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Hopf algebras of diagrams

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ABSTRACT. We investigate several Hopf algebras of diagrams related to the Quantum Field Theory of Partitions and whose product comes from **WSym** or **WQSym**, the algebras of word symmetric or quasi-symmetric functions. Bases of these algebras are indexed either by bipartite graphs (labelled or unlabelled) or by packed matrices (with integer or set coefficients). Realizations on bi-words are exhibited, and it is shown how these algebras fit well into a commutative diagram. Hopf deformations and dendriform structures are also considered for some algebras in the picture.

RÉSUMÉ. Nous étudions plusieurs algèbres de Hopf de diagrammes liées à la Théorie Quantique des Champs des Partitions et dont le produit provient de **WSym** et **WQSym**. Ces algèbres ont des bases indexées à des graphs bipartis (étiquetés ou non) ou à des matrices tassées (à cœfficients entiers ou ensemblistes). On donne des réalisations sur des bi-mots et on montre comment ces algèbres s'intègrent dans un diagramme commutatif. Pour certaines d'entre elles, on établit des structures dendriformes et l'existence de déformations préservant la structure de Hopf.

1. Introduction

The purpose of the present paper is twofold. First, we want to tighten the links between a body of Hopf algebras related to physics and the realm of noncommutative symmetric functions, although the latter domains are no longer disconnected [**10, 5, 8**]. Second, we aim at providing examples of combinatorial shifting (a generic way of deforming algebras) and expounding how **MQSym** could be considered a construction scheme including its first appearance with integers [**7**] as a special case.
Our paper is the continuation of [**3**], as we go deeper into the connections between combinatorial Hopf algebras and the Feynman diagrams of a special Field Theory introduced by Bender, Brody and Meister [**1**]. These Feynman diagrams arose in the expansion of

$$G(z) = \exp\left\{\sum_{n\geq 1}\frac{L_n}{n!}\left(z\frac{\partial}{\partial x}\right)^n\right\}\exp\left\{\sum_{m\geq 1}V_m\frac{x^m}{m!}\right\}\Bigg|_{x=0} \tag{1}$$

and are bipartite finite graphs with no isolated vertex, and edges weighted with integers. They are in bijective correspondence with packed matrices of integers up to a permutation of the columns and a permutation of the rows. The algorithm constructing the matrix from the associated diagram uses as an intermediate structure a particular packed matrix whose entries are sets. Such set matrices appear when one computes the internal product in **WSym** [**15**] and in **WQSym** [**11, 13**], then isomorphic to the Solomon-Tits algebra. In this context, it becomes natural to investigate Hopf algebras of (set) packed matrices whose product comes from **WSym** or **WQSym**.
The paper is organized as follows. In Section 2, the connection between the Quantum Field Theory of Partitions and a three-parameter deformation of the Hopf algebra **LDIAG** of labelled diagrams is explained. We introduce the shifting principle (Subsection 2.2) and give two illustrations. The first one enables to see **LDIAG** as the shifted version of an algebra of unlabelled diagrams. The second one (Subsection 2.3) explains how to carry over some constructions from algebras of integer matrices to algebras of set matrices.
In Section 3, we investigate eight Hopf algebras of matrices related to labelled or unlabelled diagrams. In particular, we exhibit realizations on biwords and show how some of these are dendriform bialgebras.

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2. Hopf algebras coming from physics

2.1. Algebras of diagrams. Many computations carried out by physicists reduce to the 'product formula', a bilinear coupling between two Taylor expandable functions, introduced by C.M. Bender, D.C. Brody, and B.K. Meister in their celebrated *Quantum field theory of partitions* (henceforth referred to as QFTP) [**1**]. For an example of such a computation derived from a partition function linked to the Free Boson Gas model, see [**17**].
To make the story short, the last expansion of the formula involves a summation over all diagrams of a certain type [**1, 4**] a labelled version of which is described below. These diagrams are bipartite graphs with multiple edges. It is in order to show that every (combinatorial) sequence of integers can be represented by Feynman diagrams subjected to suited rules that Bender, Brody and Meister [**1**] introduced QFTP as a toy model.
The case when the expansions of the two functions occurring in the product formula have constant term 1 is of special interest. The functions can then be presented as exponentials which can be regarded as "free" through the classical Bell polynomials expansion [**5**] or as coming from the integration of a Frechet one-parameter group of operators [**4**].
Working out the formal case, one sees that the coupling results in a summation without multiplicity of a certain kind of labelled bipartite graphs which are equivalent, as a data structure, to pairs of unordered partitions of the same set $\{1,2,...,n\}$. The sum can be reduced to a sum of topologically inequivalent diagrams (a monoidal basis of **DIAG**), at the cost of introducing multiplicities. Theses graphs, which can be considered as the Feynman diagrams of the QFTP, generate a Hopf algebra compatible with the product and co-addition on the multipliers. Interpreting **DIAG** as the Hopf homomorphic image of its planar counterpart, **LDIAG**, gives access to the noncommutative world and to deformations: the product is deformed by taking into account, through two variables, the number of crossings of edges involved in the superposition or the transposition of two vertices, the coproduct is obtained by interpolating. This gives the final picture of [**17**].
Labelled diagrams can be identified with their weight functions which are mappings $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ such that the supporting subgraph

$$\Gamma_\omega = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i,j) \neq 0\} \tag{2}$$

has projections *i.e.*, $pr_1(\Gamma_\omega) = [1,p]$; $pr_2(\Gamma_\omega) = [1,q]$ for some $p,q \in \mathbb{N}^+$.

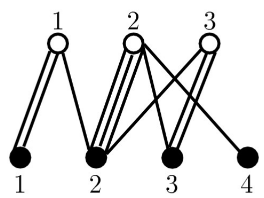


FIGURE 1. A labelled diagram of shape 3×4.

Let **ldiag** denote the set of labelled diagrams. With any element d of **ldiag**, one can associate the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$, called its multiplier, where $\alpha(d)$ (resp. $\beta(d)$) is the "white spot type" (resp. the "black spot type") *i.e.*, the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ (resp. $(\beta_i)_{i\in\mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i. For example, the multiplier of the labelled diagram of Figure 1 is $\mathbb{L}^{(0,0,2,0,1)}\mathbb{V}^{(1,1,1,0,1)}$.
One can endow **ldiag** with an algebra structure denoted by **LDIAG** where the sum is the formal sum and the product is the shifted concatenation of diagrams, *i.e.* consists in juxtaposing the second diagram to the right of the first one and then adding to the labels of the black spots (resp. of the white spots) of the second diagram the number of black spots (resp. of white spots) of the first diagram. Then the application sending a diagram to its multiplier is an algebra homomorphism.
Moreover, the black spots (resp. white spots) of diagram d can be permuted without changing the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$. The classes of labelled diagrams up to this equivalence relation (permutations of white – or black –

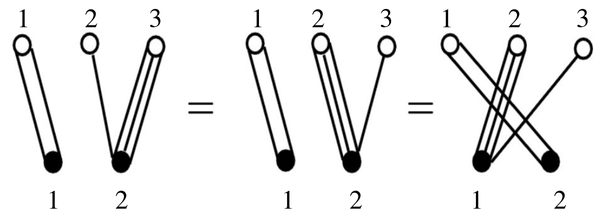


FIGURE 2. Equivalent labelled diagrams.

spots among themselves) are naturally represented by unlabelled diagrams. The set of unlabelled diagrams will be henceforth denoted by **diag**.
The set **diag** can also be endowed with an algebra structure, denoted by **DIAG**, *e.g.* as the quotient of **LDIAG** by the equivalence classes of labelled diagrams. In **DIAG**, the product of d_1 by d_2 is basic concatenation, *i.e.* simply consists in juxtaposing d_2 to the right of d_1 [**5**].

2.2. Shifted algebras and applications. A three-parameter deformation of **LDIAG**, **LDIAG**(q_c, q_s, t) has been recently constructed, which specializes to both **LDIAG** ($q_c = q_s = t = 0$) and **MQSym** ($q_c = q_s = t = 1$). This construction involves a deformation of the algebra structure, which can be seen as a particular case of the rather general principle of *shifting*. This principle will be further exemplified in the sequel of the present paper.

LEMMA 2.1. *(Shifting lemma.)*
Let $\mathscr{A} = \oplus_{\alpha \in M} \mathscr{A}_\alpha$ *be an algebra graded (as a vector space) on a commutative monoid* $(M, +)$. *Let* $s : (M, +) \mapsto (\mathrm{End}_{alg}(\mathscr{A}), \circ)$ *be an homomorphism such that the modified law, given by*

$$x *_s y := x\, s_\alpha(y) \text{ for all } x \in \mathscr{A}_\alpha,\ y \in \mathscr{A} \tag{3}$$

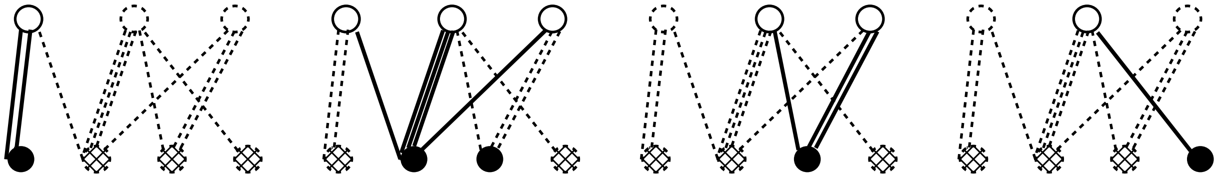
is M*-graded. Then, if* $\mathscr{A}$ *is associative, so is the deformed law* $*_s$.

Such a procedure, whenever possible, will be called the *shifting of* $\mathscr{A}$ *by the shift* s. We now recall the construction of **LDIAG**(q_c, q_s) as the shifting of another algebra of diagrams.

Let $M = \mathbb{N}^{(\mathbb{N}^+)}$ be the additive monoid of multidegrees and $M^+ = M - \{0\}$ the associated semigroup. A labelled diagram d with p white spots and q black spots can be encoded by a word $W(d) \in (M^+)^*$ as

$$W(d) = \alpha_1 \alpha_2 \cdots \alpha_q, \tag{4}$$

where, for all $i \leq p$, the i-th letter of α_j is the number of edges joining the black spot j and the white spot i (see Figure 3).



The edges adjacent to the blackspots correspond successively to the multidegrees (2) (two edges to the first white spot), $(1,3,1)$ (one edge to the first and third white spots and three edges to the second one), $(0,1,2)$ and $(0,1)$. Thus the code is $W(d) = (2)(1,3,1)(0,1,2)(0,1)$.

FIGURE 3. Coding a diagram with a word of multidegrees.

Let k be a ring, one can skew the product in $k\langle M^+\rangle$, counting crossings and superpositions as for **LDIAG**.

PROPOSITION 2.2. *Let k be a ring, $q_c, q_s \in k$, and consider the deformed graded law on $k\langle M^+\rangle$ defined by*

$$\left\{\begin{array}{rcl} 1_{(M^+)^*} * w & = & w * 1_{(M^+)^*} = w, \\ \alpha u * \beta v & = & \alpha(u * \beta v) + q_c^{|\alpha u||\beta|}\beta(\alpha u * v) + q_s^{|\alpha||\beta|} q_c^{|u||\beta|}(\alpha+\beta)(u * v), \end{array}\right. \tag{5}$$

where $\alpha, \beta \in M^+$, $u, v \in (M^+)^*$, the weight $|\alpha|$ of multidegree α is just the sum of its coordinates and the weight $|u|$ of the word $u = \alpha_1 \cdots \alpha_t$ is $|u| = \sum_{i=1}^t |\alpha_i|$.
This product is associative.

The algebra $(k\langle M^+\rangle, +, *)$ is denoted by $\mathbf{MLDIAG}(q_c, q_s)$. In the shifted version, the product amounts to performing all superpositions of black spots and/or crossings of edges, weighting them with the corresponding value q_s or q_c, powered by the number of crossings of edges.
This construction is reminiscent, up to the deformations, of Hoffman's [**12**] and its variants [**2, 9**], and also of an older one, the infiltration product in computer science [**14, 6**].

The shift going from $\mathbf{MLDIAG}(q_c, q_s)$ to itself is the following. Let $\alpha_1\alpha_2\cdots\alpha_p \in (M^+)^*$ and $n \in \mathbb{N}$. One sets

$$s_n(\alpha_1\alpha_2\cdots\alpha_p) := (0^n\alpha_1)(0^n\alpha_2)\cdots(0^n\alpha_p) \tag{6}$$

where $0^n\alpha$ is the insertion of n zeroes on the left of α. Note that $n \mapsto s_n$ is a homomorphism of monoids $(\mathbb{N}, +) \to \mathrm{End}_{alg}\big(\mathbf{MLDIAG}(q_c, q_s)\big)$.

2.3. Another application of the shifting principle. The Hopf operations of **MQSym** as described in [**7**] do not depend on the fact that the entries of the matrices are integers.
For a pointed set (X, x_0) (*i.e.*, $x_0 \in X$), let us denote by $\mathbf{MQSym}_k(X, x_0)$ the k-vector space spanned by rectangular matrices with entries in X with no line or column filled with x_0 (which plays now the rôle of zero) and the product, coproduct, unit and counit as in [**7**]. It is clear that $\mathbf{MQSym}_k(X, x_0)$ is a Hopf algebra and that the correspondence $(X, x_0) \mapsto \mathbf{MQSym}_k(X, x_0)$ is a functor from the category of pointed sets (endowed with the strict arrows, that is, the mappings $\phi\ :\ (X, x_0) \to (Y, y_0)$ such that $\phi(x_0) = y_0$ and $\phi(X - \{x_0\}) \subset Y - \{y_0\}$) to the category of k-Hopf algebras.
In the particular case when $X = 2^{(N^+)}$, that is, finite subsets of $\mathbb{N}^+$, and $x_0 = \emptyset$, one can define a shift by a translation of the elements. More precisely, for $F \in 2^{(N^+)}$ and M a $p \times q$ matrix with coefficients in $2^{(N^+)}$, one sets

$$s_n(F) = \{x + n\}_{x \in F}\ ,\ s_n(M) = \big(s_n(M[i,j])\big)_{\substack{1 \le i \le p \\ 1 \le j \le q}}\ . \tag{7}$$

One can check that the s_n define a shift on $\mathbf{MQSym}_k(2^{(N^+)}, \emptyset)$ for the grading given by

$$\mathbf{MQSym}_k(2^{(N^+)}, \emptyset)_n = \text{span of the matrices whose maximum number in the entries is } n. \tag{8}$$

For example, the vector space generated by the packed matrices whose entries partition the set $\{1, 2, \cdots n\}$ is closed by the product. This is the algebra **SMQSym**. We shall soon see that this corresponds to labelling also the edges of a labelled diagram with numbers from 1 to k.

3. Packed matrices and related Hopf algebras

3.1. The combinatorial objects. In the sequel, we will represent different kinds of diagrams using matrices to emphasize the parallel between this construction and the construction of **MQSym** ([**7**]).

3.1.1. *Set packed matrices.* Since the computations are the same in many cases, let us begin with the most general case and explain how one recovers the other cases by algebraic means. Let us consider the set **lldiag** of bipartite graphs with white and black vertices, and edges, all three labelled by intervals $[1, p]$. The diagrams **ldiag** are obtained by erasing the numbers the edges of one such element.
The set **lldiag** is in direct bijection with *set packed matrices*, that are matrices containing disjoint subsets of $[1, n]$ for some $n \in \mathbb{N}$ with no line or column filled with empty sets such that the union of all subsets is $[1, n]$ itself. The bijection consists in putting k in the cell (i, j) of the matrix if the edge labelled k connects the white dot labelled i with the black dot labelled j. Figure 4 shows an example of such a matrix.

$$\begin{pmatrix} \{3\} & \{6\} & \{2\} \\ \emptyset & \{1,5\} & \{4\} \end{pmatrix}$$

FIGURE 4. A set packed matrix.

Note that set packed matrices are in bijection with pairs of set compositions, or, ordered partition of $[1,n]$: given a set packed matrix, compute the ordered sequence of the unions of the elements in the same row (resp. column). For example, the set packed matrix of Figure 4 gives rise to the two set compositions $\{2,3,6\},\{1,4,5\}$ and $\{3\},\{1,5,6\},\{2,4\}$. Given two set compositions Π and Π', define $M_{ij} := \Pi_i \cap \Pi'_j$.
It is then easy to compute that the generating series depending on n of such matrices is given by the square of the ordered Bell numbers, that is sequence A122725 in [**16**].

3.1.2. *Integer packed matrices.* As already said, if one forgets the ordering of the edges of an element of **lldiag**, one recovers an element of **ldiag**. Its matrix representation is an *integer packed matrix*, that is, a matrix with no line or column filled with zeros. The encoding is simple: m_{ij} is equal to the number of edges between the white spot labelled i and the black spot labelled j. Note that, from the matrix point of view, it consists in replacing the subsets by their cardinality. Figure 5 shows an example of such a matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

FIGURE 5. An integer packed matrix.

The generating series of such matrices depending on n is given by sequence A120733 of [**16**].

3.1.3. *Other packed matrices.* During our study, we will also need diagrams where one forgets about the labels of the white spots, or about the labels of the black spots, or about all labels. Those three classes of diagrams are respectively in bijection with matrices up to a permutation of the rows, a permutation of the columns, and simultaneous permutations of both.

3.2. Word quasi-symmetric and symmetric functions. Let us recall briefly the definition of two already known combinatorial Hopf algebras that will be useful in the sequel.

3.2.1. **WQSym**. We use the notations of [**13**]. The word quasi-symmetric functions are the noncommutative polynomial invariants of Hivert's quasi-symmetrizing action [**11**]

$$\mathbf{WQSym}(A) := \mathbb{C}\langle A\rangle^{\mathfrak{S}(A)_{QS}}. \tag{9}$$

When A is an infinite alphabet, $\mathbf{WQSym}(A)$ is a graded Hopf algebra whose basis is indexed by set compositions, or, equivalently, packed words. Recall that packed words are words w on the alphabet $[1,k]$ where if $i \neq 1$ appears in w, then $i-1$ also appears in w. The bijection between both sets is that $w_i = j$ iff i is in the j-th part of the set composition (see Figure 6 for an example).

$$cbbadacb \qquad \longleftrightarrow \qquad [\{4,6\},\{2,3,8\},\{1,7\},\{5\}]$$

FIGURE 6. A packed word and its corresponding set composition.

By definition, **WQSym** is generated by the polynomials

$$\mathbf{WQ}_u := \sum_{\mathrm{pack}(w)=u} w, \tag{10}$$

where $u = \mathrm{pack}(w)$ is the packed word having the same comparison relations between all elements as w.
The product in **WQSym** is given by

$$\mathbf{WQ}_u \mathbf{WQ}_v = \sum_{w \in u \star_W v} \mathbf{WQ}_w, \tag{11}$$

where the convolution $u \star_W v$ of two packed words is defined by

$$u \star_W v = \sum_{\substack{w_1,w_2; w=w_1w_2 \text{ packed} \\ \mathrm{pack}(w_1)=u,\ \mathrm{pack}(w_2)=v}} w. \tag{12}$$

The coproduct is given by

$$\Delta \mathbf{WQ}_w(A) = \sum_{u,v; w\in u \Cup_W v} \mathbf{WQ}_u \otimes \mathbf{WQ}_v \tag{13}$$

where $u \Cup_W v$ denotes the *packed shifted shuffle* that is the shuffle of u and $v' = v[max(u)]$, that is the word $v'_i = v_i + max(u)$.
The dual algebra $\mathbf{WQSym}^*$ of **WQSym** is a subalgebra of the Parking quasi-symmetric functions **PQSym**. This algebra has a multiplicative basis denoted by $\mathbf{F}^w$, where the product is the shifted concatenation, that is $u.v[max(u)]$.

3.2.2. **WSym**. The algebra of word symmetric functions **WSym**, first defined by Rosas and Sagan in [**15**], where it is called the algebra of symmetric functions in noncommuting variables, is the Hopf subalgebra of **WQSym** generated by

$$\mathbf{W}_\pi := \sum_{\mathrm{sp}(u)=\pi} \mathbf{WQ}_u, \tag{14}$$

where $\mathrm{sp}(u)$ is the (unordered) set partition obtained by forgetting the order of the parts of its corresponding set composition.
Its dual $\mathbf{WSym}^*$ is the quotient of $\mathbf{WQSym}^*$

$$\mathbf{WSym}^* = \mathbf{WQSym}^*/_J \tag{15}$$

where J is the ideal generated by the polynomials $\mathbf{F}^u - \mathbf{F}^v$ with u and v corresponding to the same set partition. We denote by $F^{\mathrm{sp}(u)}$ the image of $\mathbf{F}^u$ by the canonical surjection.

4. Hopf algebras of set packed matrices

4.1. Set matrix quasi-symmetric functions. The construction of the Hopf algebra **SMQSym** over set packed matrices is a direct translation of the construction of **MQSym** ([**11, 7**]). Consider the linear subspace spanned by the elements $\mathbf{SMQ}_M$, where M runs over the set of set packed matrices. We denote by $h(M)$ the number of rows of M. Then define

$$\mathbf{SMQ}_P \mathbf{SMQ}_Q := \sum_{R \in \Cup(P,Q)} \mathbf{SMQ}_R \tag{16}$$

where the *augmented shuffle* of P and Q, $\Cup(P,Q)$ is defined as follows: let Q' be obtained from Q by adding the greatest number inside P to all elements inside Q. Let r be an integer between $\max(p,q)$ and $p+q$, where $p = h(P)$ and $q = h(Q)$. Insert rows of zeros in the matrices P and Q' so as to form matrices $\tilde{P}$ and $\tilde{Q}'$ of height r. Let R be the matrix obtained by gluing $\tilde{Q}'$ to the right of $\tilde{P}$. The set $\Cup(P,Q)$ is formed by all the matrices with no row of 0's obtained this way.
For example,

$$\begin{aligned}\mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\}\\ \{2\} & \emptyset\end{pmatrix}} \mathbf{SMQ}_{\begin{pmatrix}\{2,3,4\} & \{1\}\end{pmatrix}} &= \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \emptyset & \emptyset\\ \{2\} & \emptyset & \emptyset & \emptyset\\ \emptyset & \emptyset & \{6,7,8\} & \{5\}\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \emptyset & \emptyset\\ \{2\} & \emptyset & \{6,7,8\} & \{5\}\end{pmatrix}} \\ &+ \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \emptyset & \emptyset\\ \emptyset & \emptyset & \{6,7,8\} & \{5\}\\ \{2\} & \emptyset & \emptyset & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{3,4\} & \{1\} & \{6,7,8\} & \{5\}\\ \{2\} & \emptyset & \emptyset & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \emptyset & \{6,7,8\} & \{5\}\\ \{3,4\} & \{1\} & \emptyset & \emptyset\\ \{2\} & \emptyset & \emptyset & \emptyset\end{pmatrix}}.\end{aligned} \tag{17}$$

The coproduct $\Delta\mathbf{SMQ}_M$ is defined by

$$\Delta \mathbf{SMQ}_A = \sum_{A=\begin{pmatrix}A_1\\A_2\end{pmatrix}} \mathbf{SMQ}_{\mathrm{std}(A_1)} \otimes \mathbf{SMQ}_{\mathrm{std}(A_2)}, \tag{18}$$

where $\mathrm{std}(A)$ denotes the standardized of the matrix A, that is the matrix obtained by the substitution $a_i \mapsto i$, where $a_1 < \cdots < a_n$ are the integers appearing in A. For example,

$$\Delta \mathbf{SMQ}_{\begin{pmatrix}\{2,4\} & \{1\}\\ \{6\} & \{3,5\}\end{pmatrix}} = \mathbf{SMQ}_{\begin{pmatrix}\{2,4\} & \{1\}\\ \{6\} & \{3,5\}\end{pmatrix}} \otimes 1 + \mathbf{SMQ}_{\begin{pmatrix}\{2,3\} & \{1\}\end{pmatrix}} \otimes \mathbf{SMQ}_{\begin{pmatrix}\{3\} & \{1,2\}\end{pmatrix}} + 1 \otimes \mathbf{SMQ}_{\begin{pmatrix}\{2,4\} & \{1\}\\ \{6\} & \{3,5\}\end{pmatrix}} \tag{19}$$

Rather than checking the compatibility between the product and the coproduct, one can look for a realization of **SMQSym**, here in terms of noncommutative bi-words, that will later give useful guidelines to select or understand homomorphisms between the different algebras.

4.2. Realization of SMQSym. A noncommutative bi-word is a word over an alphabet of bi-letters $\left\langle {a_i \atop \mathbf{b}_j} \right\rangle$ with $i, j \in \mathbb{N}$ where a_i and $\mathbf{b}_j$ are letters of two distinct ordered alphabets A and $\mathbf{B}$. We will denote by $\mathbb{C}\left\langle {A \atop \mathbf{B}} \right\rangle$ the algebra of the (potentially infinite) polynomials over the bi-letters $\left\langle {a \atop \mathbf{b}} \right\rangle$ for the product $\star$ defined by

$$\left\langle {u_1 \atop \mathbf{v}_1} \right\rangle \star \left\langle {u_2 \atop \mathbf{v}_2} \right\rangle = \left\langle {u_1.u_2 \atop \mathbf{v}_1.\mathbf{v}_2[\max(\mathbf{v}_1)]} \right\rangle, \tag{20}$$

where $\mathbf{v}_1.\mathbf{v}_2[k]$ denotes the concatenation of $\mathbf{v}_1$ with the word $\mathbf{v}_2$ whose letters are shifted by k and $\max(\mathbf{v}_1)$ the maximal letter of $\mathbf{v}_1$.
For example,

$$\left\langle {142 \atop 236} \right\rangle \star \left\langle {24 \atop 31} \right\rangle = \left\langle {14224 \atop 23697} \right\rangle \tag{21}$$

LEMMA 4.1. *The product $\star$ is associative.*

One associates to each set packed matrix the bi-word such that its ith bi-letter is the coordinate in which the letter i appears in the matrix. For example,

$$\text{bi-word}\begin{pmatrix} \{2,4\} & \{1\} \\ \{6\} & \{3,5\} \end{pmatrix} = \left\langle {a_1a_1a_2a_1a_2a_2 \atop \mathbf{b}_2\mathbf{b}_1\mathbf{b}_2\mathbf{b}_1\mathbf{b}_2\mathbf{b}_1} \right\rangle. \tag{22}$$

A bi-word is said bi-packed if its two words are packed. The bi-packed of a bi-word is the bi-word obtained by packing its two words. The set packed matrices are obviously in bijection with the bi-packed bi-words. Let $\left\langle {u \atop \mathbf{v}} \right\rangle$ be a bi-packed bi-word. Then

THEOREM 4.2.

- *The algebra* **SMQSym** *can be realized on bi-words by*

$$\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle} := \sum_{\text{bipacked}\left\langle {u' \atop \mathbf{v}'} \right\rangle = \left\langle {u \atop \mathbf{v}} \right\rangle} \left\langle {u' \atop \mathbf{v}'} \right\rangle. \tag{23}$$

- **SMQSym** *is a Hopf algebra.*
- **SMQSym** *is isomorphic as a Hopf algebra to the graded endomorphisms of* **WQSym**:

$$\mathrm{End}_{gr}\mathbf{WQSym} = \bigoplus_{n\in\mathbb{N}} \mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^* \tag{24}$$

through the Hopf homomorphism

$$\phi\left(\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle}\right) = \mathbf{WQ}_u \otimes \mathbf{F}^{\mathbf{v}}. \tag{25}$$

Indeed, from the point of view of the realization, the coproduct of **SMQSym** is given by the usual trick of noncommutative symmetric functions, considering the alphabet A as an ordered sum of two mutually commuting alphabets $A' \hat{+} A''$, hence being a homomorphism for the product.

4.3. Set matrix half-symmetric functions.

4.3.1. *The Hopf algebra* **SMRSym**. Let **SMRSym** be the subalgebra of **SMQSym** generated by the polynomials $\mathbf{SMR}_{\pi_1,\Pi_2}$ indexed by a set partition π_1 and a set composition Π_2 and defined by

$$\mathbf{SMR}_{(\pi_1,\Pi_2)} := \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{SMQ}_{\Pi_1,\Pi_2}. \tag{26}$$

$$\begin{aligned}\mathbf{SMR}_{\{\{13\},\{4\},\{2\}\},[\{123\},\{4\}]} &= \mathbf{SMQ}_{\begin{pmatrix}\{1,3\} & \emptyset\\ \emptyset & \{4\}\\ \{2\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{1,3\} & \emptyset\\ \{2\} & \emptyset\\ \emptyset & \{4\}\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \{4\}\\ \{1,3\} & \emptyset\\ \{2\} & \emptyset\end{pmatrix}}\\ &+ \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \{4\}\\ \{2\} & \emptyset\\ \{1,3\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{2\} & \emptyset\\ \emptyset & \{4\}\\ \{1,3\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{2\} & \emptyset\\ \{1,3\} & \emptyset\\ \emptyset & \{4\}\end{pmatrix}}\end{aligned} \tag{27}$$

Note that a pair constituted by a set partition and a set composition is equivalent to a set packed matrix up to a permutation of its rows. Hence, the realization on bi-words follows: for example,

$$\mathbf{SMR}_{\{\{13\},\{4\},\{2\}\},[\{123\},\{4\}]} = \sum_{\substack{i_1,i_2,i_3 \text{ distinct}\\ j_1<j_2}} \left\langle \begin{matrix} a_{i_1}\, a_{i_2}\, a_{i_1}\, a_{i_3} \\ \mathbf{b}_{j_1}\, \mathbf{b}_{j_1}\, \mathbf{b}_{j_1}\, \mathbf{b}_{j_2}\end{matrix} \right\rangle \tag{28}$$

PROPOSITION 4.3.

- **SMRSym** *is isomorphic to* $\oplus \mathbf{WSym}_n \otimes \mathbf{WQSym}_n^*$.
- **SMRSym** *is a co-commutative Hopf subalgebra of* **SMQSym**.

4.3.2. *The Hopf algebra* **SMCSym**. Forgetting about the order of the columns instead of the rows leads to another Hopf algebra, **SMCSym**, a basis of which is indexed by the pairs (Π_1, π_2) (Π_1 is a set composition and π_2 is a set partition). It is the quotient of **SMQSym** by the polynomials

$$\mathbf{SMQ}_{\Pi_1,\Pi_2} - \mathbf{SMQ}_{\Pi_1,\Pi_2'} \tag{29}$$

where $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$. Note that this quotient can be brought down to the bi-words. We will denote by α the canonical surjection:

$$\alpha(\mathbf{SMQ}_{\Pi_1,\Pi_2}) =: \mathbf{SMC}_{\Pi_1,\mathrm{sp}(\Pi_2)} \tag{30}$$

The algebra **SMCSym** is spanned by the $\mathbf{SMC}_{\Pi_1,\pi_2}$ where Π_1 is a set composition and π_2 is a set partition.

PROPOSITION 4.4.

- **SMCSym** *is isomorphic to* $\oplus \mathbf{WQSym}_n \otimes \mathbf{WSym}_n^*$,
- **SMCSym** *is a Hopf algebra.*

Note that both **SMRSym** and **SMCSym** have the same Hilbert series, given by the product of ordered Bell numbers by unordered Bell numbers. This gives one new example of two different Hopf structures on the same combinatorial set since **SMCSym** is neither commutative nor cocommutative.

4.4. Set matrix symmetric functions. The algebra **SMSym** of *set matrix symmetric functions* is the subalgebra of **SMCSym** generated by the polynomials

$$\mathbf{SM}_{\pi_1,\pi_2} = \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{SMR}_{\Pi_1,\pi_2} \tag{31}$$

For example,

$$\mathbf{SM}_{\{\{1,3\},\{4\},\{2\}\},\{\{1,2,3\},\{4\}\}} = \sum_{\substack{i_1,i_2,i_3 \text{ distinct}\\ j_1<j_2}} \left\langle \begin{matrix} a_{i_1}\, a_{i_3}\, a_{i_1}\, a_{i_2} \\ \mathbf{b}_{j_1}\, \mathbf{b}_{j_1}\, \mathbf{b}_{j_1}\, \mathbf{b}_{j_2}\end{matrix} \right\rangle \tag{32}$$

THEOREM 4.5.

- **SMSym** *is a co-commutative Hopf subalgebra of* **SMCSym**.
- **SMSym** *is isomorphic as an algebra to* $\mathrm{End}_{gr}\mathbf{WSym} = \bigoplus \mathbf{WSym}_n \otimes \mathbf{WSym}_n^*$.
- **SMSym** *is isomorphic to the quotient of* **SMRSym** *by the ideal generated by the polynomials* $\mathbf{SMR}_{\pi_1,\Pi_2} - \mathbf{SMR}_{\pi_1,\Pi_2'}$ *with* $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$.

$$\begin{pmatrix} \{1,2\} & \emptyset \\ \emptyset & \{4\} \\ \{3\} & \emptyset \end{pmatrix} \qquad \begin{pmatrix} \{1,3\} & \emptyset \\ \emptyset & \{4\} \\ \{2\} & \emptyset \end{pmatrix} \tag{33}$$

FIGURE 7. An element of SA and an element not in SA.

5. Hopf algebras of packed integer matrices

5.1. Matrix quasi-symmetric functions. Let SA be the set of packed set matrices such that if one reads the entries by columns from top to bottom and from left to right, then one obtains the numbers 1 to n in the usual order. Let us consider the subalgebra $\mathbf{MQSym}'$ of **SMQSym** generated by the elements of SA.
For example,

$$\mathbf{MQ}_{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \end{pmatrix}} = \sum_{\substack{i_1<i_2 \\ j_1<j_2<j_3}} \left\langle \begin{matrix} a_{i_1}\,a_{i_1}\,a_{i_1}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2} \\ \mathbf{b}_{j_1}\mathbf{b}_{j_1}\mathbf{b}_{j_3}\mathbf{b}_{j_2}\mathbf{b}_{j_2}\mathbf{b}_{j_2}\mathbf{b}_{j_3}\mathbf{b}_{j_3}\mathbf{b}_{j_3}\mathbf{b}_{j_3} \end{matrix} \right\rangle \tag{34}$$

THEOREM 5.1. $\mathbf{MQSym}'$ *is isomorphic as a Hopf algebra to* **MQSym**.

5.2. Matrix half-symmetric functions. We reproduce the same construction as for set packed matrices. We define three algebras **MRSym** (resp. **MCSym**, **MSym**) of packed matrices up to a permutation of rows (resp. of columns, resp. of rows and columns).

5.2.1. *The Hopf algebra* **MRSym**. Let **MRSym** be the subalgebra of **MQSym** generated by the polynomials

$$\mathbf{MR}_A := \sum \mathbf{MQ}_B \tag{35}$$

where B is obtained from A by any permutation of its rows. As for **SMRSym**, the realization of **MRSym** on bi-words is automatic. For example,

$$\mathbf{MR}_{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}} = \mathbf{MQ}_{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}} + \mathbf{MQ}_{\begin{pmatrix} 0 & 3 & 4 \\ 2 & 1 & 0 \end{pmatrix}} = \sum_{\substack{i_1\neq i_2 \\ j_1<j_2<j_3}} \left\langle \begin{matrix} a_{i_1}\,a_{i_1}\,a_{i_1}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2}\,a_{i_2} \\ \mathbf{b}_{j_1}\mathbf{b}_{j_1}\mathbf{b}_{j_3}\mathbf{b}_{j_2}\mathbf{b}_{j_2}\mathbf{b}_{j_2}\mathbf{b}_{j_3}\mathbf{b}_{j_3}\mathbf{b}_{j_3}\mathbf{b}_{j_3} \end{matrix} \right\rangle. \tag{36}$$

THEOREM 5.2.

1. **MRSym** *is a co-commutative Hopf subalgebra of* **MQSym**,
2. **MRSym** *is also the subalgebra of* **SMRSym** *generated by the elements* $\mathbf{SMR}_A$ *where A is any matrix such that each element of the set composition of its columns is an interval of* $[1,n]$.

5.2.2. *The Hopf algebra* **MCSym**. We construct the algebra **MCSym** as the quotient of **MQSym** by the ideal generated by the polynomials $\mathbf{MQ}_A - \mathbf{MQ}_B$ where B can be obtained from A by a permutation of its columns.

THEOREM 5.3.

1. **MCSym** *is a commutative Hopf algebra,*
2. **MCSym** *is isomorphic as a Hopf algebra to the subalgebra of* **SMCSym** *generated by the elements* $\mathbf{SMC}_A$*, such that each element of the set partition of its columns is an interval of* $[1,n]$.

5.2.3. *Dimensions of* **MRSym** *and* **MCSym**. The dimension of the homogeneous component of degree n of **MRSym** or **MCSym** is equal to the number of packed matrices such that the sum of the entries is equal to n, up a permutation of the rows.
Let us denote by $\mathrm{PMuR}(p,q,n)$ the number of such $p\times q$ matrices. One has obviously

$$\dim \mathbf{MRSym}_n = \sum_{1\le p,q\le n} \mathrm{PMuR}(p,q,n). \tag{37}$$

The integers $\mathrm{PMuR}(p,q,n)$ can be computed through the induction

$$\mathrm{PMuR}(p,q,n) = \mathrm{MuR}(p,q,n) - \sum_{1\le k,l\le p,q} \binom{q}{l} \mathrm{PMuR}(k,l,n), \tag{38}$$

where $\mathrm{MuR}(p,q,n)$ is the number of $p\times q$ matrices up to a permutations of the rows with the sums of the entries equal to n.

Solving this induction and substituting it in equation (37), one gets

$$\dim \mathbf{MRSym}_n = \sum_{i=1}^{n+1} (-1)^{n-i} T_{n+1,i+1} \mathrm{MuR}(n,i,n), \tag{39}$$

where $T_{n,k} = \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{j+k-1}{j}$ is known as the number of minimal covers of an unlabeled n-set that cover k points of that set uniquely (see sequence A056885 of [**16**]). The generating series of the $T_{n,k}$ is

$$\sum_{i,j} T_{i,j} x^i y^j = \frac{1-x}{(1+x)(1-x-xy)}.$$

The integer $\mathrm{MuR}(n,i,n)$, computed via the Pólya enumeration theorem, is the coefficient of x^n in the cycle index $Z(G_{n,i})$, evaluated over the alphabet $1+x+\cdots+x^n+\cdots$, of the subgroup $G_{n,i}$ of $\mathfrak{S}_{in}$ generated by the permutations $\sigma.\sigma[n].\sigma[2n].\ \cdots\ .\sigma[in]$ for $\sigma \in \mathfrak{S}_n$ (here . denotes the concatenation).
This coefficient is also the number of partitions $N_{n,i}$ of n objects with i colors whose generating series is

$$\sum_n N_{n,i} x^n = \prod_k \left(\frac{1}{1-x^k} \right)^{\binom{i+k}{i}}.$$

Hence

PROPOSITION 5.4.

$$\dim \mathbf{MRSym}_n = \sum_{i=1}^{n+1} (-1)^{n-i} T_{n+1,i+1} N_{n,i}. \tag{40}$$

The first values are

$$\begin{aligned} \mathrm{Hilb}(\mathbf{MRSym}) = \mathrm{Hilb}(\mathbf{MCSym}) &= 1+t+4t^2+16t^3+76t^4+400t^5+2356t^6+15200t^7+106644t^8 \\ &\quad +806320t^9+6526580t^{10}+\cdots \end{aligned} \tag{41}$$

5.3. Matrix symmetric functions. The algebra **MSym** of *matrix symmetric functions* is the subalgebra of **MCSym** generated by

$$\sum_{\sigma \in \mathfrak{S}_n} \mathbf{MR}_B, \tag{42}$$

where B is obtained from A by any permutation of its rows.

THEOREM 5.5.

(1) **MSym** *is a commutative and co-commutative Hopf subalgebra of* **MCSym**.
(2) **MSym** *is isomorphic as a Hopf algebra to the subalgebra of* **SMSym** *generated by the elements* $\mathbf{SM}_A$*, such that each element of the set partition of its columns is an interval of* $[1,n]$.
(3) **MSym** *is isomorphic to the quotient of* **MRSym** *by the ideal generated by the polynomials* $\mathbf{MR}_A - \mathbf{MR}_B$ *where B can be obtained from A by a permutation of its columns.*

The following diagram commutes

$$\begin{tikzcd}
 & \mathbf{SMQSym} \arrow[ddr] & & & \mathbf{MQSym} \arrow[lll] \arrow[ddr] & \\
\mathbf{SMRSym} \arrow[ur] \arrow[ddr] & & & \mathbf{MRSym} \arrow[lll] \arrow[ur] \arrow[ddr] & & \\
 & & \mathbf{SMCSym} & & & \mathbf{MCSym} \arrow[lll] \\
 & \mathbf{SMSym} \arrow[ur] & & & \mathbf{MSym} \arrow[lll] \arrow[ur] &
\end{tikzcd}$$

6. Dendriform structures over SMQSym

6.1. Tridendriform structure. One defines three product rules over bi-words as follows:

(1) $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \prec \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle$ if $\max(u_1) > \max(u_2)$, and 0 otherwise.

(2) $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \circ \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle$ if $\max(u_1) = \max(u_2)$, and 0 otherwise.

(3) $\left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \succ \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix} \right\rangle \star \left\langle \begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix} \right\rangle$ if $\max(u_1) < \max(u_2)$, and 0 otherwise.

PROPOSITION 6.1. *The algebra of bi-words endowed with the three product rules $\prec, \circ$, and $\succ$ is a tridendriform algebra.*
Moreover, **SMQSym** *is stable by those three rules. So it is a tridendriform algebra.*

More precisely, one has:

(1) $\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \prec \mathbf{SMQ}_{\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle} = \sum_{\substack{w=x.y\in u\star_W u' \\ |x|=|u|;\ \max(y)<\max(x)}} \mathbf{SMQ}_{\left\langle \begin{smallmatrix} w \\ \mathbf{v}\mathbf{v}'[\max(v)] \end{smallmatrix} \right\rangle}$

(2) $\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \circ \mathbf{SMQ}_{\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle} = \sum_{\substack{w=x.y\in u\star_W u' \\ |x|=|u|;\ \max(y)=\max(x)}} \mathbf{SMQ}_{\left\langle \begin{smallmatrix} w \\ \mathbf{v}\mathbf{v}'[\max(v)] \end{smallmatrix} \right\rangle}$

(3) $\mathbf{SMQ}_{\left\langle \begin{smallmatrix} u \\ \mathbf{v} \end{smallmatrix} \right\rangle} \succ \mathbf{SMQ}_{\left\langle \begin{smallmatrix} u' \\ \mathbf{v}' \end{smallmatrix} \right\rangle} = \sum_{\substack{w=x.y\in u\star_W u' \\ |x|=|u|;\ \max(y)>\max(x)}} \mathbf{SMQ}_{\left\langle \begin{smallmatrix} w \\ \mathbf{v}\mathbf{v}'[\max(v)] \end{smallmatrix} \right\rangle}.$

COROLLARY 6.2. **SMCSym***,* **MQSym** *and* **MCSym** *are tridendriform.*

6.2. Bidendriform structures. Let us define two product rules $\ll = \prec$ and $\gg = \circ + \succ$ on bi-words. We now split the non-trivial parts of the coproduct of the **SMQ** of **SMQSym**, as

(1) $\Delta_{\ll}(\mathbf{SMQ}_A) = \sum_{\substack{A=\left(\begin{smallmatrix} B \\ C \end{smallmatrix}\right),\ A\neq B,C \\ \max(B)=\max(A)}} \mathbf{SMQ}_{\mathrm{std}(B)} \otimes \mathbf{SMQ}_{\mathrm{std}(C)}.$

(2) $\Delta_{\gg}(\mathbf{SMQ}_A) = \sum_{\substack{A=\left(\begin{smallmatrix} B \\ C \end{smallmatrix}\right),\ A\neq B,C \\ \max(C)=\max(A)}} \mathbf{SMQ}_{\mathrm{std}(B)} \otimes \mathbf{SMQ}_{\mathrm{std}(C)}.$

THEOREM 6.3. **SMQSym** *is a bidendriform bialgebra.*

Recall that **SMCSym** is the quotient of **SMQSym** by the ideal generated by $\mathbf{SMQ}_A - \mathbf{SMQ}_B$ where A and B are the same matrices up to a permutation of their columns, the row containing the maximal element is the same for any element of a given class, so that the left coproduct and the right coproduct are compatible with the quotient. Moreover, the left and right coproduct are internal within **MQSym**, so that

COROLLARY 6.4. **SMCSym** *and* **MQSym** *are bidendriform sub-bialgebras of* **SMQSym***.*

6.3. bi-words realization of $LDiag(q_c, q_s)$**.** Let us define a two-parameter generalization of **MQSym**. For this purpose, consider bi-words with parameter-commuting bi-letters depending on the biletters as follows:

$$\begin{aligned} \left\langle \begin{matrix} yx \\ zt \end{matrix} \right\rangle &= q_c \left\langle \begin{matrix} xy \\ tz \end{matrix} \right\rangle \text{ if } y > x, \\ \left\langle \begin{matrix} xx \\ zt \end{matrix} \right\rangle &= q_s \left\langle \begin{matrix} xx \\ tz \end{matrix} \right\rangle \text{ if } z < t. \end{aligned} \tag{43}$$

Let us now define the realization as a sum of bi-words of a packed integer matrix with p rows and q columns:

$$\mathbf{LD}_M := \sum_{\substack{i_1<\cdots<i_p \\ j_1<\cdots<j_q}} \prod_{a=1}^{n} \prod_{b=p}^{1} \left\langle \begin{matrix} i_a \\ j_b \end{matrix} \right\rangle^{M_{ij}}. \tag{44}$$

For example,

$$\mathbf{LD}_{\left(\begin{smallmatrix} 3 & 5 \\ 1 & 3 \end{smallmatrix}\right)} := \sum_{\substack{i_1<i_2 \\ j_1<j_2}} \left\langle \begin{matrix} i_1^5\, i_1^3\, i_2^3\, i_2 \\ j_2^5\, j_1^3\, j_2^3\, j_1 \end{matrix} \right\rangle. \tag{45}$$

We then have

THEOREM 6.5.

- *The* **LD** *generate an algebra,*
- *The specialization* $q_s = q_c = 1$ *gives back* **MQSym**.

For example, one has:

$$\mathbf{LD}_{\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}} \star \mathbf{LD}_{(1)} = \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + q_s^5 \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 1 \end{pmatrix}} + q_c^5 \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 0 \end{pmatrix}} + q_c^5 q_s^2 \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 1 \\ 1 & 4 & 0 \end{pmatrix}} + q_c^7 \mathbf{LD}_{\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 4 & 0 \end{pmatrix}} \tag{46}$$

since

$$\begin{aligned} \sum_{\substack{i_1<i_2 \\ j_1<j_2}} \left\langle \begin{matrix} i_1\, i_1\, i_2\, i_2\, i_2\, i_2\, i_2 \\ j_1\, j_1\, j_2\, j_2\, j_2\, j_2\, j_1 \end{matrix} \right\rangle \star \sum \left\langle \begin{matrix} i \\ j \end{matrix} \right\rangle = & \sum_{\substack{i_1<i_2<i_3 \\ j_1<j_2<j_3}} \left\langle \begin{matrix} i_1^2\, i_2^4\, i_2\, i_3 \\ j_1^2\, j_2^4\, j_1\, j_3 \end{matrix} \right\rangle + \sum_{\substack{i_1<i_2(=i_3) \\ j_1<j_2<j_3}} q_s^5 \left\langle \begin{matrix} i_1^2\, i_2\, i_2^4 i_2 \\ j_1^2\, j_3\, j_2^4\, j_1 \end{matrix} \right\rangle + \sum_{\substack{i_1<i_3<i_2 \\ j_1<j_2<j_3}} q_c^5 \left\langle \begin{matrix} i_1^2\, i_3\, i_2^4\, i_2 \\ j_1^2\, j_3\, j_2^4\, j_1 \end{matrix} \right\rangle \\ & + \sum_{\substack{i_1(=i_3)<i_2 \\ j_1<j_2<j_3}} q_s^2 q_c^5 \left\langle \begin{matrix} i_1\, i_1^2\, i_2^4\, i_2 \\ j_3\, j_1^2\, j_2^4\, j_1 \end{matrix} \right\rangle + \sum_{\substack{i_3<i_1<i_2 \\ j_1<j_2<j_3}} q_c^7 \left\langle \begin{matrix} i_3\, i_1^2\, i_2^4\, i_2 \\ j_3\, j_1^2\, j_2^4\, j_1 \end{matrix} \right\rangle . \end{aligned} \tag{47}$$

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