearest neighbour configurations within an orbit $\mathcal{O}_{\mathbf{t}}$. In the context of BA, however, much more important are those jumps which are induced by the Heisenberg Hamiltonian $\hat{H}$. We recall that the action of $\hat{H}$ in $\mathcal{H}^r$ is given by

$$2\hat{H}|\mathbf{j}>= \sum_{\mathbf{j}' \in Q^{(r)}_{\mathbf{j}}} (|\mathbf{j}'> - |\mathbf{j}>), \;\; \mathbf{j} \in Q^{(r)}, \tag{43}$$

where $|\mathbf{j}> \in \mathcal{H}^r$ is the quantum state of the model, corresponding to the magnetic configuration determined by the distribution $\mathbf{j} \in Q^{(r)}$ of Bethe pseudoparticles, and $Q^{(r)}_{\mathbf{j}} \subset Q^{(r)}$ is the set of all those magnetic configurations $\mathbf{j}'$ which differ from $\mathbf{j}$ at a single place $\alpha' \in \tilde{r}$ by a single lattice constant, that is

$$j'_\alpha = \begin{cases} (j_{\alpha'} \pm 1)\mathrm{mod} N & \text{for } \alpha = \alpha', \\ j_\alpha & \text{for } \alpha \neq \alpha'. \end{cases} \tag{44}$$

In the sequel we refer to $Q^{(r)}_{\mathbf{j}}$ as to the set of all *nearest neighbours* of the magnetic configuration $\mathbf{j}$ within $Q^{(r)}$. In this way, the dynamics (43) of BA imposes, in a local manner, the structure of a hypercubic lattice in the classical configuration space $Q^{(r)}$: for a generic position $\mathbf{j} \in Q^{(r)}$, its nearest neighbours $\mathbf{j}' \in Q^{(r)}_{\mathbf{j}}$ define a local coordinate system in a real r-dimensional linear space $\mathbb{R}^r$ into which $Q^{(r)}_{\mathbf{j}}$ can be embedded, with the origin at $\mathbf{j}$. Such a description reflects the r degrees of freedom for stroboscopic movements of the system $\tilde{r}$ of Bethe pseudoparticles over the ring $\tilde{N}$: a pseudoparticle $\alpha \in \tilde{r}$ can jump forward or back by one step on the crystal $\tilde{N}$, independently on other pseudoparticles $\alpha' \neq \alpha$.

The classical configuration space $Q^{(r)}$, however, is not homogeneous with respect to the degree of freedom. A look at the form (23) of a magnetic configuration reveals that some pseudoparticles can be the nearest neighbours each of other, forming thus "islands" of spin deviations in the "sea" of nodes with the spin $+ \in \tilde{2}$. Clearly, each island contributes to a single degree of freedom, since its leftmost pseudoparticle can only jump to the left, its rightmost - to the right, and all internal pseudoparticles of this island are *kinematically frozen*.

In our notation, an island of the length ξ yields an orbit $\mathcal{O}_{\mathbf{t}}$ for which the label $\mathbf{t}$, treated as the word of the length r in the alphabet $\tilde{N}$, contains a subword of consecutive letters in the form

$$t_a \underbrace{11\ldots1}_{\xi-1} t_b \subset \mathbf{t}, \;\; t_a \neq 1, t_b \neq 1, \tag{45}$$

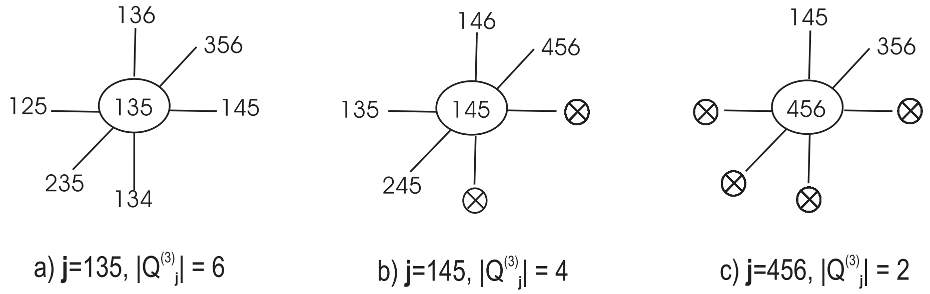

Figure 3: Examples for the set $Q_{\mathbf{j}}^{(r)}$ of nearest neighbours of $\mathbf{j}$ in the configuration space $Q^{(r)}$ for $N = 6$, $r = 3$. Lines connect the central node $\mathbf{j}$ with all its nearest neighbours . The nodes $\mathbf{j}$ are encircled. The cross inside a circle denotes transitions which are forbiden due to hard - core condition. The cases a, b, and c correspond to the local dimension $F = 3$, 2, and 1, respectively.

with $b = a + \xi$. More generally, the structure of islands of spin deviations is defined as a sequence of lengths

$$\xi = (\xi_1, \xi_2, \ldots, \xi_F), \ \sum_{\gamma=1}^{F} \xi_\gamma = r. \tag{46}$$

Bethe pseudoparticles, which correspond to the subword (45), satisfy the relations

$$\begin{array}{ll} (j_\alpha + 1) \bmod N = j_{(\alpha+1) \bmod r}, & a < \alpha < b - 1, \\ (j_\alpha - 1) \bmod N = j_{(\alpha-1) \bmod r}, & a + 1 < \alpha < b, \end{array} \tag{47}$$

so that the only moves of Bethe pseudoparticles of the island given by Eq. (46) are

$$j_{a+1} \longmapsto (j_{a+1} - 1) \,\mathrm{mod} N, \ j_{b-1} \longmapsto (j_{b-1} + 1) \,\mathrm{mod} N, \tag{48}$$

which result in a single degree of freedom, or a one-dimensional (local) subspace in $Q^{(r)}$.

The structure of islands imposes thus a decomposition

$$Q^{(r)} = \bigcup_{F=1}^{r} Q^{(r,F)} \tag{49}$$

of the classical configuration space into subsets

$$Q^{(r,F)} = \{\mathbf{j} \in Q^{(r)} \,|\, |Q_{\mathbf{j}}^{(r)}| = 2F\}, \tag{50}$$

consisting of orbits with exactly $F \leq r$ islands of spin deviations (with arbitrary lengths, satisfying Eq. (46)). The case $F = r$ is *generic*, whereas subsets $Q^{(r,F)}$ with $F < r$ correspond to *boundaries* of the classical configuration space with a lower dimension (cf. Fig. 3). Thus the integer F, defined in Eq. (46) as the number of islands of spin deviations for a magnetic configuration within an orbit $\mathcal{O}_{\mathbf{t}}$, plays also the role of *local dimension* for all elements of the boundary $Q^{(r,F)}$ (such that $\mathcal{O}_{\mathbf{t}} \subset Q^{(r,F)}$) of the classical configuration space $Q^{(r)}$. The local dimension F varies within the range

$$1 \leq F \leq r, \tag{51}$$

and satisfies

$$r' \leq F, \tag{52}$$

where r' determines the total spin $S = N/2 - r'$ (cf. Eq. (7)).

We are going to demonstrate in the sequel that just these boundaries indicate unambiguously the origin of strings - the building blocks for exact BA solutions.

3 The combinatorics of strings.

1. String configurations.

As we have mentioned in the Introduction, the space of all quantum states of the system $\tilde{r}$ of Bethe pseudoparticles decomposes into subspaces $\mathcal{H}^{rS}$, corresponding to the total spin $S = N/2 - r', 0 \leq r' \leq r$. The space $\mathcal{H}^{rS}$ carries the irrep Δ^{λ} of the symmetric group Σ_N, with $\lambda = \{N - r', r'\}$. A *string configuration* ν is, by the definition, a partition of the integer r', $\nu \vdash r'$. Each part of the partition ν is referred to as a *string*, and its size (the number of boxes in the Young diagram of ν) - the *length* of the string. Denoting by m_l the number of strings of the length l in ν, we have

$$\sum_l l m_l = r'. \tag{53}$$

Each string (lv), $v \in \tilde{m}_l = \{1, 2, \ldots, m_l\}$, is equipped with a nonnegative integer L, called its *rigging*. The rigging L varies within the range

$$0 \leq L \leq P_l, \tag{54}$$

where

$$P_l = N - 2Q_l, \tag{55}$$

with Q_l being the number of boxes in the first l *columns* of the Young diagram of the string configuration ν. The integer P_l is referred to as the number of *holes* for strings of the length l. The pair $\nu\mathcal{L}$, where

$$\mathcal{L} = \{L_{lv} | l = 1, 2, \ldots; \;\; v \in \tilde{m}_l\}, \tag{56}$$

is the set of all riggings of ν, is referred to as a *rigged string configuration*. All m_l strings of the length l are distinguished by their riggings only, so that they can be arranged in $\nu\mathcal{L}$ in the nondecreasing order (from the top to the bottom of the $l \times m_l$ - rectangle incorporated in the Young diagram $\nu \vdash r'$), that is

$$0 \le L_{lv} \le L_{lv'} \le P_l \ \text{ for } 1 \le v < v' \le m_l, \ l = 1, 2, \ldots \tag{57}$$

Such a counting implies that the set $z(\nu) = \{\nu\mathcal{L}\}$ of all rigged string configurations corresponding to a given $\nu \vdash r'$ has the cardinality

$$|z(\nu)| = \prod_l \binom{P_l \quad + \quad m_l}{m_l}, \tag{58}$$

and, moreover,

$$\dim \Delta^\lambda = \sum_{\nu \vdash r'} |z(\nu)|. \tag{59}$$

For example, for $N = 5$ and $\lambda = \{32\}$, we have $\nu \vdash 2$, and thus either $\nu = \{2\}$, or $\nu = \{1^2\}$. The corresponding rigged string configurations are given by sets

$$\mathrm{z}(\{2\}) = \left\{ \begin{array}{|c|c|}\hline 0 & \\ \hline\end{array} \ \ 1 \ , \ \begin{array}{|c|c|}\hline 1 & \\ \hline\end{array} \ \ 1 \right\}, \tag{60}$$

$$z(\{1^2\}) = \left\{ \begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \end{array} , \ \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \end{array} , \ \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \end{array} \right\}, \tag{61}$$

with riggings L placed inside the row corresponding to a string, and the number P_l of holes - to the right of the $l \times m_l$ rectangle of m_l l-strings (here $l = 2$ and 1 for $\nu = \{2\}$ and $\{1^2\}$, respectively). The counting (59) of dimensions reads here $2 + 3 = 5 = \dim \Delta^{\{32\}}$.

For a slightly more elaborated example of $N = 7$, and $\lambda = \{43\}$, we have

$$z(\{3\}) = \left\{ \begin{array}{|c|c|c|}\hline 0 & & \\ \hline\end{array} \ \ 1 \ , \ \begin{array}{|c|c|c|}\hline 1 & & \\ \hline\end{array} \ \ 1 \right\}, \tag{62}$$

$$\begin{aligned} & z(\{21\}) = \\ & \left\{ \begin{array}{|c|c|}\hline 0 & \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \ \begin{array}{|c|c|}\hline 0 & \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \ \begin{array}{|c|c|}\hline 0 & \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \ \begin{array}{|c|c|}\hline 0 & \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \right. \\ & \left. \begin{array}{|c|c|}\hline 1 & \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \ \begin{array}{|c|c|}\hline 1 & \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \ \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} , \ \begin{array}{|c|c|}\hline 1 & \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ \begin{array}{c}1\\3\end{array} \right\}, \end{aligned} \tag{63}$$

$$z(\{1^3\}) = \left\{ \begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 0 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \\ \ \end{array} , \ \begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 1 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \\ \ \end{array} , \ \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline 1 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \\ \ \end{array} , \ \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline\end{array} \ \begin{array}{c}1\\ \ \\ \ \end{array} \right\}, \tag{64}$$

with the counting $2+8+4 = \dim \Delta^{\{43\}}$.

These definitions, looking somehow strange at the first sight, have found their beautiful and deep physical interpretation within the midst of exact solutions of BA equations. Namely, each rigged string configuration $\nu\mathcal{L}$ classifies an exact eigenstate of the Heisenberg Hamiltonian $\hat{H}$ in each space $\mathcal{H}^r$, $r' \leq r \leq N/2$ (cf. Eq. (43). The eigenstate $|\nu\mathcal{L} >$ is a superposition of magnetic configurations $|f >$, $f \in Q^{(r)}$, and can be obtained as a solution of highly non-linear Bethe equations. It is amasing that this state is fully characterised by the combinatoric data $\nu\mathcal{L}$, which can be simply interpreted in terms of strings. In this combinatoric interpretation, a string of the length l is a collection of l consecutive nodes $j, j+1, \ldots, j+l-1$, each occupied by a spin deviation, followed by next l nodes $j+l, j+l+1, \ldots, j+2l-1$, occupied by a spin $''+'' \in \tilde{2}$. In other words, an l-string is a *composite* system, or a molecule consisting of $2l$ consecutive nodes, first l spins $''-'' \in \tilde{2}$, and then l spins $''+'' \in \tilde{2}$. In the case when there is no other strings, then the remaining $P_l = N - 2l$ nodes of the crystal $\tilde{N}$ play the role of the "sea of holes" for the string configuration $\nu = \{l\}$. The rigging of this l-string is $L = j - 1$, which coincides with the number of holes to the left of the molecule representing this string. All possible riggings of this l-string correspond to various locations of the molecule inside the *open* chain $\tilde{N}$, with distinguished initial and final node. In other words, the left boundary node j of such a string should satisfy $1 \leq j \leq N - 2l$. Clearly, the number of such locations is $|z(\{l\})| = P_l + 1 = N - 2l + 1$, in accordance with Eq. (58). More complex string configurations require some combinatoric extension which we describe in the next paragraph.

2. Bijections of Robinson-Schensted (RS) and Kerov, Kirillov and Reshetikhin (KKR).

In fact, such an apparently simplified, but surprisingly adequate picture of an exact BA solution emerges by virtue of existence of two bijections. The first is Robinson-Schensted correspondence between the set $\tilde{2}^{\tilde{N}}$ of all magnetic configurations and the set b_{irr} of an irreducible basis of the duality of Weyl, and the second - the bijection of Kerov, Kirillov and Reshetikhin between the set b_{irr} and appropriate rigged string configurations. The irreducible basis of the duality of Weyl, written in the form

$$b_{irr} = \bigcup_{0 \leq r' \leq N/2} SYT(\{N-r', r'\}) \times WT(\{N-r', r'\}, \tilde{2}), \tag{65}$$

where $SYT(\lambda)$ is the set of all standard Young tableaux of the shape λ in the alphabet $\tilde{N}$ of nodes, and $WT(\lambda, \tilde{2})$ is the set of all Weyl tableaux (that is, semistandard Young tableaux) of the same shape λ in the alphabet $\tilde{2}$ of spins, with $\lambda = \{N - r', r'\}$, mediates thus between the initial basis $\tilde{2}^{\tilde{N}}$ for quantum computations, and the final basis of rigged string configurations which classify the exact BA eigenstates. Here we consider the *composition* of these two bijections, which runs immediately from magnetic configurations to rigged string configurations, omitting thus the pairs of Young and Weyl tableaux of the duality of Weyl. Moreover, we focus on a restriction of this composite bijection to an arbitrary single orbit $\mathcal{O}_\mu$, $\mu = \{N-r, r\}$, of the symmetric group Σ_N, that is to the classical configuration space $Q^{(r)}$ of the system of r Bethe pseudopaticles on the ring $\tilde{N}$. Thus $Q^{(r)}$ serves as the source of the bijection, and the target is the basis of those exact BA eigenstates which span the space $\mathcal{H}^r = lc_{\mathbb{C}} Q^{(r)}$. We thus describe the composite RSKKR bijection

$$\rho : Q^{(r)} \longrightarrow RC(N, r), \tag{66}$$

where

$$RC(N, r) = \bigcup_{r'=0}^{r} RC(N, r, r'), \tag{67}$$

and

$$RC(N, r, r') = \bigcup_{\nu \vdash r'} z(\nu). \tag{68}$$

Here $RC(N, r)$, $RC(N, r, r')$, and $z(\nu)$ denotes the set of all rigged string configurations on the magnetic ring $\tilde{N}$ which correspond to the system of r Bethe pseudoparticles, all those with the total spin $S = N/2 - r'$ and its projection $M = N/2 - r$, and all riggings of the string configuration $\nu \vdash r'$, respectively. The completeness of exact BA solutions reads

$$lc_{\mathbb{C}}\, RC(N, r, r') = \mathcal{H}^{rS}, \;\; S = N/2 - r', \tag{69}$$

and

$$lc_{\mathbb{C}}\, RC(N, r) = \mathcal{H}^r. \tag{70}$$

We proceed to describe the RSKKR bijection in detail. To this aim, we introduce the notion of a *minimal compensated subword* of a magnetic configuration $f \in Q^{(r)}$. Let

$$v = f(j) f(j+1) \dots f(j+2A-1), \;\; 2A \leq N - j + 1, \tag{71}$$

be a subword of f, consisting of 2A letters of consecutive nodes. Let F^α be the subword of v, consisting of its first α letters, i. e.

$$F^\alpha = f(j) f(j+1) \dots f(j+\alpha-1), \;\; \alpha = 1, 2 \dots, 2A, \tag{72}$$

and let $w(F^\alpha) = \{w_1, w_2\}$ be the weight of the word F^α, so that w_i is the number of occurences of the letter $i \in \tilde{2}$ in F^α, and thus $w_1 + w_2 = \alpha$. By the definition, v is called a minimal compensated subword of f if

$$w_1 \begin{cases} < w_2 & \text{for} \quad \alpha < 2A, \\ = w_2 = A & \text{for} \quad \alpha = 2A. \end{cases} \tag{73}$$

Thus each of A Bethe pseudoparticles entering the word (that is, each $f(j') = {}''\!-\!''$for j' within the range defined by Eq. (71)) is "compensated" (individually!)

by a node j'', $j' < j'' \leq j{+}2A$, occupied by the spin $''+''$, and located *to the right* of it. As the result, a minimal compensated word of the length 2A corresponds to an ensemble of A pairs of nodes (in general *not* adjacent), each pair consisting of a Bethe pseudoparticle, followed by a node occupied by the spin $''+''$.

It is evident that each magnetic configuration f can be decomposed, in a unique way, into consecutive subwords

$$f = u_1 v_1 u_2 v_2 \dots v_a u_{a+1} \tag{74}$$

such that $v_1, v_2, \dots, v_a$ are minimal compensated subwords, whereas $u_1, u_2, \dots, u_{a+1}$ are formed consecutively from the remaining letters of f (some u_i might have zero length, when it happens that v_{i-1} and v_i are adjacent). The weights of subwords

$$v(f) = v_1 v_2 \dots v_a, \;\; u(f) = u_1 u_2 \dots u_{a+1} \tag{75}$$

are given by

$$w(v(f)) = (r', r'), \;\; w(u(f)) = \{N - r' + r, r - r'\}, \tag{76}$$

with

$$r' = \sum_{i=1}^{a} A_i, \tag{77}$$

and satisfy the sum rule

$$w(v(f)) + w(u(f)) = \{N - r, r\}. \tag{78}$$

The main algorithm which determines the RSKKR bijection can be expressed transparently in terms of paths. Let us consider the set of all points in a (j, l)-plane, given by vectors with non-negative integral coefficients, placed inside and at boundaries of the area restricted by the j - axis and the line $l = j$. A *path* is defined recursively with respect to $j = 1, 2, \dots, N$, as a continuous line consisting of consecutive segments such that (i) the segment at the step $j - 1$ joins the point $(j-1, l)$, given by the recursive assumption, with one of the two points: either $(j, l+1)$, corresponding to the step up, or $(j, l-1)$ - to the step down, (ii) the path starts at $(0, 0)$, so that the first segment ends at $(1, 1)$. We denote a path p as a sequence of steps

$$p = (p_1, p_2, \dots, p_N), \;\; p_j \in \{\text{up, down}\}. \tag{79}$$

Now we are ready to determine the key ingredient of the composite bijection. Namely, the path $p(f)$, corresponding to a magnetic configuration f in the RSKKR bijection is given by

$$p_j(f) = \left\{ \begin{array}{lll} \text{up} & \text{if} & f(j) = ''-'', \\ & \text{or if} & f(j) = ''+'' \quad \text{and } f(j) \in u(f), \\ \text{down} & \text{if} & f(j) = ''+'' \quad \text{and } f(j) \in v(f), \end{array} \right\}, j \in \tilde{N}. \tag{80}$$

Thus all Bethe pseudoparticles in a configuration f correspond to steps up in the path $p(f)$, whereas the nodes with $f(j) = ''+''$ correspond to the step either up or down, depending if they belong to the subword $u(f)$ or $v(f)$, respectively.

A path $p(f)$ in the (j, l) - plane looks thus like a contour of a mountain ridge, consisting of N steps up or down, joined together, with local cuspidal maxima and minima. Each subword u_α, $\alpha = 1, 2, \ldots, a+1$, corresponds to a straight line consisting of $|u_\alpha|$ steps up, whereas each subword v_α, $\alpha = 1, 2, \ldots, a$, takes on the shape of a *multipyramid* of the length $2A_\alpha$ whose initial and terminal heights coincide, the intermediate heights are all higher, and some of them reach local maxima (peaks). Within the combinatoric interpretation based on the RSKKR bijection, each straight line u_α belongs to the "sea of holes" for all strings, whereas a multipyramid v_α constitutes a collection of overlapping strings. We proceed to interpret such a multipyramid.

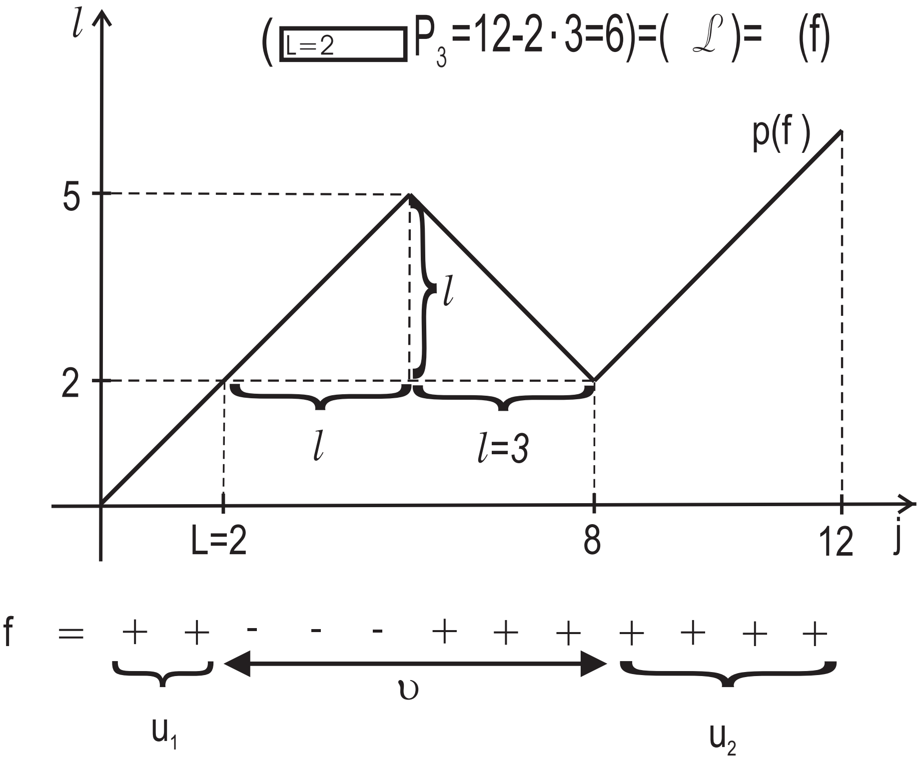

Figure 4: The path $p(f)$ corresponding to a single 3-string for $N = 12$, $r = l = 3$, drawn from the magnetic configuration f according to Eq. (80). The words $u_1 = ++$ and $u_2 = + + ++$ correspond to the sea of holes, and the interior word $v = - - - + ++$ is a minimal compensated subword, corresponding to the 3-string. The 3-string is represented in the path $p(f)$ as a pyramid with the base $2l = 6$ and the height $l = 3$. Location of the pyramid in the sea of holes is determined by the rigging L, $0 \leq L \leq P_3 = N - 2l = 6$ (actually, $L = 2$). The corresponding rigged string configuration $\nu\mathcal{L} = \rho(f)$ is presented in detail above the path p(f)

As we have already mentioned, an isolated l-string consists of l consecutive Bethe pseudoparticles, followed by next l consecutive nodes with the spin $''+''$ (cf. Fig. 4). The corresponding multipyramid constitutes simply a *pyramid* of the height l. Such a pyramid can be located in the sea of holes according to the rigging L, $0 \leq L \leq N - 2l$. One says that the l-pyramid can move, step by step forward or back, on the sea of holes, within the range of its rigging L. Similarly, each of two or more pyramids can move within its range of riggings, obeying two simple *rules of navigation*, resulting from kinematical restrictions which emerge from geometry of the classical configuration space $Q^{(r)}$.
Firstly, two l-strings are mutually inpenetrable, that is, they behave like hard-core objects. It is due to the fact that the strings cannot exchange their positions in the one-dimensional ring $\tilde{N}$ since the order of the queue of Bethe pseudoparticles is fixed.
Secondly, for $l < l'$, the l-string, treated as a pyramid, can slide over both slopes of the l'-string until the heights of their tops coincide (cf. Fig. 5). In fact, the first rule can be looked at as a special case of the second.

As a result, each multipyramid $v(f)$ represents a collection of overlapping strings. An example presented in Fig. 5 clearly shows that the longest string (or strings) within a multipyramid cannot move, but strings of smaller lengthes can make some steps on slopes of larger ones. All strings can move on the sea represented by the word $u(f)$. Now we are in a position to define a rigging L_{lv} of an arbitrary string (l, v) on the path $p(f)$, as *the number of steps to the left*, allowed by the rules formulated above. The range P_l of rigging for those strings, or the number of l-holes, is also readily deduced from these rules to be given by Eq. (55), with $2Q_l$ being interpreted as the number of forbidden nodes due to the "hard-core" requirement: an l-string cannot penetrate all l'-strings with $l' \leq l$, and can slide over both slopes of an l'-string when $l' > l$, making $l' - l$ steps on each side until the corresponding tops achieve equal heights. Thus

$$Q_l = \sum_{l'} \min(l', l) m_{l'}, \tag{81}$$

which coincides with the combinatoric definition of Q_l given previously in Chap. (3.1).

3. Minimal and maximal riggings of a string configuration.

The combined RSKKR bijection ρ, denoted by Eqs. (66)-(68), and specified essentially by the formula (80) and the decomposition (68) of a magnetic configuration f into multipyramids v_α of strings, separated by segments u_α of sea of holes, prescribes to each such magnetic configuration $f \in Q^{(r)}$ a unique rigged string configuration $\nu\mathcal{L} = \rho(f) \in RC(N, r)$. The latter is deduced from the path $p(f)$. This bijection separates, for each $f \in Q^{(r)}$, the nodes $j \in \tilde{N}$ into those bounded into strings (the letters in the alphabet $\tilde{2}$ of spins which enter the subword $v(f)$), and unbounded nodes which form the sea of holes (the letters which enter $u(f)$). The weights of these subwords, $w(v(f))$ and $w(u(f))$, given by Eqs. (76) and (77), define, for a given N and r, the integer r' which determines the total spin $S = N/2 - r'$ of the eigenstate $(v\mathcal{L})(f)$, as well as the total length of all strings forming this eigenstate, so that $\nu \vdash r'$. The distribution of strings among multipyramids v_α, taken together with sizes of segments u_α of the sea of holes, allows one to determine uniquely all riggings $L_{lv} \in \mathcal{L}$ (we do not present all these details here; they are given, e. g., in Warnaar [6]).

For a given string configuration $\nu \vdash r'$ one distinguishes two extreme riggings: the maximal for which $L_{lv} = P_l$, $l = 1, 2, \ldots;\ \ v \in \tilde{m}_l$ (each rigging is maximal), and minimal for which $L_{lv} = 0$, $l = 1, 2, .., v \in \tilde{m}_l$. For the minimal rigging, all strings gather tightly at the left boundary node $j = 1$, and are ordered from left to right according to non-decreasing lengthes l, whereas the maximal rigging corresponds to a mirrow reflection of strings, gathered tightly at right boundary $j = N$, ordered according to non-increasing lengthes. An example of minimal, maximal, and an intermediate rigging for $N = 30$ and $\nu = \{4^2 21^2\}$ is shown in Fig. 6.

We display some typical features of rigged string configurations on an example of all riggings for the case $N = 9$, $\nu = \{21^2\}$ (cf. Fig. 7). A rigged string configuration $\nu\mathcal{L} = \rho(f)$ has now the form

$$\nu\mathcal{L} \quad = \quad \begin{array}{|c|c|} \hline L & \\ \hline L_1 \\ \cline{1-1} L_2 \\ \cline{1-1} \end{array} \quad \begin{array}{ll} 1 & 0 \leq L \leq 1, \\ 3 & 0 \leq L_1 \leq L_2 \leq 3, \end{array} \tag{82}$$

so that L is the rigging of the 2-string, and $L_1 \leq L_2$ are the riggings of the two 1-strings, the latter counted from the left to the right in the open chain $\tilde{N} = \tilde{9}$. All riggings $\nu\mathcal{L}$ are shown together with the corresponding paths $p(f)$ and magnetic configurations $f = \rho^{-1}(\nu\mathcal{L})$ (cf. (66)). This case corresponds to $r' = 4$ Bethe pseudoparticles, coupled into a 2-string ($m_2 = 1$) of the form

$$v_2 = - - ++ \tag{83}$$

and two ($m_1 = 2$) 1-strings, each of the form

$$v_1 = - + . \tag{84}$$

The total number of holes for the 2-string is $P_2 = 9 - 2 \cdot 4 = 1$, whereas for each of the 1-strings is $P_1 = 9 - 2 \cdot 3 = 3$. Thus the total number of different riggings is

$$|z(\{21^2\})| = \left(\begin{array}{c} P_1 + m_1 \\ m_1 \end{array} \right) \left(\begin{array}{c} P_2 + m_2 \\ m_2 \end{array} \right) = \left(\begin{array}{c} 3 + 2 \\ 2 \end{array} \right) \left(\begin{array}{c} 1 + 1 \\ 1 \end{array} \right) = 20. \tag{85}$$

All these riggings are arranged in Fig. 7 into two triangles, corresponding to the rigging $L = 0$ and $L = 1$ of the 2-string. This Figure demonstrates that each rigging $\mathcal{L} = \{L, L_1, L_2\}$, admissible by the selection rules (82), can be uniquely realized by a word f composed of a few minimal compensated subwords and the sea of holes, represented here by a single word

$$u = +. \tag{86}$$

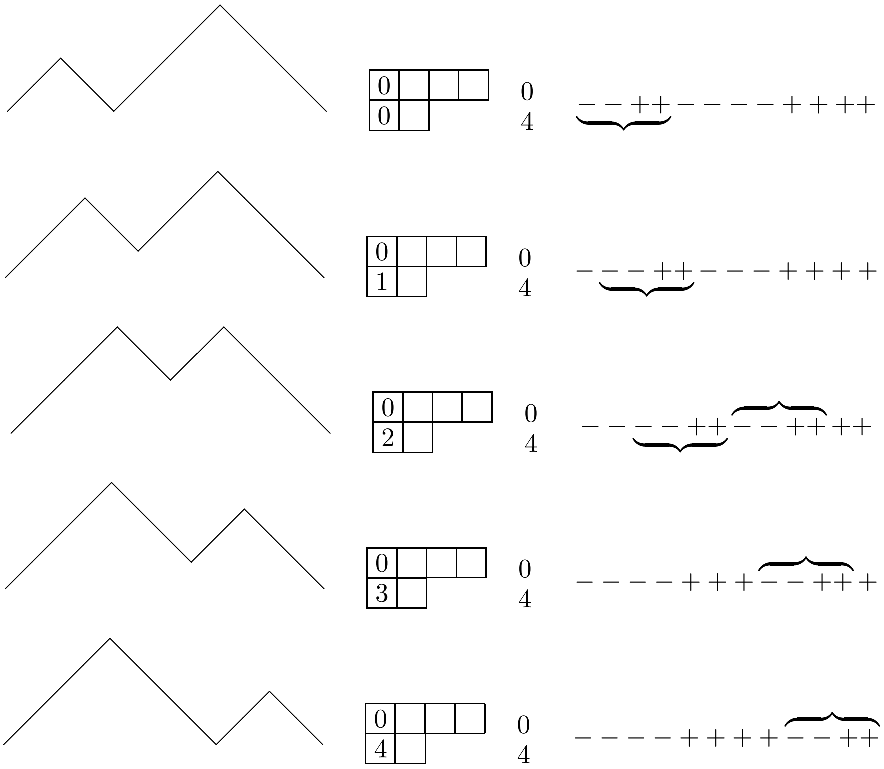

Figure 5: Sliding of pyramids: the 2-string slides over two slopes of the 4-string. The Figure presents the case $N = 12$, $r = 6$, of all $|z(\{42\})| = \binom{0+1}{1}\binom{4+1}{1} = 5$ riggings of the string configuration $\nu = \{4\,2\}$. The first, the second, and the third column of the picture presents the 5 paths $p(f)$, the corresponding rigged string configurations $\nu\mathcal{L} = \rho(f)$, and the initial magnetic configurations f of the Robinson - Schensted algorithm, respectively. The rigged string configurations are ordered consecutively according to the rigging $0 \leq L_2 \leq 4$ of the 2-string, from the minimal configuration $L_2 = 0$ in the top, to the maximal $L_2 = 4$ at the bottom of the Figure. The first three steps ($0 \leq L_2 \leq 2$) correspond to sliding of the 2-string on the left slope of the 4-string, and the last three ($2 \leq L_2 \leq 4$) - on the right slope. Note that the strings are distinguishable by tops of their pyramids in each case with the exception of the third step when these tops have the same height $l = 3$ on the path. In this case, the multipyramid can be interpreted either as $--(--++)--++++$ (the 2-string on the left slope of the 4-string), or as $----++(--++)++$ (the 2-string on the right slope). The maximal and minimal riggings correspond to separation of strings (each string corresponds to a separate compensated subword, $v = --++$, or $v' = ----++++$), whereas intermediate riggings describe here overlapping strings, represented by a single minimal compensated subword.

The minimal and maximal riggings correspond to magnetic configurations

$$f_{\rm min} = v_1 v_1 v_2 u, \tag{87}$$

and

$$f_{\rm max} = u v_2 v_1 v_1, \tag{88}$$

correspond respectively to the lowest leftmost picture of the first triangle, and to the highest rightmost picture of the second triangle. Some riggings correspond merely to the interchange of strings with the sea of holes, that is to the change of the position of the subword u in f, and some other - also to a sliding of 1-string on the 2-string. There are two extra minimal compensated subwords, namely

$$v_3 = -\underbrace{-+}\underbrace{-+}+ = -(-+)-++ = --+(-+)+ \tag{89}$$

and

$$v_4 = -\underbrace{-+}\underbrace{-+}\underbrace{-+}+. \tag{90}$$

The subword v_3 corresponds to a sliding of one of 1-string on the 2-string, at the stage of equal heights of their tops on the corresponding path (that is, the rigging (001), the nearest neighbour of the minimal rigging). In this case one cannot distinguish, which maximum on the path corresponds to a particular string, since both assignments are equivalent. Similarly the subword v_4 corresponds to sliding of both 1-strings on the 2-string, and the latter can be assigned to *any* of the three maxima of the same height; however the order of two 1-strings cannot be changed. It is also worth to observe that the constant rigging does *not* imply a constant position of the corresponding maximum on the path. For example, the rigging L of the 2-string is constant on each triangle, but the position of the peak changes by several steps due to "amoebic" motions resulted from sliding of both 1-strings from one side of the 2-string to another.

4 Behaviour of RSKKR preimages of strings under translations

A rigged string configuration $\nu\mathcal{L}$ labels an exact eigenstate of the Heisenberg Hamiltonian (43) in an appropriate space $\mathcal{H}^{rS}$, and as such, transforms irreducibly under the translation $A(C_N)$, according to a definite quasimomentum k from the Brillouin zone of the crystal $\tilde{N}$. Existence of the RSKKR bijection ρ, however, legitimates the question of behaviour of the preimage $f = \rho^{-1}(\nu\mathcal{L})$ of this eigenstate under such a translation. Moreover, a clear recognition of this question yields a transparent picture of distribution of exact BA eigenstates over the classical configuration space.

Formally, we proceed to consider some properties of the mapping $\mathcal{K}_N : RC(N,r) \longrightarrow RC(N,r)$, defined by

$$\mathcal{K}_N = \rho^{-1} \circ A(C_N) \circ \rho. \tag{91}$$

In other words, $\mathcal{K}_N$ is the push-forward of the shift $A(C_N) : Q^{(r)} \longrightarrow Q^{(r)}$ from the classical configuration space $Q^{(r)}$ to the corresponding set $RC(N, r)$ of exact BA solutions along the RSKKR bijection $\rho : Q^{(r)} \longrightarrow RC(N, r)$. We start with a simple case of a single l-string, i. e. $\nu = \{l\}$.

Clearly, the origin of an l-string, interpreted as the preimage of the bijection ρ applied to the classical configuration space $Q^{(l)}$, is the orbit

$$\mathcal{O}_{\mathbf{t}} = Q^{(l,1)} \subset Q^{(l)}, \; \mathbf{t} = (\underbrace{11 \ldots 1}_{l-1} \; N - l + 1), \tag{92}$$

corresponding to the boundary $F = 1$ of a single island of l consecutive Bethe pseudoparticles. By an application of bijection ρ on this orbit, one readily gets

$$\rho(\mathbf{t}, j) = \begin{cases} \overbrace{\boxed{j-1 \qquad}}^{l} \quad N - 2l \quad \text{for} \quad 1 \leq j \leq N - 2l + 1 \, (\text{regime a}), \\ \overbrace{\boxed{j-1 \quad}}^{l'} \quad N - 2l' \quad \text{for} \quad N - 2l + 2 \leq j \leq N - l \, (\text{regime b}), \\ \emptyset \quad \text{for} \quad j = N - l + 1 \, (\text{regime c}), \\ \overbrace{\boxed{0 \qquad}}^{l'} \quad N - 2l' \quad \text{for} \quad N - l + 2 \leq j \leq N \, (\text{regime d}), \end{cases} \tag{93}$$

with

$$l' = \pm(N + 1 - l - j), \tag{94}$$

and the sign + and - applied for j within the regime b and d, respectively. (cf. Sec. 3.1 for the notation for rigged string configurations $\nu\mathcal{L} = \rho(\mathbf{t}, j)$). These formulas decompose the one-dimensional boundary $Q^{(l,1)}$ of $Q^{(l)}$ into four regimes, denoted in Eq. (93) by a, b, c, d. The regime a corresponds to the whole set $z(\{l\})$ of all riggings of a single l-string. Further translations of magnetic configurations on this C_N-orbit result in shortening of the length l to l', step by step, with the maximal rigging $L_{max} = P_{l'} = N - 2l'$, $1 \leq l' \leq l$ (the regime b). The magnetic configuration $(\mathbf{t}, j)$ for $j = N - l + 1$ (the only configuration in the regime c) corresponds to degenerated ferromagnetic vacuum, i. e.

$$\mathcal{H}^{lN/2} = lc_{\mathbb{C}} \, \rho((11 \ldots 1, N - l + 1), N - l + 1). \tag{95}$$

Note that all Bethe pseudoparticles in the preimage of this vacuum are located at the rightmost end $j = N$ of the chain $\tilde{N}$. The rest of the orbit (the regime d) corresponds to increasing of the length l' of the string from 1 to $l - 1$, with the minimal rigging $L_{min} = 0$. In this way, we arrive at a simple picture in which a (combinatoric) l-string is an extended object of the size $2l$ which itinerates along the orbit $\mathcal{O}_{\mathbf{t}}$ under consecutive C_N-translations from the leftmost node $j = 1$ to the rightmost $j = N$, making $P_l = N - 2l$ steps between consecutive $P_l + 1$

positions in the regime a. Further translations result in decline of the length l' of the string within the regime b, with accompanying emanation of $l-l'$ free Bethe pseudoparticles to the sea of holes, and thus with increasing the total spin S from $\frac{N}{2}-l+1$ to $\frac{N}{2}-1$. The string decreases completely at the step $j = N-l+1$ (the only point of the regime c), which corresponds to $S = N/2$, and all l Bethe pseudoparticles in the sea of holes. The remaining $l-1$ translations (the regime d) describe arising of the string, with the increasing length $1 \leq l' < l$, at the leftmost node $j = 1$ and thus with the minimal rigging, $S = N/2$ and all l Bethe pseudoparticles in the sea of holes. Such a behaviour resemblies a circle of the sun over the sky, with the travel from east to west, sunset, night, and sunrise corresponding to the regime a, b, c, and d, respectively. An example of such a circle is presented in detail in Fig. 8.

More complex cases can be treated in an essentially the same way. Fig. 9 presents the travel of the coupled system of a 1-string and a 3-string (with the string configuration $\nu = \{31\}$) whose tops are separated by 3 lattice units, over the ring of $N = 10$ nodes along the orbit $\mathbf{t} = (1216)$. The corresponding multipyramid

$$v = --+---+++ \tag{96}$$

has the size $2(1 + 3) = 8$, and can be thus located on the open chain of 10 nodes in three different positions. RSKKR preimages of these positions are given by magnetic configurations $(\mathbf{t}, j)$ with $j \in \{1, 2, 3\}$. For the next three nodes, $j \in \{4, 5, 6\}$, the right slope of the 3-string disappears at the terminal node of the chain, so this string is shortened to 2-string ($j = 4$), 1-string ($j = 5$), and vanishes completely at $j = 6$. Each such shortening results in extension of the sea of unbounded nodes by 2: one "+" which arises at the beginning of the chain, and one "-", which is no longer bound to the shortened string. Then, at the nodes $j \in \{7, 8\}$ a string arises at the initial node (1-string and 2-string for $j = 7$ and 8, respectively). In the meantime, the 1-string reaches the terminal node for $j = 8$, then disappears at $j = 9$, and arises again at $j = 10$ as the neighbour of the 2-string. The next step, corresponding to $j = 1$, creates the multipiramid (96) at its leftmost position.

It is evident from these two examples how one can predict the behaviour of rigged string configuration $\nu\mathcal{L} = \rho(f)$, $f \in Q^{(r)}$, along an orbit $\mathbf{t}$ of the translation group C_N. In general, the ridge represented by the path $p(f)$ should be translated by one units to the right in the (l, j)-plane, together with annihilation of the last segment $p(f(N))$ of the path, and an appropriate reconstruction of the left - hand side of this path. To describe this reconstruction in detail, one has to distinguish three cases: (i) $f(N)$="+" and $f(N) \in u(f) \equiv$ the last node is occupied by the spin "+", and belongs to the sea of holes; (ii) $f(N)$="+" and $f(N) \in v(f) \equiv$ the last node is "+" and belongs to the (last) minimal compensated subword $v_a(f) \subset v(f)$; (iii) $f(N)$="-". It is worth to observe that the only possibility in the case (iii) is $f(N) \in u$, by virtue of the combinatoric definition of a string. In other words, a string is annihilated under consecutive C_N - translations, starting from its *right* slope (and thus $f(N)$="+"), so that its left slope never reaches the rightmost node $j = N$ of the chain $\tilde{N}$, and thus

$f(N) \in v$ is excluded. We proceed to describe these three cases in more detail.

The reconstruction of the path $p(f)$ is perhaps the most simple in the case (i). Here, the node $f(N)$, with the spin "+" in the sea of holes, has to be put under a single shift at the first place of the new, shifted magnetic configuration $f' = f \circ C_N^{-1}$. Thus we have

$$u'(f) = u(f),\ v'(f) = v(f), \tag{97}$$

where the prime denotes appropriate path characteristics of the shifted magnetic configuation f' along the notation of Eqs. ((74) - (75)), i. e. $u'(f) = u(f')$, $v'(f) = v(f')$. It follows that the path $p(f')$ is obtained from $p(f)$ by a simple shift by one step along the diagonal $l = j$, with the replacement of the last step "up" by the first step, which is also "up". Clearly, all relative positions between various multipyramids remain unchanged, and therefore

$$\nu' = \nu, \tag{98}$$

i. e. the string configuration of $\mathcal{K}_N(\nu\mathcal{L}) = \nu'\mathcal{L}'$ remains unchanged. Hence, the occupation numbers for all l-string and l-holes remain the same,

$$P_l' = P_l,\ m_l' = m_l,\ l = 1, 2, \ldots, \tag{99}$$

whereas all riggings increase by one step,

$$L_{lv}' = L_{lv} + 1,\ l = 1, 2, \ldots;\ \ v \in \tilde{m}_l. \tag{100}$$

The case (ii), i. e. $f(N) = ''+'' \in v(f)$, consists, in general, in the dissociation of the *last* multipyramid $v_a(f)$ (cf. Eq. (74)) such that

$$v_a(f) = ''-''\, v_a'(f)''+'',\ v_i'(f) = v_i(f) \text{ for } i < a, \tag{101}$$

and correspondingly

$$u'(f) = ''+''\, u(f)''-''\,. \tag{102}$$

It corresponds to diminishing the segment of the path $p(f)$, corresponding to $v_a(f)$, by one vertical step, and deleting one step at each end. Note that $v_a'(f)$, as defined by Eq. (101), is a compensated word, which is not necesarily minimal. For example, if

$$v_a(f) = - - - + + - + + = ''-''\, v_a'(f)''+'', \tag{103}$$

then

$$v_a'(f) = (- - + +)(- +) \tag{104}$$

is the product of two minimal compensated subwords (cf. Fig. 10). The dissociation (101) of the multipyramid $v_a(f)$ to $v_a'(f)$ determines the string which is shortened by one box. We refer to it as to the edge string, and denote its length in $v_a(f)$ by l_e.

$$l_e = \max\{l | lv \in v_a,\ L_{lv} = P_l\}. \tag{105}$$

It follows from the construction that the edge string is singular, i. e. with maximal rigging ($L_{l_e m_{l_e}} = P_{l_e}$), and, moreover, it is a string of the maximal length within the rigged string configuration $\nu\mathcal{L} = \rho(f)$. The new string configuration ν' is obtained from ν by deleting the last box from the edge string (cf. Fig. 11). The difference

$$\nu \setminus \nu' = \Box \tag{106}$$

is a skew Young diagram, consisting of the crossed box of Fig. 10a. It follows that in the case (ii) the numbers of strings change according to

$$m'_l = \begin{cases} m_{l_e} - 1 & \text{for} \quad l = l_e, \\ m_{l_e - 1} + 1 & \text{for} \quad l = l_e - 1, \quad \text{(case (ii))}, \\ m_l & \text{otherwise}, \end{cases} \tag{107}$$

whereas the numbers of l-holes (the ranges of riggings) are

$$P'_l = \begin{cases} P_l + 2 & \text{for} \quad l \geq l_e, \\ & \qquad\qquad\qquad \text{(case (ii))}, \\ P_l & \text{for} \quad 1 \leq l < l_e. \end{cases} \tag{108}$$

The riggings of all strings increase by one step, with the exception of the edge one which remains singular, i. e.

$$L'_{lv} = \begin{cases} L_{lv} + 1 & \text{for} \quad (lv) \neq \text{the edge string}, \\ P_{l'_e} = P_{l_e - 1} & \text{for} \quad \text{the edge string}. \end{cases} \tag{109}$$

The case (iii), i. e. $f(N) = {}''-'' \in u(f)$, corresponds to the *growth* of the edge string at the *left* side of the chain. In this case, the edge string does not necessarily arise from the first multipyramid $v_1(f)$, and thus we have to start from a more extended description. Let

$$|u_1| = |u_2| = \cdots = |u_{b-1}| = 0, \; |u_b| \geq 1, \tag{110}$$

i. e. let the canonical form (74) of the word f be such that the first non-vanishing subword of the sea is u_b, with $1 \leq b \leq a$. Then the (left) edge string, whose length has to be increased by one step due to the shift C_N, is enclosed somewhere in the subword

$$v_B = v_1 v_2 \ldots v_{b-1} \subset v(f). \tag{111}$$

The shift C_N yields the heightening of the corresponding segment of the path $p(f)$ by one vertical step according to the formula

$$v'_B = {}'' -'' \, v_B \, {}''+'' \tag{112}$$

which determines the (left) edge string as the (one of the) longest strings of the word v'_B with the minimal rigging, i. e.

$$l_e = \max\{l | lv \in v'_B, \, L'_{lv} = 0\}. \tag{113}$$

Note that v'_B is a minimal compensated subword in f' ($v'_B = v_1(f')$), even though v_B can be a product of some minimal multipyramids. Observe also that for $b = 1$, $v_B = \varnothing$ is the empty word, and $v'_B = '' - +''$ is the new edge 1-string, arising at the left end of the chain. Now, the new string configuration ν' is obtained from ν by adding a single box to the edge string, so that

$$\nu' \setminus \nu = \square \tag{114}$$

Now we have

$$m'_l = \begin{cases} m_{l_e} - 1 & \text{for} \quad l = l_e, \\ m_{l_e+1} + 1 & \text{for} \quad l = l_e + 1, \quad \text{(case (iii))}, \\ m_l & \text{otherwise}, \end{cases} \tag{115}$$

$$P'_l = \begin{cases} P_l - 2 & \text{for} \quad l > l_e, \\ & \qquad\qquad \text{(case (iii))}, \\ P_l & \text{for} \quad l \leq l_e, \end{cases} \tag{116}$$

and

$$L'_{lv} = \begin{cases} L_{lv} + 1 & \text{for} \quad (lv) \neq \text{the edge string}, \\ 0 & \text{for} \quad \text{the edge string}. \end{cases} \tag{117}$$

The total spin $S' = \frac{N}{2} - r'$ changes in the cases described above according to the formula

$$S' = \begin{cases} S & \text{in the case (i)}, \\ S - 1 & \text{in the case (ii)}, \\ S + 1 & \text{in the case (iii)}. \end{cases} \tag{118}$$

5 Geography of strings in the classical configuration space

We have already demonstrated the way in which the combined RSKKR bijection $\rho : Q^{(r)} \longrightarrow RC(N, r)$ imposes a close relation between the geometry of the classical configuration space $Q^{(r)}$ and the distribution of various admissible rigged string configurations. The combinatoric definition of an l-string as an extended object on the chain $\tilde{N}$, consisting of $2l$ consecutive nodes inside the open chain, implies an immediate connection between the structure of C_N-orbits on $Q^{(r)}$ and the corresponding rigged strings. The string configuration ν of the RSKKR image $\rho(f)$ of the magnetic configuration $f = (\mathbf{t}, j) \in \mathcal{O}_{\mathbf{t}} \subset Q^{(r,F)}$ is essentially defined by the structure $\xi = (\xi_1, \xi_2, \ldots, \xi_F)$ of islands of spin deviations on the C_N-orbit $\mathcal{O}_{\mathbf{t}}$, and the number j of position on this orbit.

Let us start with the simplest case when f is of the form

$$f = u_1 v_1 u_2 v_2 \ldots u_F v_F u_{F+1}, \tag{119}$$

with

$$v_i = \underbrace{- - \cdots -}_{\xi_i}\underbrace{+ + \cdots +}_{\xi_i}, \; i \in \tilde{F}, \tag{120}$$

being an isolated ξ_i-string. Here, the string do not overlap, but form separate pyramids on the path $p(f)$. Separation between i-th and $(i+1)$-th pyramid is provided by the segment u_{i+1} of the sea of holes. The case $|u_i| = 0$, $i \in \tilde{F}/\{1\}$ is admitted; then the corresponding pyramids are adjacent. The case $|u_{F+1}| = 0$ means that the last pyramid, v_F, reaches the rightmost node of the open chain and thus becomes the (right) edge string - in the next $\mathcal{K}_N$-shift it becomes schortened by one box. The case $u_1 = 0$ is also admitted - it means that the string v_1 is the (left) edge one (and has already reached its maximal length). In this case all r Bethe pseudoparticles are bound into strings, $\nu \vdash r$ and thus $S = N/2 - r$, so that $\nu\mathcal{L} = \rho(f)$ is a highest weight eigenstate. It is clear that in such cases the string configuration ν is determined immediately from the structure ξ of islands of spin deviations, namely

$$m_l = |\{i \in \tilde{F} | \xi_i = l\}|. \tag{121}$$

In a general case, the distances between consecutive islands are not long enough which yields an overlap of consecutive strings and formation of multipyramids, corresponding to minimal cpmpensated subwords. Moreover, there can be a shortage of "+" nodes at the right end of the open chain, so that some Bethe pseudoparticles remain unbound into strings, and form the right segment of the sea of holes, of the length $r - r'$. In this case $\nu \vdash r'$, $S = N/2 - r'$, and thus $\nu\mathcal{L} = \rho(f)$ is an eigenstate of a lower weight. The string configuration ν for this case can be resolved unambiguously (but not quite trivially) by a careful determination of content of strings for each constituent multipyramid v_i along the prescription of Warnaar []. The strings can have various lengths, and thus the total spin S, which varies at most by one unit under a single shift $\mathcal{K}_N$ (cf. Eq. 118), can achieve a considerable range of variation along an orbit $\mathcal{O}_{\mathbf{t}}$. For example, the orbit corresponding to a single island of the length r, i. e. $\mathbf{t} = (11\ldots1\,N-r)$, admits all values of the total spin. It is worth to observe that the total number of strings,

$$q(f) = \sum_{l \geq 1} m_l, \tag{122}$$

can be deduced immediately from the shape of the path $p(f)$ - it is just equal to the number of cuspidal maxima *inside* the open chain $\tilde{N}$. Clearly the variation of $q(f)$ along an orbit $\mathcal{O}_{\mathbf{t}}$ is much smaller than that of S, namely

$$q(\mathbf{t}, j) \in \{F, F-1\} \text{ for } \mathcal{O}_{\mathbf{t}} \subset Q^{(r,F)}. \tag{123}$$

It follows from the observation that at most one island can approach the rightmost end of the open chain, and thus not to contribute to a string, at each step along the orbit $\mathcal{O}_{\mathbf{t}}$. Moreover, the number of occurences that $q(\mathbf{t}, j) = F - 1$ along an orbit $\mathcal{O}_{\mathbf{t}}$ (possibly κ - tuply rarefied) is

$$|\{j \in \{1, 2, \ldots, N/\kappa\} | q(\mathbf{t}, j) = F - 1\} = F/\kappa. \tag{124}$$

In other words, each of F strings on an orbit $\mathcal{O}_{\mathbf{t}} \subset Q^{(r,F)}$ disappears at exactly one step of the itinerary along this orbit. In general, strings can be labelled by their parent islands of spin deviations, and traced by appropriate cuspidal maxima on path $p(f)$. These considerations clealy indicate, by an immediate construction, that each string originates in an island of consecutive Bethe pseudopaticles on the classical configuration space. The length l of the string, originated from the i-th island, cannot exceed the size ξ_i of this island, but can be smaller, i. e.

$$0 \leq l \leq \xi_i. \tag{125}$$

Evidently, a string cannot arise from two or more separated islands. In particular, the generic stratum $Q^{(r,r)}$ $(F = r)$ of the classical configuration space yields only 1-strings, i. e.

$$\rho(Q^{(r,r)}) \subset \bigcup_{r'=1}^{r} z(1^{r'}). \tag{126}$$

We demonstrate the geography of strings on the classical configuration space for $r = 3$ Bethe pseudoparticles on the ring of $N = 9$ nodes (Tables 1 and 2). The classical configuration space $Q^{(r)}$, consisting here of $|Q^{(3)}| = 84$ magnetic configurations, decomposes into subsets $Q^{(3,f)}$, $F = 1, 2$, and 3. Table 1 provides a complete map $\rho : Q^{(3)} \longrightarrow RC(9, 3)$, arranged into 3 subtables, each for a particular local dimension F. Rows of each subtable are labelled by $j \in \tilde{9}$, columns - by the C_9-orbits $\mathbf{t} = (t_1, t_2, t_3)$, and the entries are $\rho(\mathbf{t}, j) = \nu\mathcal{L}$. One can observe the change of the rigged string configurations along C_9-orbits (each column of a table), and along the skeleton $\mathbf{t}$ (each row). One sees at the first sight that the 3-strings occur in the one-dimensional boundary $Q^{(3,1)}$ only, the 2-strings enter both $Q^{(3,1)}$ (as the edge strings for the case $r' = 2$), and the two-dimensional boundary $Q^{(3,2)}$ (for $r' = 2$ and $r' = 3$), and the 1-strings are present everywhere. The generic stratum $Q^{(3,3)}$, however, consists of 1-strings only. It is easy to make further observations concerning the distribution of a particular set $z(\nu)$, the variation of riggings, positions of the edge strings etc., but we do not enter all such details here. We give in Table 2 a complete array of cordinates of intersections $|Q^{(r,F)} \cap z(\nu)|$ which presents the distribution of riggings of each string configuration $\nu \vdash r'$, $0 \leq r' \leq 3$, over appropriate pieces of the classical configuration space $Q^{(3)}$. The sum rules presented in this Table demonstrate completeness of the set RC(9,3) of exact BA solutions for this case.

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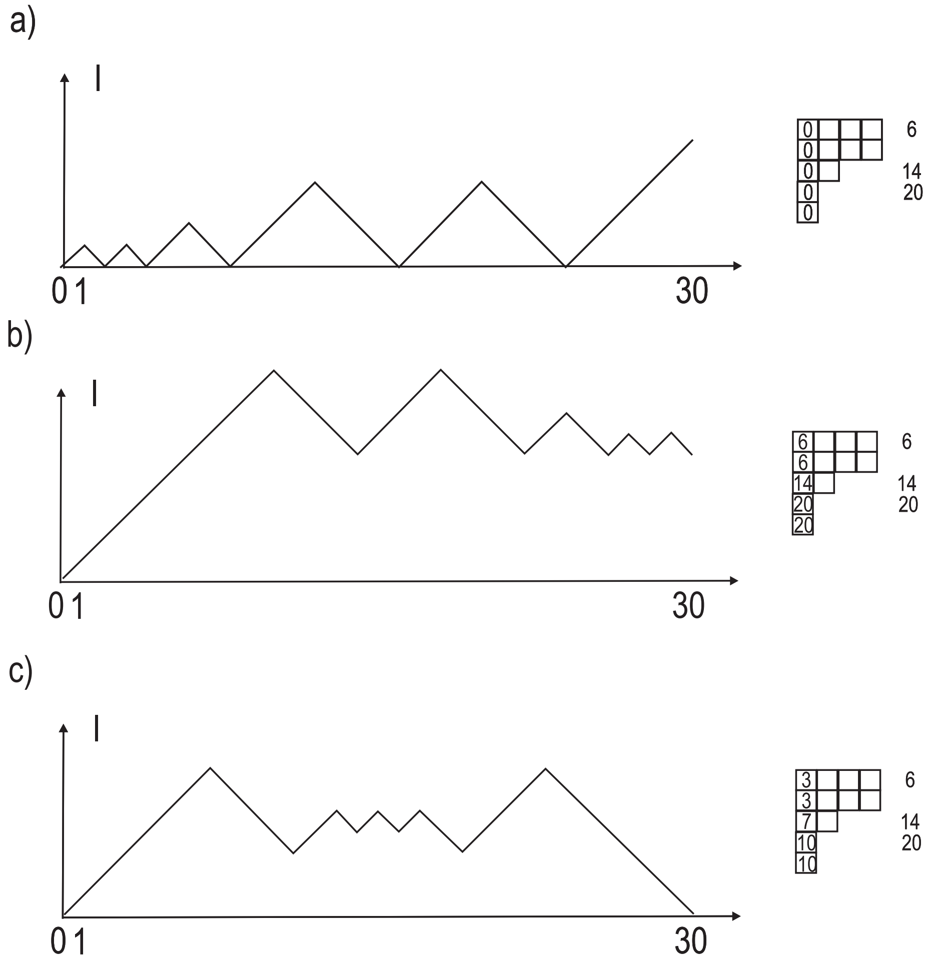


Figure 6: Specified riggings for the string configuration $\nu = \{4^2 2 1^2\}$ for $N = 30$: a) the minimal, b) the maximal, c) all strings achieve their half - maximal riggings.

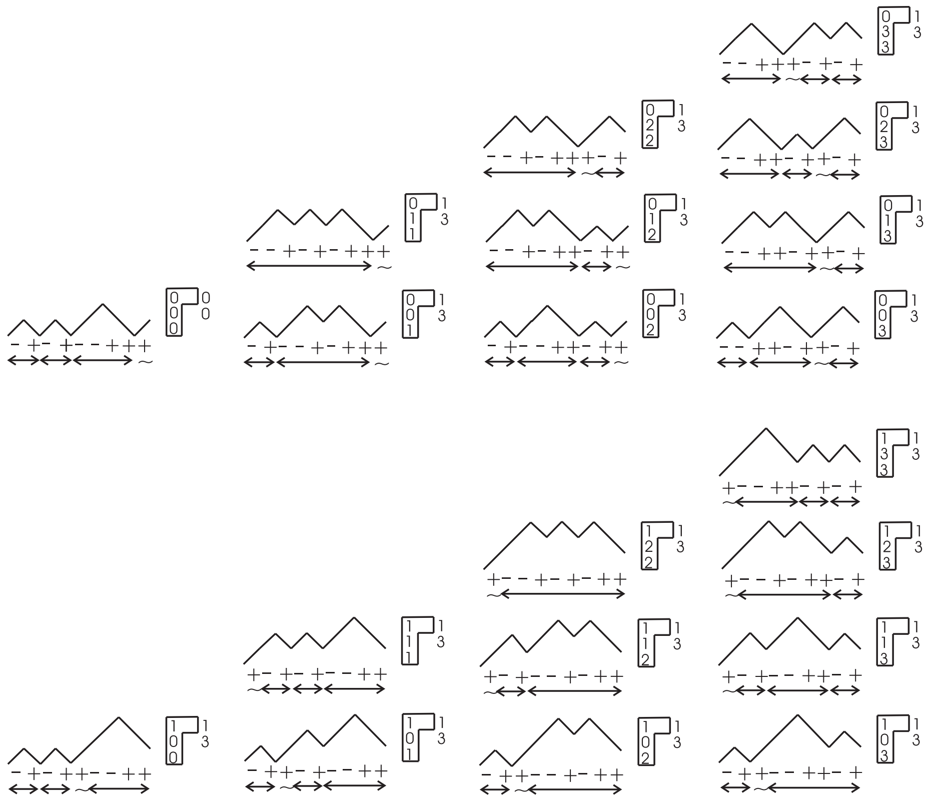


Figure 7: The set of all riggings $\nu\mathcal{L}$ of the string configuration $\nu = \{21^2\}$ for $N = 9$. All rigged string configurations are arranged into two triangles, corresponding to $L = 0$ and $L = 1$. Each $\nu\mathcal{L}$, given by Eq. (82), is accompanied from the left by the corresponding path $p(f)$ (Eqs. (79)-(80)), together with the magnetic configuration $f = \rho^{-1}(\nu\mathcal{L})$ below (cf. (66)). Arrows below the magnetic configuration f indicate minimal compensated subwords, and the (single) wavy line - the sea of holes for the 2-string. Within each triangle, the rigging L_1 of the first 1-string, and L_2 of the second, increases with the step one, respectively horisontally and vertically.

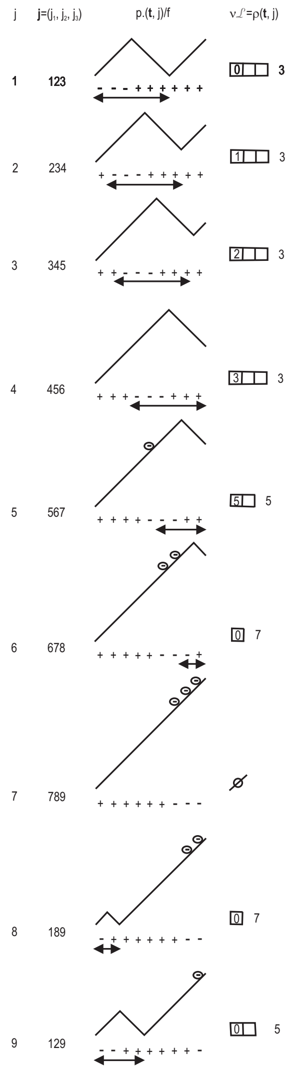


Figure 8: Travel of the 3-string on the C_9-orbit $\mathbf{t} = (117)$ for $N = 9$, $r = l = 3$. The first column provides the consecutive number j in the natural order of the C_9-orbit, the second - the corresponding magnetic configuration $\mathbf{j} = (j_1, j_2, j_3)$, the third - the path $p(\mathbf{t}, j)$, together with the correcponding magnetic configuration below, and the fourth - the rigged string configuration $\nu\mathcal{L} = \rho(\mathbf{t}, j)$. The regime a $(1 \leq j \leq 4)$ corresponds to the travel of the 3-string on the orbit $\mathcal{O}_{\mathbf{t}}$ from the left to the right end of the open chain $\tilde{9}$, the regime b $(5 \leq j \leq 6)$ presents the "sunset" of the 3-string, resulting in consecutive shortening of its length and the corresponding change of riggings, the regime c $(j = 7)$ denotes "the night", i. e. a complete disappearance of the 3-string, and the regime d $(8 \leq j \leq 9)$ provides the "sunrise" of the 3-string, by consecutive increase of its length. In each magnetic configuration f (below the path $p(\mathbf{t}, j)$), each string is enclosed by a solid line; the rest of nodes corresponds to the sea of holes. The length of the sea of holes to the left of a string is the rigging of this string. The sea of holes consist of either nodes with the sign "+" (the regime a, $P_3 = 3$), or nodes with both "+" and "-"; in the latter case all "-" nodes are placed to the right of unbounded "+" nodes. Such unbounded "-" nodes result from a decay or shortening of a string; they are additionally denoted by the sign "-" over an appropriate step of the path $p(\mathbf{t}, j)$. In particular, the case of $j = 7$ (the regime c, the "night") corresponds to a complete lack of strings, so that the sea fills in the whole crystal $\tilde{9}$ (with three "-" nodes at the end); such a case corresponds to the ferromagnetic vacuum ($S_{\max} = 9/2$) counterpart in the space $\mathcal{H}^3$.

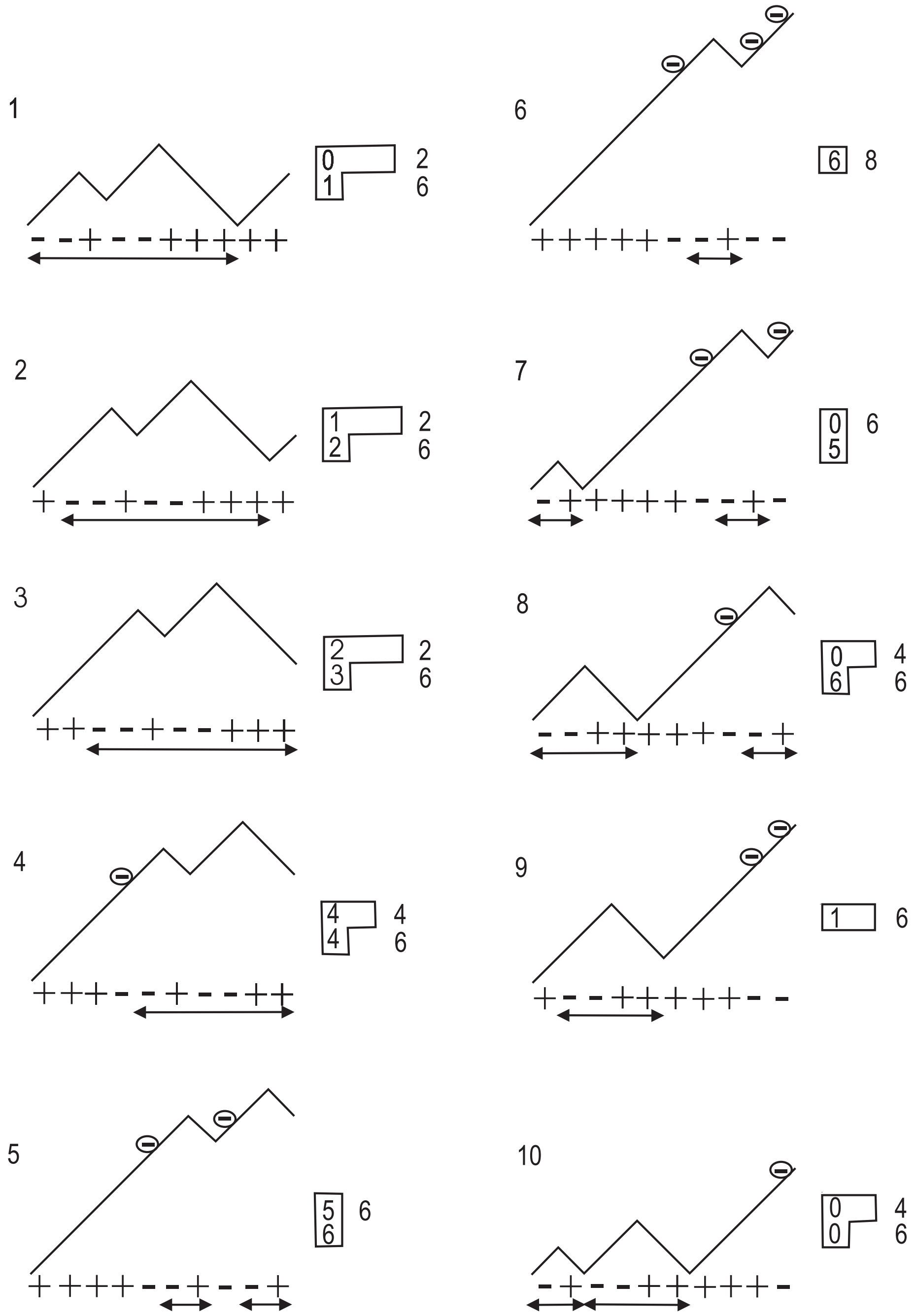


Figure 9: The RSKKR bijection for the C_{10}-orbit $\mathbf{t} = (1216)$ ($N = 10$, $r = 4$).

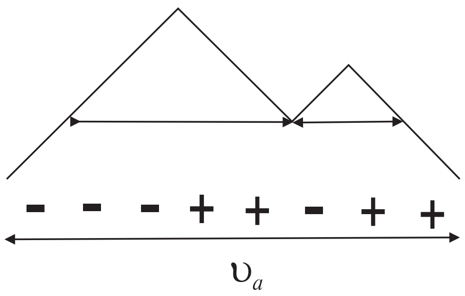


Figure 10:

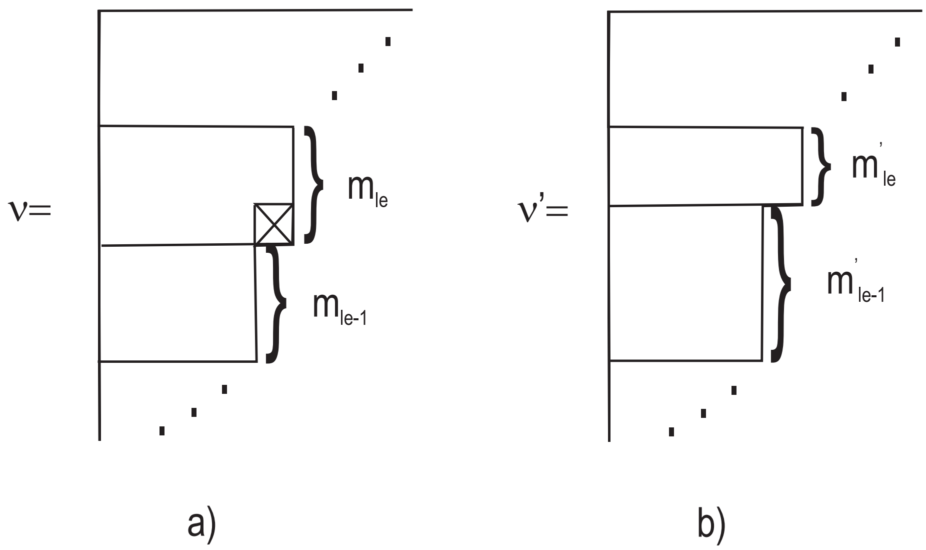
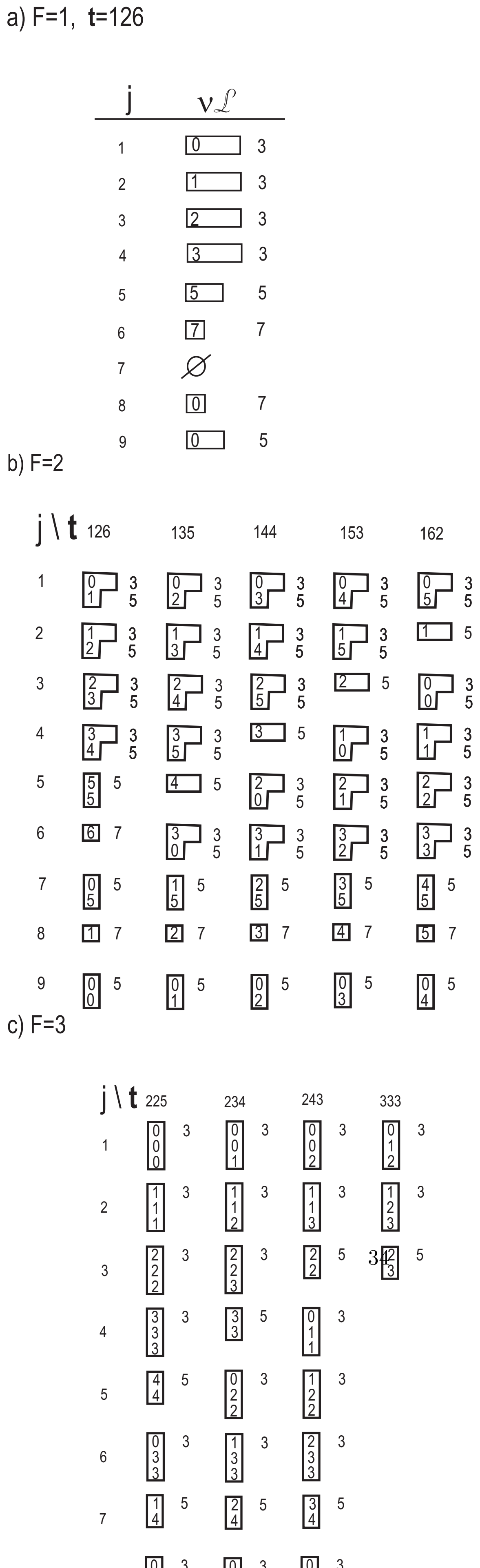


Figure 11: The change of string configuration under the shortening of the string (the case (ii), $f(N) ='' +'' \in v(f)$). a) the string configuration ν, coresponding to f; the edge string, of the length l_e, is designated by the cross in its last box; the shortening consists in removing this box, b) the string configuration ν', corresponding to $f' = f \circ C_N^{-1}$. Both figures display the numbers of l_e-strings and (l_e^{-1}) strings.


a) F=1, t=126
j
νℒ
b) F=2
j \ t
126
135
144
153
162
c) F=3
j \ t
225
234
243
333

r'	ν	F=1	F=2	F=3	$\lvert z(\nu)\rvert$	dim $\Delta^{\{9-r',r'\}}$
0	∅	1	0	0	1	1
1	7	2	6	0	8	8
2	5	2	4	0	6	27
	5	0	11	10	21	
3	3	4	0	0	4	
	3 5	0	24	0	24	48
	3	0	0	20	20	
		$\lvert Q^{(3,1)}\rvert=9$	$\lvert Q^{(3,2)}\rvert=45$	$\lvert Q^{(3,3)}\rvert=30$		$84=\lvert Q^{(3)}\rvert$

Table 2: Distribution of strings over the classical configuration space $Q^{(r)}$ for $N = 9$. The column F gives the integer $|Q^{(r,F)} \cap z(\nu)$.