Hopf Algebra Structure of a Model Quantum Field Theory

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Abstract. Recent elegant work[1] on the structure of Perturbative Quantum Field Theory (PQFT) has revealed an astonishing interplay between analysis(Riemann Zeta functions), topology (Knot theory), combinatorial graph theory (Feynman Diagrams) and algebra (Hopf structure). The difficulty inherent in the complexities of a fully-fledged field theory such as PQFT means that the essential beauty of the relationships between these areas can be somewhat obscured. Our intention is to display some, although not all, of these structures in the context of a simple zero-dimensional field theory; i.e. quantum theory of non-commuting non-field-dependent operators. The combinatorial properties of these boson creation and annihilation operators, which is our chosen example, may be described by graphs [2, 3], analogous to the Feynman diagrams of PQFT, which we show possess a Hopf algebra structure[4]. We illustrate these ideas by means of simple solvable models, e.g. the partition function for a free boson gas and for a superfluid bose system. Finally, we sketch the relationship between the Hopf algebra of our simple model and that of the PQFT algebra.

1 Partition Function Integrand

Consider the Partition Function Z of a Quantum Statistical Mechanical System

$$Z = \mathrm{Tr}\exp(-\beta H)\,. \tag{1}$$

whose hamiltonian is H ($\beta \equiv 1/kT$, k=Boltzmann's constant T=absolute temperature). We may evaluate the trace over any complete set of states; we choose the (over-)complete set of coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_n (z^n/\sqrt{n!}){a^\dagger}^n|0\rangle \tag{2}$$

where $a^\dagger$ is the boson creation operator satisfying $[a.a^\dagger] = 1$ and for which the completeness or *resolution of unity* property is

$$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = I \equiv \int d\mu(z)|z\rangle\langle z|. \tag{3}$$

The simplest, and generic, example is the free single-boson hamiltonian $H = \epsilon a^\dagger a$ for which the appropriate trace calculation is

$$\begin{aligned} Z &= \frac{1}{\pi}\int d^2z\langle z|\exp\bigl(-\beta\epsilon a^\dagger a\bigr)|z\rangle = \\ &= \frac{1}{\pi}\int d^2z\langle z| : \exp\bigl(a^\dagger a(e^{-\beta\epsilon}-1)\bigr) : |z\rangle. \end{aligned} \tag{4}$$

Here we have used the following well-known relation [**?**, **?**] for the *forgetful* normal ordering operator $:f(a,a^\dagger):$ which means "normally order the creation and annihilation operators in f *forgetting* the commutation relation $[a,a^\dagger]=1$."[1]

We may write the Partition Function in general as

$$Z(x) = \int F(x,z)\,d\mu(z) \tag{5}$$

thereby defining the Partition Function Integrand (PFI) $F(x,z)$. We have explicitly written the dependence on $x \equiv -\beta$, the inverse temperature, and ϵ, the energy scale in the hamiltonian.

[1] Of course, this procedure may alter the value of the operator to which it is applied.

2 Combinatorial aspects: Bell numbers

The generic free-boson example Eq.(5)above may be rewritten to show the connection with certain well-known combinatorial numbers. Writing $y = |z|^2$ and $x = -\beta\epsilon$, Eq.(5) becomes

$$Z = \int_0^\infty dy \exp\bigl(y(e^x - 1)\bigr) . \qquad (6)$$

This is an integral over the classical *exponential generating function* for the *Bell polynomials*

$$\exp\bigl(y\bigl(e^x - 1\bigr)\bigr) = \sum_{n=0}^{\infty} B_n(y) \frac{x^n}{n!} \qquad (7)$$

where the Bell number is $B_n(1) = B(n)$, the number of ways of putting n different objects into n identical containers (some may be left empty). Related to the Bell numbers are the *Stirling numbers of the second kind* $S(n, k)$, which are defined as the number of ways of putting n different objects into k identical containers, leaving none empty. From the definition we have $B(n) = \sum_{k=1}^{n} S(n, k)$. The foregoing gives a combinatorial interpretation of the partition function integrand $F(x, y)$ as the exponential generating function of the Bell polynomials.

2.1 Graphs

We now give a graphical representation of the Bell numbers. Consider labelled lines which emanate from a white dot, the origin, and finish on a black dot, the vertex. We shall allow only one line from each white dot but impose no limit on the number of lines ending on a black dot. Clearly this simulates the definition of $S(n, k)$ and $B(n)$, with the white dots playing the role of the distinguishable objects, whence the lines are labelled, and the black dots that of the indistinguishable containers. The identification of the graphs for 1,2 and 3 lines is given in Figure 1. We have concentrated on the Bell number sequence and its associated graphs since, as we shall show, there is a sense in which this sequence of graphs is *generic*. That is, we can represent *any* combinatorial sequence by the same sequence of graphs as in the Figure

Figure 1. Graphs for $B(n)$, $n = 1, 2, 3$.

1, with suitable vertex multipliers (denoted by the V terms in the same figure). Consider a general partition function

$$Z = \mathrm{Tr}\exp(-\beta H) \tag{8}$$

where the Hamiltonian is given by $H = \epsilon w(a, a^\dagger)$, with w a string (= sum of products of positive powers) of boson creation and annihilation operators. The partition function integrand F for which we seek to give a graphical expansion, is

$$Z(x) = \int F(x, z)\, d\mu(z)\,, \tag{9}$$

where

$$\begin{aligned} F(x,z) &= \langle z|\exp(xw)|z\rangle = \qquad\qquad (x=-\beta\epsilon) \\ &= \sum_{n=0}^{\infty}\langle z|w^n|z\rangle\,\frac{x^n}{n!} = \\ &= \sum_{n=0}^{\infty} W_n(z)\,\frac{x^n}{n!} = \\ &= \exp\left(\sum_{n=1}^{\infty} V_n(z)\,\frac{x^n}{n!}\right), \end{aligned} \tag{10}$$

with obvious definitions of W_n and V_n. The sequences $\{W_n\}$ and $\{V_n\}$ may each be recursively obtained from the other [**?**]. This relates the sequence of multipliers $\{V_n\}$ of Figure 1 to the Hamiltonian of Eq.(8). The lower limit 1 in the V_n summation is a consequence of the normalization of the coherent state $|z\rangle$.

3 Hopf Algebra structure

We briefly describe the Hopf algebra $\mathcal{L}_{\text{Bell}}$ which the diagrams of Figure 1 define.

1. Each distinct diagram is an individual basis element of $\mathcal{L}_{\text{Bell}}$; thus the dimension is infinite. (Visualise each diagram in a "box".) The sum of two diagrams is simply the two boxes containing the diagrams. Scalar multiples are formal; for example, they may be provided by the V coefficients.

2. The identity element e is the empty diagram (an empty box).

3. Multiplication is the juxtaposition of two diagrams within the same "box". $\mathcal{L}_{\text{Bell}}$ is generated by the *connected* diagrams; this is a consequence of the Connected Graph Theorem[5]. Since we have not here specified an order for the juxtaposition, multiplication is commutative.

4. The coproduct $\Delta\mathcal{L}_{\text{Bell}} \to \mathcal{L}_{\text{Bell}} \times \mathcal{L}_{\text{Bell}}$ is defined by

$$\begin{aligned} \Delta(e) &= e\times e \quad \text{(unit } e) \\ \Delta(x) &= x\times e + e\times x \quad \text{(generator } x) \\ \Delta(AB) &= \Delta(A)\Delta(B) \quad \text{otherwise} \end{aligned}$$

so that Δ is an algebra homomorphism.

References

[1] A readable account may be found in Dirk Kreimer's "Knots and Feynman Diagrams", Cambridge Lecture Notes in Physics, CUP (2000).

[2] We use the graphical description of Bender, Brody and Meister, J.Math.Phys. 40, 3239 (1999) and arXiv:quant-ph/0604164

[3] This approach is extended in Blasiak, Penson, Solomon, Horzela and Duchamp, J.Math.Phys. 46, 052110 (2005) and arXiv:quant-ph/0405103

[4] A preliminary account of the more mathematical aspects of this work may be found in arXiv: cs.SC/0510041

[5] GW Ford and GE Uhlenbeck,Proc. Nat. Acad. **42**, 122,1956.