QUASI-SHUFFLE PRODUCTS

Michael E. Hoffman

Abstract. Given a locally finite graded set A and a commutative, associative operation on A that adds degrees, we construct a commutative multiplication $*$ on the set of noncommutative polynomials in A which we call a quasi-shuffle product; it can be viewed as a generalization of the shuffle product ш. We extend this commutative algebra structure to a Hopf algebra $(\mathfrak{A}, *, \Delta)$; in the case where A is the set of positive integers and the operation on A is addition, this gives the Hopf algebra of quasi-symmetric functions. If rational coefficients are allowed, the quasi-shuffle product is in fact no more general than the shuffle product; we give an isomorphism exp of the shuffle Hopf algebra $(\mathfrak{A}, ш, \Delta)$ onto $(\mathfrak{A}, *, \Delta)$. Both the set L of Lyndon words on A and their images $\{\exp(w) \mid w \in L\}$ freely generate the algebra $(\mathfrak{A}, *)$. We also consider the graded dual of $(\mathfrak{A}, *, \Delta)$. We define a deformation $*_q$ of $*$ that coincides with $*$ when $q = 1$ and is isomorphic to the concatenation product when q is not a root of unity. Finally, we discuss various examples, particularly the algebra of quasi-symmetric functions (dual to the noncommutative symmetric functions) and the algebra of Euler sums.

1. Introduction. Let k be a subfield of $\mathbf{C}$, and let A be a locally finite graded set. If we think of the graded noncommutative polynomial algebra $\mathfrak{A} = k\langle A\rangle$ as a vector space over k, we can make it commutative k-algebra by giving it the shuffle multiplication ш, defined inductively by

$$aw_1 \,ш\, bw_2 = a(w_1 \,ш\, bw_2) + b(aw_1 \,ш\, w_2)$$

for $a, b \in A$ and words w_1, w_2. The commutative k-algebra $(\mathfrak{A}, ш)$ is in fact a polynomial algebra on the Lyndon words in $\mathfrak{A}$ (as defined in §2 below). If we define

$$\Delta(w) = \sum_{uv=w} u \otimes v,$$

then $(\mathfrak{A}, ш, \Delta)$ becomes a commutative (but not cocommutative) Hopf algebra, usually called the shuffle Hopf algebra; and its graded dual is the concatenation Hopf algebra (see [15], Chapter 1).

Recently another pair of dual Hopf algebras has inspired much interest. The Hopf algebra **Sym** of noncommutative symmetric functions, introduced in [7], has as its graded dual the Hopf algebra of quasi-symmetric functions [5,13]. In a recent paper of the author [12], the algebra of quasi-symmetric functions arose via a modification of the shuffle product, which suggested a connection between the two pairs of Hopf

1991 *Mathematics Subject Classification.* Primary 16W30, 16W50, 16S80; Secondary 05E05.

Key words and phrases. Hopf algebra, shuffle algebra, quasi-symmetric functions, Lyndon word, noncommutative symmetric functions, quantum shuffle product.

algebras. In fact, the Hopf algebra of quasi-symmetric functions (over k) is known to be isomorphic to the shuffle Hopf algebra on a countably infinite set of generators (with one in each positive degree). It is the purpose of this paper to study this Hopf algebra isomorphism in a more general setting. (We emphasize that we are working over a subfield k of $\mathbf{C}$; if we instead work over $\mathbf{Z}$, there is no such isomorphism–the integral algebra of quasi-symmetric functions is a polynomial algebra [3,17], but the integral shuffle algebra is not [11].)

More explicitly, our construction is as follows. Suppose also that for any generators $a, b \in A$ there is another generator $[a, b]$ so that the operation $[\cdot, \cdot]$ is commutative, associative, and adds degrees. If we define a "quasi-shuffle product" $*$ by

$$aw_1 * bw_2 = a(w_1 * bw_2) + b(aw_1 * w_2) + [a, b](w_1 * w_2),$$

then $(\mathfrak{A}, *)$ is a commutative and associative k-algebra (Theorem 2.1 below). In fact, as we show in §3, $(\mathfrak{A}, *, \Delta)$ is a Hopf algebra, which we call the quasi-shuffle Hopf algebra corresponding to A and $[\cdot, \cdot]$. This construction gives the quasi-symmetric functions in the case where A consists of one element z_i in each degree $i > 0$, with $[z_i, z_j] = z_{i+j}$; this and other examples are discussed in §6. We give an explicit isomorphism exp from the shuffle Hopf algebra on the generating set A onto any quasi-shuffle Hopf algebra with the same generating set (Theorems 2.5 and 3.3). This allows us to show that any quasi-shuffle algebra on A is the free polynomial algebra on Lyndon words in $\mathfrak{A}$ (Theorem 2.6). In §4 we take (graded) duals, giving an isomorphism $\exp^*$ from the concatenation Hopf algebra to the dual of $(\mathfrak{A}, *, \Delta)$.

In §5 we consider a q-deformation $*_q$ of the quasi-shuffle product, generalizing the quantum shuffle product as defined in [4] (see also [10,16]). This product coincides with the quasi-shuffle product $*$ when $q = 1$, but is noncommutative when $q \neq 1$; when q is not a root of unity, we use the theorem of Varchenko [20] to prove that the algebra $(\mathfrak{A}, *_q)$ is isomorphic to the concatenation algebra on A (Theorem 5.4). In this case, if we declare the elements of A primitive, we get a Hopf algebra $(\mathfrak{A}, *_q, \Delta_q)$ isomorphic to the concatenation Hopf algebra.

A construction equivalent to the quasi-shuffle algebra, but (in effect) not assuming commutativity of the operation $[\cdot, \cdot]$, was developed independently by F. Fares [6]. The author thanks A. Joyal for bringing it to his attention.

2. The algebra structure. As above we begin with the graded noncommutative polynomial algebra $\mathfrak{A} = k\langle A\rangle$ over a subfield $k \subset \mathbf{C}$, where A is a locally finite set of generators (i.e. for each positive integer n the set A_n of generators in degree n is finite). We write $\mathfrak{A}_n$ for the vector space of homogeneous elements of $\mathfrak{A}$ of degree n. We shall refer to elements of A as letters, and to monomials in the letters as words. For any word w we write $\ell(w)$ for its length (the number of letters it contains) and $|w|$ for its degree (the sum of the degrees of its factors). The unique word of length 0 is 1, the empty word.

Now define a new multiplication $*$ on $\mathfrak{A}$ by requiring that $*$ distribute over addition, that $1 * w = w * 1 = w$ for any word w, and that, for any words w_1, w_2 and letters a, b,

$$aw_1 * bw_2 = a(w_1 * bw_2) + b(aw_1 * w_2) + [a, b](w_1 * w_2), \tag{1}$$

where $[\cdot, \cdot] : \bar{A} \times \bar{A} \to \bar{A}$ ($\bar{A} = A \cup \{0\}$) is a function satisfying

S0. $[a, 0] = 0$ for all $a \in \bar{A}$;

S1. $[a,b]=[b,a]$ for all $a,b\in\bar{A}$;
S2. $[[a,b],c]=[a,[b,c]]$ for all $a,b,c\in\bar{A}$; and
S3. Either $[a,b]=0$ or $|[a,b]|=|a|+|b|$ for all $a,b\in A$.

Theorem 2.1. $(\mathfrak{A},*)$ *is a commutative graded k-algebra.*

Proof. It is enough to show that the operation $*$ is commutative, associative, and adds degrees. For commutativity, it suffices to show $w_1*w_2=w_2*w_1$ for any words w_1 and w_2. We proceed by induction on $\ell(w_1)+\ell(w_2)$. Since there is nothing to prove if either w_1 or w_2 is empty, we can assume there are letters a,b so that $w_1=au$ and $w_2=bu$. Then (1) together with the induction hypothesis gives

$$w_1*w_2-w_2*w_1=[a,b](u*v)-[b,a](v*u),$$

and the right-hand side is zero by the induction hypothesis and (S1). Similarly, for associativity it is enough to prove $w_1*(w_2*w_3)=(w_1*w_2)*w_3$ for any words w_1,w_2, and w_3: this follows from induction on $\ell(w_1)+\ell(w_2)+\ell(w_3)$ using (1) and (S2). Finally, to show $*$ adds degrees, induct on $\ell(w_1)+\ell(w_2)$ using (1) and (S3) to prove that $|w_1*w_2|=|w_1|+|w_2|$ for any words w_1,w_2.

If $[a,b]=0$ for all $a,b\in A$, then $(\mathfrak{A},*)$ is the shuffle algebra as usually defined (see e.g. [15]) and we write ш for the multiplication instead of $*$. Suppose now that the set A of letters is totally ordered. Then lexicographic ordering gives a total order on the words: we put $u<uv$ for any nonempty word v, and $w_1aw_2<w_1bw_3$ for any letters $a<b$ and words w_1,w_2, and w_3. We call a word $w\neq 1$ of $\mathfrak{A}$ Lyndon if $w<v$ for any nontrivial factorization $w=uv$. The following result is proved in Chapter 6 of [15]; it was first obtained by Radford [14].

Theorem 2.2. *The shuffle algebra* $(\mathfrak{A},\text{ш})$ *is the free polynomial algebra on the Lyndon words.*

We shall define an isomorphism $\exp:(\mathfrak{A},\text{ш})\to(\mathfrak{A},*)$. To do so, we must first develop some notation relating to the operation $[\cdot,\cdot]$ and compositions. Define inductively $[S]\in\bar{A}$ for any finite sequence S of elements of A by setting $[a]=a$ for $a\in A$, and $[a,S]=[a,[S]]$ for any $a\in A$ and sequence S of elements of A.

Proposition 2.3. *(i) If $[S]=0$, then $[S']=0$ whenever S is a subsequence of S';*
(ii) $[S]$ does not depend on the order of the elements of S (i.e., it depends only on the underlying multiset of S);
(iii) For any sequences S_1 and S_2, $[S_1\sqcup S_2]=[[S_1],[S_2]]$, where $S_1\sqcup S_2$ denotes the concatenation of sequences S_1 and S_2;
(iv) If $[S]\neq 0$, then the degree of S is the sum of the degrees of the elements of S.

Proof. (i),(ii),(iii),(iv) follow from (S0),(S1),(S2),(S3) respectively.

A composition of a positive integer n is a sequence $I=(i_1,i_2,\dots,i_k)$ of positive integers such that $i_1+i_2+\cdots+i_k=n$. We call $n=|I|$ the weight of I and $k=\ell(I)$ its length; we write $\mathcal{C}(n)$ for the set of compositions of n, and $\mathcal{C}(n,k)$ for the set of compositions of n of length k. For $I\in\mathcal{C}(n,k)$ and $J\in\mathcal{C}(k,l)$, the composition $J\circ I\in\mathcal{C}(n,l)$ is given by

$$J\circ I=(i_1+\cdots+i_{j_1},i_{j_1+1}+\cdots+i_{j_1+j_2},\dots,i_{j_1+\cdots+j_{l-1}+1}+\cdots+i_k).$$

If $K = J \circ I$ for some J, we call I a refinement of K and write $I \succcurlyeq K$. Compositions act on words via $[\cdot,\cdot]$ as follows. For any word $w = a_1a_2\cdots a_n$ and composition $I = (i_1,\dots,i_l) \in \mathcal{C}(n)$, set

$$I[w] = [a_1,\dots,a_{i_1}][a_{i_1+1},\dots,a_{i_1+i_2}]\cdots[a_{i_1+\cdots+i_{l-1}+1},\dots,a_n].$$

(This is really an action in the sense that $I[J[w]] = I \circ J[w]$.)

Now let $\exp : \mathfrak{A} \to \mathfrak{A}$ be the linear map with $\exp(1) = 1$ and

$$\exp(w) = \sum_{(i_1,\dots,i_l)\in\mathcal{C}(\ell(w))} \frac{1}{i_1!\cdots i_l!}(i_1,\dots,i_l)[w]$$

for any nonempty word w (so, e.g. $\exp(a_1a_2a_3) = a_1a_2a_3 + \frac{1}{2}[a_1,a_2]a_3 + \frac{1}{2}a_1[a_2,a_3] + \frac{1}{6}[a_1,a_2,a_3]$). There is an inverse log of exp given by

$$\log(w) = \sum_{(i_1,\dots,i_l)\in\mathcal{C}(\ell(w))} \frac{(-1)^{\ell(w)-l}}{i_1\cdots i_l}(i_1,\dots,i_l)[w]$$

for any word w, and extended to $\mathfrak{A}$ by linearity; this follows by taking $f(t) = e^t - 1$ in the following lemma.

Lemma 2.4. *Let $f(t) = a_1t + a_2t^2 + a_3t^3 + \cdots$ be a function analytic at the origin, with $a_1 \neq 0$ and $a_i \in k$ for all i, and let $f^{-1}(t) = b_1t + b_2t^2 + b_3t^3 + \cdots$ be the inverse of f. Then the map $\Psi_f : \mathfrak{A} \to \mathfrak{A}$ given by*

$$\Psi_f(w) = \sum_{I\in\mathcal{C}(\ell(w))} a_{i_1}a_{i_2}\cdots a_{i_l}I[w]$$

for words w, and extended linearly, has inverse $\Psi_f^{-1} = \Psi_{f^{-1}}$ given by

$$\Psi_{f^{-1}}(w) = \sum_{I\in\mathcal{C}(\ell(w))} b_{i_1}b_{i_2}\cdots b_{i_l}I[w].$$

Proof. It suffices to show that $\Psi_{f^{-1}}(\Psi_f(w)) = w$ for any word w of length $n \geq 1$ (Note that $\Psi_f(\Psi_{f^{-1}}(w)) = w$ is then automatic, since Ψ_f and $\Psi_{f^{-1}}$ can be thought of as linear maps of the vector space with basis $\{I[w] \mid I \in \mathcal{C}(n)\}$.) Now for any $K = (k_1,\dots,k_l) \in \mathcal{C}(n)$, the coefficient of $K[w]$ in $\Psi_{f^{-1}}(\Psi_f(w))$ is

$$\sum_{J\circ I=K} b_{j_1}b_{j_2}\cdots b_{j_l}a_{i_1}a_{i_2}\cdots a_{i_{|J|}}. \tag{2}$$

We must show that (2) is 1 if K is a sequence of n 1's, and 0 otherwise. To see this, let $t_1, t_2, \dots$ be commuting variables. Then (2) is the coefficient of $t_1^{k_1}t_2^{k_2}\cdots t_l^{k_l}$ in

$$t_1t_2\cdots t_l = f^{-1}(f(t_1))f^{-1}(f(t_2))\cdots f^{-1}(f(t_l)).$$

Theorem 2.5. exp *is an isomorphism of* $(\mathfrak{A}, ш)$ *onto* $(\mathfrak{A}, *)$ *(as graded* k*-algebras).*

Proof. From the lemma, exp is invertible. Also, it follows from 2.3(iv) that exp preserves degree. To show exp a homomorphism it suffices to show $\exp(w \text{ ш } v) = \exp(w) * \exp(v)$ for any words w, v. Let $w = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$. Evidently both $\exp(w \text{ ш } v)$ and $\exp(w) * \exp(v)$ are sums of rational multiples of terms

$$[S_1 \sqcup T_1][S_2 \sqcup T_2] \cdots [S_l \sqcup T_l] \tag{3}$$

where the S_i and T_i are subsequences of $a_1, \dots, a_n$ and $b_1, \dots, b_m$ respectively such that

i. for each i, at most one of S_i, T_i is empty; and
ii. the concatenation $S_1 \sqcup S_2 \sqcup \cdots \sqcup S_l$ is the sequence $a_1, \dots, a_n$, and similarly the T_i concatenate to give the sequence $b_1, \dots, b_m$.

Now the term (3) arises in $\exp(w) * \exp(v)$ in only one way, and its coefficient is

$$\frac{1}{(\text{length}\, S_1)!(\text{length}\, S_2)! \cdots (\text{length}\, S_l)!(\text{length}\, T_1)!(\text{length}\, T_2)! \cdots (\text{length}\, T_l)!}.$$

On the other other hand, (3) can arise in $\exp(w \text{ ш } v)$ from

$$\binom{\text{length}\, S_1 \sqcup T_1}{\text{length}\, S_1}\binom{\text{length}\, S_2 \sqcup T_2}{\text{length}\, S_2} \cdots \binom{\text{length}\, S_l \sqcup T_l}{\text{length}\, S_l} = \frac{(\text{length}\, S_1 \sqcup T_1)! \cdots (\text{length}\, S_l \sqcup T_l)!}{(\text{length}\, S_1)! \cdots (\text{length}\, S_l)!(\text{length}\, T_1)! \cdots (\text{length}\, T_l)!}$$

distinct terms of the shuffle product $w \text{ ш } v$, and after application of exp each such term acquires a coefficient of

$$\frac{1}{(\text{length}\, S_1 \sqcup T_1)! \cdots (\text{length}\, S_l \sqcup T_l)!}.$$

It follows from Theorems 2.2 and 2.5 that $(\mathfrak{A}, *)$ is the free polynomial algebra on the elements $\{\exp(w) \mid w \text{ is a Lyndon word}\}$. In fact the following is true.

Theorem 2.6. $(\mathfrak{A}, *)$ *is the free polynomial algebra on the Lyndon words.*

Proof. It suffices to show that any word w can be written as a $*$-polynomial of Lyndon words. We proceed by induction on $\ell(w)$. If $\ell(w) = 1$ the result is immediate, since every letter is a Lyndon word. Now let $\ell(w) > 1$: by Theorem 2.5 there are Lyndon words $w_1, \dots, w_n$ and a polynomial P so that

$$w = P(\exp(w_1), \dots, \exp(w_n))$$

in $(\mathfrak{A}, *)$. Note that since $\log(w) = P(w_1, \dots, w_n)$ in $(\mathfrak{A}, ш)$, we can assume every term of $P(w_1, \dots, w_n)$ (as a ш-polynomial) has length at most $\ell(w)$, since the shuffle product preserves lengths. But then in $(\mathfrak{A}, *)$,

$$w - P(w_1, \dots, w_n) = P(\exp(w_1), \dots, \exp(w_n)) - P(w_1, \dots, w_n)$$

must consist of terms of length less than $\ell(w)$, and so is expressible in terms of Lyndon words by the induction hypothesis.

By the preceding result, the number of generators of $(\mathfrak{A}, *)$ in degree n is the number L_n of Lyndon words of degree n. This number can be calculated from Poincaré series

$$A(x) = \sum_{n \geq 0} (\dim \mathfrak{A}_n) x^n = \frac{1}{1 - \sum_{n \geq 1} (\text{card}\, A_n) x^n}$$

of $\mathfrak{A}$ as follows.

Proposition 2.7. *The number L_n of Lyndon words in $\mathfrak{A}_n$ is given by*

$$L_n = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)c_d,$$

where the numbers c_n are defined by

$$x\frac{d}{dx}\log A(x) = \sum_{n\geq 1} c_n x^n$$

for $A(x)$ as above.

Proof. In view of Theorems 2.2 and 2.6, we must have

$$A(x) = \prod_{n\geq 1}(1-x^n)^{-L_n}.$$

The conclusion then follows from taking logarithms, differentiating, and using the Möbius inversion formula.

3. The Hopf algebra structure. For basic definitions and facts about Hopf algebras see [18]. We define a comultiplication $\Delta : \mathfrak{A}\otimes\mathfrak{A} \to \mathfrak{A}$ and counit $\epsilon : \mathfrak{A} \to k$ by

$$\Delta(w) = \sum_{uv=w} u\otimes v$$

and

$$\epsilon(w) = \begin{cases} 1, & w = 1 \\ 0, & \text{otherwise} \end{cases}$$

for any word w of $\mathfrak{A}$. Then $(\mathfrak{A}, \Delta, \epsilon)$ is evidently a (non-cocommutative) coalgebra. In fact the following result holds.

Theorem 3.1. *$\mathfrak{A}$ with the $*$-multiplication and Δ-comultiplication is a bialgebra.*

Proof. It suffices to show that ϵ and Δ are $*$-homomorphisms. The statement for ϵ is obvious; to show $\Delta(w_1)*\Delta(w_2) = \Delta(w_1 * w_2)$ for any words w_1, w_2 use induction on $\ell(w_1)+\ell(w_2)$. Since the result is immediate if w_1 or w_2 is 1, we can write $w_1 = au$ and $w_2 = bv$ for letters a, b and words u, v. Adopting Sweedler's sigma notation [18], we write

$$\Delta(u) = \sum u_{(1)}\otimes u_{(2)}, \quad \text{and} \quad \Delta(v) = \sum v_{(1)}\otimes v_{(2)}.$$

Then from the definition of Δ,

$$\Delta(w_1) = \sum au_{(1)}\otimes u_{(2)} + 1\otimes au \quad \text{and} \quad \Delta(w_2) = \sum bv_{(1)}\otimes v_{(2)} + 1\otimes bv,$$

so that $\Delta(w_1)*\Delta(w_2)$ is

$$\sum(au_{(1)}*bv_{(1)})\otimes(u_{(2)}*v_{(2)}) + \sum au_{(1)}\otimes(u_{(2)}*bv) + \sum bv_{(1)}\otimes(au*v_{(2)}) + 1\otimes(au*bv).$$

Using (1), this is

$$\sum a(u_{(1)} * bv_{(1)}) \otimes (u_{(2)} * v_{(2)}) + \sum b(au_{(1)} * v_{(1)}) \otimes (u_{(2)} * v_{(2)}) +$$
$$\sum [a,b](u_{(1)} * v_{(1)}) \otimes (u_{(2)} * v_{(2)}) + \sum au_{(1)} \otimes (u_{(2)} * bv) + \sum bv_{(1)} \otimes (au * v_{(2)})$$
$$+ 1 \otimes a(u * w_2) + 1 \otimes b(w_1 * v) + 1 \otimes [a,b](u * v),$$

or, applying the induction hypothesis,

$$(a \otimes 1)(\Delta(u) * \Delta(w_2)) + 1 \otimes a(u * w_2) + (b \otimes 1)(\Delta(w_1) * \Delta(v)) + 1 \otimes b(w_1 * v)$$
$$+ ([a,b] \otimes 1)\Delta(u * v) + 1 \otimes [a,b](u * v),$$

which can be recognized as $\Delta(w_1 * w_2) = \Delta(a(u * w_2) + b(w_1 * v) + [a,b](u * v))$.

Since both $*$ and Δ respect the grading, it follows automatically that $\mathfrak{A}$ is a Hopf algebra (cf. Lemma 2.1 of [5]). In fact there are two explicit formulas for the antipode, whose agreement is of some interest.

Theorem 3.2. *The bialgebra $\mathfrak{A}$ has antipode S given by*

$$S(w) = \sum_{(i_1,\dots,i_l)\in\mathcal{C}(n)} (-1)^l a_1 \cdots a_{i_1} * a_{i_1+1} \cdots a_{i_1+i_2} * \cdots * a_{i_1+\cdots+i_{l-1}+1} \cdots a_n$$
$$= (-1)^n \sum_{I\in\mathcal{C}(n)} I[a_n a_{n-1} \cdots a_1]$$

for any word $w = a_1 a_2 \cdots a_n$ of $\mathfrak{A}$.

Proof. We can compute S recursively from $S(1) = 1$ and

$$S(w) = -\sum_{k=0}^{n-1} S(a_1 \cdots a_k) * a_{k+1} \cdots a_n \tag{4}$$

for a word $w = a_1 \cdots a_n$. The first formula for S then follows easily by induction on n. For the the second formula, we also proceed by induction on n, following the proof of Proposition 3.4 of [5]. For $w = a_1 \cdots a_n$, $n > 0$, the induction hypothesis and (4) give $S(w)$ as

$$\sum_{k=0}^{n-1} \sum_{(i_1,\dots,i_l)\in\mathcal{C}(k)} (-1)^{k+1} (i_1,\dots,i_l)[a_k a_{k-1} \cdots a_1] * a_{k+1} \cdots a_n =$$
$$\sum_{k=0}^{n-1} \sum_{(i_1,\dots,i_l)\in\mathcal{C}(k)} (-1)^{k+1} [a_k, a_{k-1}, \dots, a_{k-i_1+1}] \cdots [a_{i_l}, \dots, a_1] * a_{k+1} \cdots a_n$$

Now the first factor of each term of the $*$-product in the inner sum is, from consideration of (1), one of three generators: $[a_k, \dots, a_{k-i_1+1}]$, $[a_{k+1}, a_k, \dots, a_{k-i_1+1}]$, or a_{k+1}. We say the term is of type k in the first case, and of type $k+1$ in the latter two cases. Now consider a word that appears in the expansion of $S(w)$. If it has type $i \le n-1$, then it occurs for both $k = i$ and $k = i-1$, and the two occurrences will cancel. The only words that do not cancel are those of type n, which occur only for $k = n-1$: these will all carry the coefficient $(-1)^n$, and give the second formula for $S(w)$.

Remark. In the case of the shuffle algebra (i.e., where $[\cdot,\cdot]$ is identically zero), the second formula for the antipode is simply $S(w) = (-1)^{\ell(w)}\bar{w}$. Cf. [15, p. 35].

Theorem 3.3. $\exp : \mathfrak{A} \to \mathfrak{A}$ *is a Hopf algebra isomorphism of* $(\mathfrak{A}, ш, \Delta)$ *onto* $(\mathfrak{A}, *, \Delta)$.

Proof. We have already shown that exp is an algebra homomorphism. It suffices to show that $\exp \circ \epsilon(w) = \epsilon \circ \exp(w)$ and $\Delta \circ \exp(w) = (\exp \otimes \exp) \circ \Delta(w)$ for any word w. The first equation is immediate, and the second follows since both sides are equal to

$$\sum_{uv=w} \sum_{\substack{(i_1,\dots,i_k)\in\mathcal{C}(\ell(u)) \\ (j_1,\dots,j_l)\in\mathcal{C}(\ell(v))}} \frac{1}{i_1!\cdots i_k!} I[u] \otimes \frac{1}{j_1!\cdots j_l!} J[v].$$

4. Duality. The graded dual $\mathfrak{A}^* = \bigoplus_{n\ge 0} \mathfrak{A}_n^*$ has a basis consisting of elements w^*, where w is a word of $\mathfrak{A}$: the pairing $(\cdot,\cdot) : \mathfrak{A} \otimes \mathfrak{A}^* \to k$ is given by

$$(u, v^*) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

Then the transpose of Δ is the concatenation product $\operatorname{conc}(u^* \otimes v^*) = (uv)^*$, and the transpose of ш is the comultiplication δ defined by

$$\delta(w^*) = \sum_{\text{words } u,v \text{ of } \mathfrak{A}} (u \text{ ш } v, w^*) u^* \otimes v^*.$$

Since $(\mathfrak{A}, ш, \Delta)$ is a Hopf algebra, so is its graded dual $(\mathfrak{A}^*, \operatorname{conc}, \delta)$, which is called the concatenation Hopf algebra in [15]. Dualizing $(\mathfrak{A}, *, \Delta)$, we also have a Hopf algebra $(\mathfrak{A}^*, \operatorname{conc}, \delta')$, where δ' is the comultiplication defined by

$$\delta'(w^*) = \sum_{\text{words } u,v \text{ of } \mathfrak{A}} (u * v, w^*) u^* \otimes v^*.$$

Then from our earlier results we have the following.

Theorem 4.1. *There is a Hopf algebra isomorphism* $\exp^*$ *from* $(\mathfrak{A}^*, \operatorname{conc}, \delta')$ *to* $(\mathfrak{A}^*, \operatorname{conc}, \delta)$.

$\exp^*$ is the transpose of exp: explicitly, $\exp^*$ is the endomorphism of $(\mathfrak{A}^*, \operatorname{conc})$ with

$$\exp^*(a^*) = \sum_{n\ge 1} \frac{1}{n!} \sum_{(n)[w]=a} w^* = \sum_{n\ge 1} \sum_{[a_1,\dots,a_n]=a} \frac{1}{n!} (a_1 \cdots a_n)^*$$

for $a \in A$. It has inverse $\log^*$ given by

$$\log^*(a^*) = \sum_{n\ge 1} \frac{(-1)^{n-1}}{n} \sum_{(n)[w]=a} w^*, \quad a \in A. \tag{5}$$

The set of Lie polynomials in $\mathfrak{A}^*$ is the smallest sub-vector-space of $\mathfrak{A}^*$ containing the set of generators $\{a^* \mid a \in A\}$ and closed under the Lie bracket

$$[P, Q]_{\text{Lie}} = PQ - QP.$$

Since the Lie polynomials are exactly the primitives for δ [15, Theorem 1.4], we have the following result.

Theorem 4.2. *The primitives for δ' are elements of the form $\log^* P$, where P is a Lie polynomial.*

We note that $(\mathfrak{A}^*, \mathrm{conc}, \delta')$ has antipode

$$S^*(w^*) = \sum_{v \in \mathcal{P}(\bar{w})} (-1)^{\ell(v)} v^*,$$

where $\bar{w}$ is the reverse of w (i.e. $\bar{w} = a_n a_{n-1} \cdots a_1$ if $w = a_1 a_2 \cdots a_n$) and $\mathcal{P}(w) = \{v \mid I[v] = w \text{ for some } I \in \mathcal{C}(\ell(v))\}$.

5. q-deformation. We now define a deformation of $(\mathfrak{A}, *)$. We again start with the noncommutative polynomial algebra $\mathfrak{A} = k\langle A \rangle$ and define, for $q \in k$, a new multiplication $*_q$ by requiring that $*_q$ distribute over addition, that $w *_q 1 = 1 *_q w = w$ for any word w and that

$$aw_1 *_q bw_2 = a(w_1 *_q bw_2) + q^{|aw_1||b|} b(aw_1 *_q w_2) + q^{|w_1||b|}[a, b](w_1 *_q w_2) \tag{6}$$

for any words w_1, w_2 and letters a, b.

Theorem 5.1. *$(\mathfrak{A}, *_q)$ is a graded k-algebra, which coincides with $(\mathfrak{A}, *)$ when $q = 1$.*

Proof. The argument is similar to that for Theorem 2.1. It is easy to show that $|w_1 *_q w_2| = |w_1| + |w_2|$ for any words w_1, w_2 by induction on $\ell(w_1) + \ell(w_2)$. To show the operation $*_q$ associative, it suffices to show that $w_1 *_q (w_2 *_q w_3) = (w_1 *_q w_2) *_q w_3$ for any words w_1, w_2, and w_3, which we do by induction on $\ell(w_1) + \ell(w_2) + \ell(w_3)$. We can assume $w_i = a_i u_i$ for letters a_i and words u_i, $i = 1, 2, 3$. Then $w_1 *_q (w_2 *_q w_3)$ is

$$\begin{gathered} a_1(u_1 *_q a_2(u_2 *_q w_3)) + q^{|w_1||a_2|} a_2(w_1 *_q (u_2 *_q w_3)) + q^{|u_1||a_2|}[a_1, a_2](u_1 *_q (u_2 *_q w_3)) \\ + q^{|w_2||a_3|} a_1(u_1 *_q a_3(w_2 *_q u_3)) + q^{|w_2||a_3| + |w_1||a_3|} a_3(w_1 *_q (w_2 *_q u_3)) + \\ q^{|w_2||a_3| + |u_1||a_3|}[a_1, a_3](u_1 *_q (w_2 *_q u_3)) + q^{|u_2||a_3|} a_1(u_1 *_q [a_2, a_3](u_2 *_q u_3)) + \\ q^{|u_2||a_3| + |w_1||a_2 a_3|}[a_2, a_3](w_1 *_q (u_2 *_q u_3)) + q^{|u_2||a_3| + |u_1||a_2 a_3|}[a_1, a_2, a_3](u_1 *_q (u_2 *_q u_3)), \end{gathered}$$

while $(w_1 *_q w_2) *_q w_3$ is

$$\begin{gathered} a_1((u_1 *_q w_2) *_q w_3) + q^{|w_1 w_2||a_3|} a_3(a_1(u_1 *_q w_2) *_q u_3) + q^{|u_1 w_2||a_3|}[a_1, a_3]((u_1 *_q w_2) *_q u_3) \\ + q^{|w_1||a_2|} a_2((w_1 *_q u_2) *_q w_3) + q^{|w_1||a_2| + |w_1 w_2||a_3|} a_3(a_2(w_1 *_q u_2) *_q u_3) + \\ q^{|w_1||a_2| + |w_1 u_2||a_3|}[a_2, a_3]((w_1 *_q u_2) *_q u_3) + q^{|u_1||a_2|}[a_1, a_2]((u_1 *_q u_2) *_q w_3) + \\ q^{|u_1||a_2| + |w_1 w_2||a_3|} a_3([a_1, a_2](u_1 *_q u_2) *_q u_3) + q^{|u_1||a_2| + |u_1 u_2||a_3|}[a_1, a_2, a_3]((u_1 *_q u_2) *_q u_3). \end{gathered}$$

Applying the induction hypothesis, the difference is

$$\begin{gathered} a_1(u_1 *_q (a_2(u_2 *_q w_3) + q^{|w_2||a_3|} a_3(w_2 *_q u_3) + q^{|u_2||a_3|}[a_2, a_3](u_2 *_q u_3))) \\ + q^{(|w_2| + |w_1|)|a_3|} a_3(w_1 *_q (w_2 *_q u_3)) - a_1((u_1 *_q w_2) *_q w_3) \\ -q^{|w_1 w_2||a_3|} a_3((a_1(u_1 *_q w_2) + q^{|w_1||a_2|} a_2(w_1 *_q u_2) + q^{|u_1||a_2|}[a_1, a_2](u_1 *_q u_2)) *_q u_3), \end{gathered}$$

which by application of (6) and the induction hypothesis is seen to be zero.

Remark. The author arrived at the definition (6) as follows. Knowing the first two terms on the right-hand side from the definition of the quantum shuffle product, he tried an arbitrary power of q on the third term, and found that the resulting product was only associative when the exponent is as in (6). Shortly afterward he discussed this with J.-Y. Thibon, who directed him to [19], where the rule (6) appears in the special case of the quasi-symmetric functions (see Example 1 below).

Of course, for $q \neq 1$ the algebra $(\mathfrak{A}, *_q)$ is no longer commutative. For each fixed q, there is a homomorphism Φ_q of graded associative k-algebras from the concatenation algebra $(\mathfrak{A}, \mathrm{conc})$ to $(\mathfrak{A}, *_q)$ defined by

$$\Phi_q(a_1 a_2 \cdots a_n) = a_1 *_q a_2 *_q \cdots *_q a_n$$

for letters $a_1, a_2, \dots, a_n$; we call q generic if Φ_q is an isomorphism (i.e., if it is surjective). To give an explicit formula for Φ_q, we introduce some notation. For a permutation σ of $\{1, 2, \dots, n\}$, let $\iota(\sigma) = \{(i, j) \mid 1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)\}$ be the set of inversions of σ, and let $C(\sigma)$ be the descent composition of σ, i.e. the composition $(i_1, i_2, \dots, i_l) \in \mathcal{C}(n)$ with

$$\sigma(i_1 + \cdots + i_{j-1} + 1) < \sigma(i_1 + \cdots + i_{j-1} + 2) < \cdots < \sigma(i_1 + \cdots + i_j)$$

for $j = 1, 2, \dots, l$ and l minimal. (Equivalently, $C(\sigma) = (i_1, \dots, i_l)$ is the composition such that the associated subset $\{i_1, i_1 + i_2, \dots, i_1 + \cdots + i_{l-1}\}$ of $\{1, 2, \dots, n-1\}$ is the descent set of σ, i.e. the set of $1 \le i \le n-1$ such that $\sigma(i) > \sigma(i+1)$.)

Lemma 5.2. *For any letters $a_1, a_2, \dots, a_n$,*

$$\Phi_q(a_1 a_2 \cdots a_n) = \sum_{\text{permutations } \sigma} q^{\sum_{(i,j) \in \iota(\sigma)} |a_i||a_j|} \sum_{I \succcurlyeq C(\sigma)} I[a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}]$$

Proof. We proceed by induction on n, the case $n = 2$ being immediate. Assuming the induction hypothesis, we have

$$\Phi_q(a_1 \cdots a_{n+1}) = \sum_{(\sigma, I) \in P(n)} q^{\sum_{(i,j) \in \iota(\sigma)} |a_i||a_j|} I[a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}] * a_{n+1}$$

where $P(n)$ is the set of ordered pairs (σ, I) such that σ is a permutation of $\{1, 2, \dots, n\}$ and $I \succcurlyeq C(\sigma)$. For $(\sigma, I) \in P(n)$ with $I = (i_1, i_2, \dots, i_l)$ and $0 \le k \le l$, let σ'_k be the permutation of $\{1, 2, \dots, n+1\}$ given by

$$\sigma'_k(j) = \begin{cases} \sigma(j), & j \le i_1 + \cdots + i_k \\ n+1, & j = i_1 + \cdots + i_k + 1 \\ \sigma(j-1), & j > i_1 + \cdots + i_k + 1. \end{cases}$$

Also, for $0 \le k \le l$ let $I'_k = (i_1, \dots, i_k, 1, i_{k+1}, \dots, i_l)$, and for $1 \le k \le l$ let $I''_k = (i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_l)$; evidently $(\sigma'_k, I'_k), (\sigma'_k, I''_k) \in P(n+1)$ for all k. By iterated application of (6) we have

$$I[a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}] * a_{n+1} = q^{\sum_{i=1}^{n} |a_i||a_{n+1}|} a_{n+1} I[a_{\sigma(1)} \cdots a_{\sigma(n)}] +$$
$$\sum_{k=1}^{l} q^{\sum_{j=i_1+\cdots+i_k+1}^{n} |a_{\sigma(j)}||a_{n+1}|} \left(I'_k[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}] + I''_k[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}] \right).$$

Hence $\Phi_q(a_1 \cdots a_{n+1})$ is the sum over $(\sigma, I) \in P(n)$ of

$$q^{\sum_{(i,j)\in\iota(\sigma'_0)} |a_i||a_j|} I'_0[a_{\sigma'_0(1)} \cdots a_{\sigma'_0(n+1)}] + \sum_{k=1}^{l} q^{\sum_{(i,j)\in\iota(\sigma'_k)} |a_i||a_j|} \left(I'_k[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}] + I''_k[a_{\sigma'_k(1)} \cdots a_{\sigma'_k(n+1)}] \right)$$

and the conclusion follows by noting that every $(\tau, J) \in P(n+1)$ can be written uniquely as one of (σ'_k, I'_k) or (σ'_k, I''_k) for some $(\sigma, I) \in P(n)$.

In the case $q = 0$, our formula for $\Phi_q(w)$ reduces to

$$\Phi_0(w) = \sum_{I \in \mathcal{C}(\ell(w))} I[w] = (-1)^{\ell(w)} S(\bar{w}),$$

and by applying Lemma 2.4 with $f(t) = t/(1-t)$ we see that Φ_0 has inverse Φ_0^{-1} given by

$$\Phi_0^{-1}(w) = \sum_{k=1}^{\ell(w)} \sum_{I \in \mathcal{C}(\ell(w),k)} (-1)^{\ell(w)-k} I[w].$$

For any word $w = a_1 a_2 \cdots a_n$, let V_w be the vector space over k with basis $\{a_{\tau(1)} \cdots a_{\tau(n)} \mid \text{permutations } \tau\}$, and let $\phi_{w,q} : V_w \to V_w$ be Φ_q followed by projection onto V_w. Then $\phi_{w,q}$ is given by

$$\phi_{w,q}(a_{\tau(1)} \cdots a_{\tau(n)}) = \sum_{\text{permutations } \sigma} q^{\sum_{(i,j)\in\iota(\sigma)} |a_{\tau(i)}||a_{\tau(j)}|} a_{\sigma\tau(1)} \cdots a_{\sigma\tau(n)},$$

and we have the following result.

Lemma 5.3. *The linear map* $\phi_{w,q}$ *as defined above has determinant*

$$\prod_{k=2}^{n} \prod_{\substack{k\text{-sets} \\ S \subset \{1,\dots,n\}}} \left(1 - q^{2\sum_{i,j\in S} |a_i||a_j|}\right)^{(n-k+1)!(k-2)!}.$$

Proof. Following [4], we use Varchenko's theorem [20] on determinants of bilinear forms on hyperplane arrangements. To apply the result of [20], we consider the set of hyperplanes in $\mathbf{R}^n$ given by $\mathcal{H}_{ij} = \{(x_1, \dots, x_n) \mid x_i = x_j\}$. To the hyperplane $\mathcal{H}_{ij}$ we assign the weight $\operatorname{wt} \mathcal{H}_{ij} = q^{|a_i||a_j|}$. The edges (nontrivial intersections) of this arrangement are indexed by subsets $S \subset \{1, 2, \dots, n\}$ with two or more elements: the edge E_S corresponding to the set S is

$$\bigcap \{\mathcal{H}_{ij} \mid i, j \in S\} = \{(x_1, \dots, x_n) \mid x_i = x_j \text{ for all } i, j \in S\}.$$

The edge E_S has weight

$$\operatorname{wt} E_S = \prod_{i,j\in S} \operatorname{wt} \mathcal{H}_{ij} = q^{\sum_{i,j\in S} |a_i||a_j|}.$$

The domains (connected components) for this hyperplane arrangement are indexed by permutations: $C_\sigma = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \mid x_1 < x_2 < \cdots < x_n\}$. Then the quadratic form B on this arrangement given by

$$B(C_\sigma, C_\tau) = \prod_{\substack{\text{hyperplanes } \mathcal{H}_{ij} \\ \text{separating } C_\sigma \text{ and } C_\tau}} \operatorname{wt} \mathcal{H}_{ij} = \prod_{(i,j)\in\iota(\sigma\tau^{-1})} q^{|a_{\tau(i)}||a_{\tau(j)}|}$$

has the same matrix as $\phi_{w,q}$. Hence, by Theorem 1.1 of [20] we have

$$\det \phi_{w,q} = \prod_{\text{edges } E} (1 - \operatorname{wt}(E)^2)^{n(E)p(E)},$$

where the product is over the edges of the hyperplane arrangement, and $n(E)$ and $p(E)$ are numbers defined in §2 of [20]. It is easy to see from the definitions that $n(E_S) = (n-k+1)!$ and $p(E_S) = (k-2)!$ for the edge E_S corresponding to a k-set $S \subset \{1, \dots, n\}$, so the conclusion follows.

Theorem 5.4. *Any $q \in k$ that is not a root of unity is generic (i.e., Φ_q is an isomorphism when q is not a root of unity).*

Proof. Suppose q is not a root of unity. We shall show that $\Phi_q^{-1}(w)$ exists for any word w by induction on $\ell(w)$. Using Lemma 5.2 and the induction hypothesis, to find $\Phi_q^{-1}(a_1 \cdots a_n)$ it suffices to find an element u such that

$$\Phi_q(u) = a_1 a_2 \cdots a_n + \text{terms of length} < n.$$

But we can do this by taking $u = \phi_{w,q}^{-1}(w)$, and $\phi_{w,q}$ is invertible by Lemma 5.3.

If q is generic, we can define a comultiplication Δ_q on $\mathfrak{A}$ by requiring that all letters be Δ_q-primitives and that Δ_q be a $*_q$-homomorphism, i.e. that $\Delta_q(a) = a \otimes 1 + 1 \otimes a$ for all $a \in A$ and $\Delta_q(u *_q v) = \Delta_q(u) *_q \Delta_q(v)$ for all $u, v \in \mathfrak{A}$. This makes $(\mathfrak{A}, *_q, \Delta_q)$ a Hopf algebra. In fact, as we see in the next result, it is isomorphic to the concatenation Hopf algebra $(\mathfrak{A}, \operatorname{conc}, \delta)$, where

$$\delta(w) = \sum_{\text{words } u,v \text{ of } \mathfrak{A}} (u \sqcup\!\sqcup v, w^*) u \otimes v.$$

Theorem 5.5. *For generic q, Φ_q is a Hopf algebra isomorphism from $(\mathfrak{A}, \operatorname{conc}, \delta)$ to $(\mathfrak{A}, *_q, \Delta_q)$.*

Proof. Since q is generic, Φ_q is an algebra isomorphism. It suffices to show that $(\Phi_q \otimes \Phi_q) \circ \delta = \Delta_q \circ \Phi_q$ on a set of generators: but this follows because they agree on the primitives (elements of A), which generate $\mathfrak{A}$ under conc.

In the next result we record a formula for $\Delta_q(ab)$ when q is generic. This may be compared with the corresponding formula in Example 5.2 of [4].

Proposition 5.6. *Let $a, b, c \in A$. For q generic,*

$$\Delta_q(ab) = ab \otimes 1 + 1 \otimes ab + \frac{1}{1 + q^{|a||b|}}(a \otimes b + b \otimes a).$$

Proof. Apply Δ_q to the equation

$$ab = (1 - q^{2|a||b|})^{-1}(a *_q b - q^{|a||b|} b *_q a) - (1 - q^{|a||b|})^{-1}[a, b].$$

A formula for $\Delta_q(abc)$ can be derived by applying Δ_q to

$$abc = \left(\phi^{-1}_{abc,q}\right)_{\mathrm{id},\mathrm{id}} a *_q b *_q c + \left(\phi^{-1}_{abc,q}\right)_{\mathrm{id},(12)} b *_q a *_q c + \cdots + \text{terms of length } \leq 2,$$

but it is too complicated to give here (it contains twenty terms).

For the cases $q = 1$ and q not a root of unity, we have defined a Hopf algebra $(\mathfrak{A}, *_q, \Delta_q)$ with all elements of A primitive. It would be of interest to define such a Hopf algebra structure for all q.

6. Examples. As we have already remarked, if $[a, b] = 0$ for all generators $a, b \in A$ then $(\mathfrak{A}, *) = (\mathfrak{A}, ш)$ is the shuffle algebra as described in Chapter 1 of [15] (Note, however, that the grading may be different). The q-shuffle product $\odot_q$ as defined in [4, §4] is the operation $*_q = ш_q$ in this case. This algebra may also be obtained as a special case of the constructions of Green [10] and Rosso [16] involving quantum groups. To identify Green's "quantized shuffle algebra" with our construction, take the "datum" to be our generating set A, with bilinear form $a \cdot b = |a||b|$ for $a, b \in A$; then Green's algebra $G(k, q, A, \cdot)$ [10, p. 284] is our $(\mathfrak{A}, ш_q)$, except that Green's algebra is $\mathbf{N}A$-graded rather than $\mathbf{N}$-graded. To obtain our algebra from Rosso's "exemple fondamental" [16, §2.1], take V to be the vector space over k generated by $A = \{e_1, e_2, \dots\}$, and let $q_{ij} = q^{|e_i||e_j|}$. Here are some other examples.

Example 1. Let $A_n = \{z_n\}$ for all $n \geq 1$ and $[z_i, z_j] = z_{i+j}$. Then $(\mathfrak{A}, *)$ is just the algebra $\mathfrak{H}^1$ as presented in [12]. As is proved there (Theorem 3.4 ff.), the map ϕ defined by

$$\phi(z_{i_1} z_{i_2} \cdots z_{i_k}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k}$$

is an isomorphism of $\mathfrak{H}^1$ onto the algebra of quasi-symmetric functions over k (denoted QSym_k in [13]). For each $n \geq 0$, the monomial quasi-symmetric functions $M_{(i_1,\dots,i_k)} = \phi(z_{i_k} \cdots z_{i_1})$, where $(i_1, \dots, i_k) \in \mathcal{C}(n)$, form a vector-space basis for $\mathfrak{A}_n$. For our purposes it is more convenient to identify $M_{(i_1,\dots,i_k)}$ with $z_{i_1} \cdots z_{i_k}$: under this identification (which is also an isomorphism), the notation used above is simplified by the observation that, for compositions $I \in \mathcal{C}(n, k)$ and $J \in \mathcal{C}(k)$, $J[M_I] = M_{J \circ I}$. So, e.g., $S(M_I) = (-1)^{\ell(I)} \sum_{\bar{I} \succeq J} M_J$, where $\bar{I}$ is the reverse of I. If we let $\mathcal{L}$ denote the set of I such that M_I corresponds to a Lyndon word, then Theorem 2.6 says that $\{M_I \mid I \in \mathcal{L}\}$ generates $\mathfrak{A} = \mathrm{QSym}_k$ as an algebra. The Hopf algebra structure is that described in [5,13]; the two formulas for its antipode are discussed in [5, §3].

For the *integral* Hopf algebra QSym of quasi-symmetric functions, $\{M_I \mid I \in \mathcal{C}(n)\}$ is a $\mathbf{Z}$-module basis for the elements of degree n, but $\{M_I \mid I \in \mathcal{L}\}$ is not an algebra basis. Nevertheless, from [3,17] QSym has an algebra basis $\{M_I \mid I \in \mathcal{L}^{\mathrm{mod}}\}$, where $\mathcal{L}^{\mathrm{mod}}$ is the set of "modified Lyndon" or "elementary unreachable" compositions, i.e. concatenation powers of elements of $\mathcal{L}$ whose parts have greatest common factor 1. (There is a bijection of $\mathcal{L}$ onto $\mathcal{L}^{\mathrm{mod}}$ given by sending $(i_1, \dots, i_l)$ to the dth concatenation power of $(\frac{i_1}{d}, \dots, \frac{i_l}{d})$, where d is the greatest common factor of $i_1, \dots, i_k$.) Of course exp cannot be defined over $\mathbf{Z}$ because of denominators.

Another algebra basis for QSym_k is given by $\{P_I \mid I \in \mathcal{L}\}$, where $P_I = \exp(M_I)$. (These are exactly the elements whose duals $P_I^* = \log^*(M_I^*)$ are introduced in [13, §2] as a basis for the dual QSym_k^*; cf. equations (2.12) of [13] and (5) above.) Since exp is a Hopf algebra isomorphism, we have the formulas

$$P_I * P_J = \sum_{K \in I ш J} P_K, \quad \Delta(P_K) = \sum_{I \sqcup J = K} P_I \otimes P_J, \quad \text{and} \quad S(P_I) = (-1)^{\ell(I)} P_{\bar{I}},$$

where, for compositions I and J, $I ш J$ is the multiset of compositions obtained by "shuffling" I and J (e.g. $(1,2)$ ш $(2) = \{(2,1,2),(1,2,2),(1,2,2)\}$), and $I \sqcup J$ is the concatenation of I and J.

Following Gessel [8], there is still another basis $\{F_I \mid I \in \mathcal{L}\}$ for QSym_k, where $F_I = \sum_{J \succcurlyeq I} M_J$. (Then $M_I = \sum_{J \succcurlyeq I} (-1)^{\ell(J)-\ell(I)} F_J$, and since the coefficients are integral $\{F_I \mid I \in \mathcal{L}^{\mathrm{mod}}\}$ is a basis for QSym). The expansion of the product $F_I * F_J$ in terms of the F_K can be described using permutations and their descent compositions; see [19] or [13]. Dualizing Proposition 3.13 and Corollary 3.16 of [7] (see below), we have

$$\Delta(F_K) = \sum_{I \sqcup J = K} F_I \otimes F_J + \sum_{I \vee J = K} F_I \otimes F_J \quad \text{and} \quad S(F_I) = (-1)^{|I|} F_{I^\sim},$$

where $I \vee J = (i_1, \dots, i_{k-1}, i_k + j_1, j_2, \dots, j_l)$ for nonempty compositions $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$, and $I^\sim$ is the conjugate composition of I (as defined in [7, §3.2]). By dualizing Corollary 4.28 of [7] we have a formula for F_I in terms of the P_I:

$$F_I = \sum_{|J|=|I|} \mathrm{phr}(I,J) \frac{P_J}{\Pi(J)}.$$

Here $\Pi(I)$ is the product of the parts of the composition I, and $\mathrm{phr}(I,J)$ is as defined in [7, §4.9]: for compositions I and $J = (j_1, \dots, j_s)$ of the same weight, let $I = I_1 \bullet I_2 \bullet \cdots \bullet I_s$ be the unique decomposition of I such that $|I_i| = j_i$ for $1 \le i \le s$ and each symbol $\bullet$ is either $\sqcup$ or $\vee$; then

$$\mathrm{phr}(I,J) = \prod_{i=1}^{s} \frac{(-1)^{\ell(I_i)-1}}{\binom{|I_i|-1}{\ell(I_i)-1}}.$$

The dual Hopf algebra QSym_k^* is described in [13, §2]; it is also the algebra **Sym** of noncommutative symmetric functions as defined in [7]. (The coproduct δ' of §4 corresponds to the coproduct denoted γ in [13] and [7].) The M_I are dual to the "products of complete homogeneous symmetric functions" S^I (i.e., $(M_I, S^J) = \delta_{IJ}$), while the "products of power sums of the second kind" Φ^I are dual to the elements $P_I/\Pi(I)$ (see [7, §3] for definitions). The F_I are dual to the "ribbon Schur functions" R_I [7, §6].

The deformation $(\mathfrak{A}, *_q)$ is the algebra of quantum quasi-symmetric functions as defined in [19]. The multiplication rule for "quantum quasi-monomial functions" as given in [19, p. 7345] can be recognized as (6).

Example 2. For a fixed positive integer r, let $A_n = \{z_{n,i} \mid 0 \le i \le r-1\}$ and $[z_{n,i}, z_{m,j}] = z_{n+m,i+j}$, where the second subscript is to be understood mod r. By

Theorem 2.6, $(\mathfrak{A}, *)$ is the polynomial algebra on the Lyndon words in the $z_{i,j}$; by Proposition 2.7, the number of Lyndon words in $\mathfrak{A}_n$ is

$$L_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right)(r+1)^d$$

for $n \geq 2$ (and $L_1 = r$). In this case, we call the Hopf algebra $(\mathfrak{A}, *, \Delta)$ the Euler algebra $\mathfrak{E}_r$. Of course $\mathfrak{E}_1$ is the preceding example (We write z_i for $z_{i,0}$ if $r = 1$); in general there is a homomorphism $\pi_r : \mathfrak{E}_r \to \mathfrak{E}_1$ given by $\pi_r(z_{i,j}) = z_i$. The map $\phi : \mathfrak{E}_r \to \mathbf{C}[[t_1, t_2 \dots]]$ with

$$\phi(z_{i_1,j_1} z_{i_2,j_2} \cdots z_{i_k,j_k}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} e^{\frac{2\pi i}{r}(n_1 j_1 + \cdots + n_k j_k)} t_{n_1}^{i_1} \cdots t_{n_k}^{i_k} \tag{7}$$

is an isomorphism of $\mathfrak{E}_r$ onto a subring of $\mathbf{C}[[t_1, t_2 \dots]]$ (for proof see §7 below.) If we define $\psi_r : \mathbf{C}[[t_1, t_2 \dots]] \to \mathbf{C}[[t_1, t_2 \dots]]$ by

$$\psi_r(t_i) = \begin{cases} 0, & r \nmid i \\ t_j, & i = rj \end{cases}$$

(Note ψ_r takes $\mathrm{QSym}_k \subset \mathbf{C}[[t_1, t_2 \dots]]$ isomorphically onto itself!), then $\psi_r \circ \phi = \phi \circ \pi_r$. The sets L of Lyndon words in the $z_{i,j}$ and $\{\exp(w) \mid w \in L\}$ are both algebra bases for $\mathfrak{E}_r$, corresponding to the bases $\{M_I \mid I \in \mathcal{L}\}$ and $\{P_I \mid I \in \mathcal{L}\}$, respectively, of Example 1. If we set $\hat{w} = \sum_{v \in \mathcal{P}(w)} v$, where $\mathcal{P}(w)$ is as defined at the end of §4, then there is a a basis $\{\hat{w} \mid w \in L\}$ corresponding to $\{F_I \mid I \in \mathcal{L}\}$. Note, however, that while π_r maps words to the M_I and exponentials of words to the P_I (exp commutes with π_r), in general $\pi_r(\hat{w})$ is not of the form F_I.

The dual $\mathfrak{E}_r^*$ of the Euler algebra is the concatenation algebra on elements $z_{i,j}^*$, with coproduct δ' as described in §4. The transpose of π_r is the homomorphism $\pi_r^* : \mathfrak{E}_1^* \to \mathfrak{E}_r^*$ with $\pi_r^*(z_i^*) = \sum_{j=1}^{r-1} z_{i,j}^*$.

The motivation for the Euler algebra $\mathfrak{E}_r$ comes from numerical series of the form

$$\sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{\epsilon_1^{n_1} \epsilon_2^{n_2} \cdots \epsilon_k^{n_k}}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \tag{8}$$

where the ϵ_i are rth roots of unity and $i_1, i_2, \dots, i_k$ are positive integers (with $\epsilon_1 i_1 \neq 1$, for convergence). In fact (8) is $\lim_{n\to\infty} \phi_n(z_{i_1,j_1} \cdots z_{i_k,j_k})(1, 2, \dots, \frac{1}{n})$, where ϕ_n is as defined in §7 and the j_s are chosen appropriately, so the algebra of such series can be seen as a homomorphic image of (a subalgebra of) $\mathfrak{E}_r$. These series are called "Euler sums" in [1,2] and "values of multiple polylogarithms at roots of unity" in [9]; in the case $r = 1$ the corresponding series are known as "multiple harmonic series" [12] or "multiple zeta values" [21].

Example 3. Fix a positive integer m and let $A_n = \{z_n\}$ for $n \leq m$ and $A_n = \emptyset$ for $n > m$. Define

$$[z_i, z_j] = \begin{cases} z_{i+j} & \text{if } i + j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{A}, *)$ is the algebra of quasi-symmetric functions on variables $t_1, t_2, \dots$ subject to the relations $t_i^{m+1} = 0$ for all i.

Example 4. Let $P(n)$ be the set of partitions of n and let $A_n = \{z_\lambda \mid \lambda \in P(n)\}$. Define $[z_\lambda, z_\mu] = z_{\lambda\cup\mu}$, where $\lambda \cup \mu$ is the union λ and μ as multisets. Then $(\mathfrak{A}, *)$ can be thought of as the algebra of quasi-symmetric functions in the variables $t_{i,j}$, where $|t_{i,j}| = j$, in the following sense. For a partition $\lambda = (n_1, \dots, n_l)$, let $t_i^\lambda = t_{i,n_1} \cdots t_{i,n_l}$. Then any monomial in the $t_{i,j}$ can be written in the form $t_{i_1}^{\lambda_1} \cdots t_{i_k}^{\lambda_k}$, and we call a formal power series quasi-symmetric when the coefficients of any two monomials $t_{i_1}^{\lambda_1} \cdots t_{i_k}^{\lambda_k}$ and $t_{j_1}^{\lambda_1} \cdots t_{j_k}^{\lambda_k}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ are the same.

7. The Euler algebra as power series. Fix a positive integer r, and let $\mathfrak{E}_r$ and $\pi_r : \mathfrak{E}_r \to \mathfrak{E}_1$ be as in Example 2. We shall show $\mathfrak{E}_r$ can be imbedded in the formal power series ring $\mathbf{C}[[t_1, t_2, \dots]]$. For positive integers n, define a map $\phi_n : \mathfrak{E}_r \to \mathbf{C}[t_1, \dots, t_n]$ as follows. Let ϕ_n send $1 \in \mathfrak{E}_r$ to $1 \in \mathbf{C}[t_1, \dots, t_n]$ and any nonempty word $w = z_{i_1,j_1} z_{i_2,j_2} \dots z_{i_k,j_k}$ to the the polynomial

$$\sum_{n\ge n_1>n_2>\cdots>n_k\ge 1} \omega^{j_1n_1+j_2n_2+\cdots+j_kn_k} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k},$$

where $\omega = e^{\frac{2\pi i}{r}}$ (If $k > n$, the sum is empty and we assign it the value 0). Extend ϕ_n to $\mathfrak{E}_r$ by linearity. If we make $\mathbf{C}[t_1, \dots, t_n]$ a graded algebra by setting $|t_i| = 1$, then ϕ_n preserves the grading. Also, it is immediate from the definition that

$$\phi_n(z_{p,i}w) = \sum_{n\ge m>1} \omega^{im} t_m^p \phi_{m-1}(w) \tag{9}$$

for any nonempty word w.

Theorem 7.1. *For any n, $\phi_n : \mathfrak{E}_r \to \mathbf{C}[t_1, \dots, t_n]$ is a homomorphism of graded k-algebras.*

Proof. It suffices to show $\phi_n(w_1 * w_2) = \phi_n(w_1)\phi_n(w_2)$ for words w_1, w_2. This can be done by induction on $\ell(w_1)+\ell(w_2)$, following the argument of [12, Theorem 3.2] (and using equation (9) above in place of equation (*) of [12]).

Lemma 7.2. *For $0 \le j_1, j_2, \dots, j_m \le r-1$, let $c_{j_1,j_2,\dots,j_m} \in \mathbf{Q}$ be such that*

$$\sum_{j_1=0}^{r-1}\sum_{j_2=0}^{r-1}\cdots\sum_{j_m=0}^{r-1} c_{j_1,j_2,\dots,j_m}\omega^{n_1j_1+n_2j_2+\cdots+n_mj_m} = 0$$

for all $mr \ge n_1 > n_2 > \cdots > n_m \ge 1$, where $\omega = e^{\frac{2\pi i}{r}}$ as above. Then all the $c_{j_1,j_2,\dots,j_m}$ are zero.

Proof. We use induction on m. For $m = 1$ the hypothesis is

$$\sum_{j=1}^{r-1} c_j\omega^{nj} = 0 \quad \text{for all} \quad 1 \le n \le r,$$

which is evidently equivalent to having the equality for $0 \le n \le r-1$. But then the conclusion follows from the nonsingularity of the Vandermonde determinant of the quantities $1, \omega, \omega^2, \dots, \omega^{r-1}$.

Now let $m > 1$, and fix $(m-1)r \geq n_2 > n_3 > \cdots > n_m \geq 1$. Then the hypothesis says

$$\sum_{j_1=0}^{r-1} \left(\sum_{j_2=0}^{r-1} \cdots \sum_{j_m=0}^{r-1} c_{j_1,j_2,\ldots,j_m} \omega^{n_2 j_2 + \cdots + n_m j_m} \right) \omega^{n_1 j_1} = 0 \quad \text{for } (m-1)r < n_1 \leq mr.$$

This is evidently equivalent to having the equality hold for all $1 \leq n_1 \leq r$: but then we are in the situation of the preceding paragraph and so

$$\sum_{j_2=0}^{r-1} \cdots \sum_{j_m=0}^{r-1} c_{j_1,j_2,\ldots,j_m} \omega^{n_2 j_2 + \cdots + n_m j_m} = 0,$$

from which the conclusion follows by the induction hypothesis.

Theorem 7.3. *The homomorphism ϕ_{nr} is injective through degree n.*

Proof. Suppose $u \in \ker \phi_{nr}$ has degree $\leq n$. Without loss of generality we can assume u is homogeneous, and in fact that $\pi_r(u)$ is a multiple of $z_{i_1} z_{i_2} \cdots z_{i_m}$ for $m \leq n$. Then u has the form

$$u = \sum_{j_1=0}^{r-1} \sum_{j_2=0}^{r-1} \cdots \sum_{j_m=0}^{r-1} c_{j_1,j_2,\ldots,j_m} z_{i_1,j_1} z_{i_2,j_2} \cdots z_{i_m,j_m},$$

and $u \in \ker \phi_{nr}$ implies that

$$\sum_{j_1=0}^{r-1} \sum_{j_2=0}^{r-1} \cdots \sum_{j_m=0}^{r-1} c_{j_1,j_2,\ldots,j_m} \omega^{n_1 j_1 + n_2 j_2 + \cdots + n_m j_m} = 0$$

for all $nr \geq n_1 > n_2 > \cdots > n_m \geq 1$. But then $u = 0$ by the lemma.

For $m \geq n$, there is a restriction map $\rho_{m,n} : \mathbf{C}[t_1, \ldots, t_m] \to \mathbf{C}[t_1, \ldots, t_n]$ sending t_i to t_i for $1 \leq i \leq n$ and t_i to zero for $i > n$. Let $\mathfrak{P}$ be the inverse limit of the $\mathbf{C}[t_1, \ldots, t_n]$ with respect to these maps (in the category of graded algebras); $\mathfrak{P}$ is a subring of $\mathbf{C}[[t_1, t_2, \ldots]]$. The ϕ_n define a homomorphism $\phi : \mathfrak{E}_r \to \mathfrak{P}$, and the following result is evident.

Theorem 7.4. *The homomorphism ϕ is injective, and satisfies equation (7).*

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Mathematics Department, U.S. Naval Academy, Annapolis, Md. 21402
E-mail address: meh@nadn.navy.mil