Hidden symmetry and Hopf algebra

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In celebration of fecund
Pepín Cariñena's 60th birthday

Abstract

We spell two conundrums, one of physical and another of mathematical nature, and explain why one helps to elucidate the other.

1 Introduction I: Hopf algebra cohomology

Let us start by the mathematical conundrum. The two main classical examples of Hopf algebras, respectively cocommutative and commutative, are the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ and the algebra $\mathcal{R}(G)$ of representative functions on a group G. For definiteness, consider both over the complex numbers. On $\mathcal{U}(\mathfrak{g})$ the coproduct $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ is defined first on elements $X \in \mathfrak{g}$ by

$$\Delta(X) := X_{(1)} \otimes X_{(2)} = X \otimes 1 + 1 \otimes X,$$

and then extended to all of $\mathcal{U}(\mathfrak{g})$ multiplicatively. The output of Δ is invariant under exchange of the two copies of $\mathcal{U}(\mathfrak{g})$ in its image: this is cocommutativity. For the second, $\mathcal{R}(G)$ is the space of functions $f : G \to \mathbb{C}$ whose translates $x \mapsto f(xt)$, for all $t \in G$, generate a finite-dimensional subalgebra of the commutative algebra of continuous functions $C(G)$ under ordinary multiplication. Then also $\mathcal{R}(G)$ is endowed with a coproduct in which

$$\mathcal{R}(G) \otimes \mathcal{R}(G) \ni \Delta f \quad \text{is given by} \quad \Delta f(x,y) := \big(f_{(1)} \otimes f_{(2)}\big)(x,y) := f(xy);$$

which is not cocommutative, unless G is abelian. There is of course a functor going back from commutative Hopf algebras to groups.

When $\mathfrak{g}$ is the tangent Lie algebra of a Lie group G, it is sometimes asserted that both previous constructions are mutually dual. Reality is richer: although $\mathcal{U}(\mathfrak{g})$ is certainly in duality to $\mathcal{R}(G)$, there is a bigger dual space, the Sweedler dual $\mathcal{R}^\circ(G)$ of $\mathcal{R}(G)$, which is still a Hopf algebra, and includes in particular the (also cocommutative) *group algebra* $\mathbb{C}G$; $\Delta g = g\otimes g$ holds for 'pure' elements $g \in \mathbb{C}G$. In fact, $\mathcal{R}^\circ(G)$ is a semidirect product of $\mathbb{C}G$ and $\mathcal{U}(\mathfrak{g})$. We touch here at the general situation, as any cocommutative Hopf algebra is a semidirect product of a group algebra and an enveloping algebra [1, 2]. For the general background on Hopf algebras and matters of notation, besides [2] we refer to [3]; we denote by 1 the unit in H and the augmentation homomorphism by η. Going to cohomology, is stands to reason that the cohomology of enveloping algebras will contain the same information as the theory of Lie algebra extensions, and that of commutative Hopf algebras as the theory of group cocycles. But this is not quite what happens! Let us follow Majid [4] now. With id the identity map of H onto itself, define four maps from $H \otimes H$ to $H \otimes H \otimes H$ by:

$$\Delta_0(\cdot) = 1 \otimes (\cdot); \quad \Delta_3(\cdot) = (\cdot) \otimes 1; \quad \Delta_1 = \Delta \otimes \mathrm{id}; \quad \Delta_2 = \mathrm{id} \otimes \Delta.$$

Let χ be an invertible (in the algebra) element of $H \otimes H$. This is a 2-cochain in general. Then its coboundary:

$$H \otimes H \otimes H \ni \partial\chi := \Delta_0(\chi)\Delta_2(\chi)\Delta_1(\chi^{-1})\Delta_3(\chi^{-1}) =: \partial_+\chi\partial_-\chi^{-1}.$$

An 2-cocycle *for* H is a 2-cochain such that $\partial\chi = 1$. We compute:

$$\begin{aligned} &(1 \otimes \chi)(\mathrm{id} \otimes \Delta)\chi(\Delta \otimes \mathrm{id})\chi^{-1}(\chi^{-1} \otimes 1) = 1, \\ &\text{that is} \quad (1 \otimes \chi)(\mathrm{id} \otimes \Delta)\chi = (\chi \otimes 1)(\Delta \otimes \mathrm{id})\chi. \end{aligned} \tag{1.1}$$

Now, for $H = \mathcal{R}(G)$, we recognize a group 2-cocycle, that is a nowhere vanishing function χ on $G \times G$ such that

$$\chi(g_1, g_2)\chi(g_1g_2, g_3) = \chi(g_1, g_2g_3)\chi(g_2, g_3); \quad \forall g, g_1, g_2, g_3 \in G.$$

We have recovered the standard theory of group 2-cocycles, allowing to construct group extensions, and well known to physicists —we require as well unitality of χ, that is $(\eta\otimes\mathrm{id})\chi = (\mathrm{id} \otimes \eta)\chi = 1$, guaranteeing $\chi(g, 1_G) = \chi(1_G, g) = 1$.

There naturally exists a dual theory of cocycles *on* Hopf algebras that, when applied to $\mathcal{U}(\mathfrak{g})$, reproduces the results of Lie algebra cohomology. For Lie algebras like $\mathfrak{P}$, the one of the Poincaré group, which is well known to be inextensible, such dual procedure gives nothing; and of course the same is true of the previous theory of cocycles *for* Hopf algebras when applied to the component of the identity of the Poincaré group. But, what about coming back to the framework of (1.1) and trying to apply it to the *noncommutative* Hopf algebra $\mathcal{U}(\mathfrak{P})$? Or, for that matter, what about trying to apply the theory of cocycles *on* Hopf algebras to $\mathcal{R}(G)$? Well, it is not true that the theory of n-cocycles *for* Hopf algebras, when used on objects that are not commutative; or the dual theory of n-cocycles *on* Hopf algebras, when used on non-cocommutative algebras, always lead to proper cohomologies. But this was never to stop quantum group theorists; and, lo and behold, for 2-cocycles there is no difficulty, indicating that a sort of generalized symmetry is present. A 2-cocycle *for* $\mathcal{U}(\mathfrak{g})$ is precisely what they call a *twist*. We briefly review how twists permit to deform the coproduct. Let H be a cocommutative Hopf algebra and χ the twist. Consider $\Delta_\chi(a) := \chi\Delta(a)\chi^{-1}$.

This gives a new coproduct on H. First of all, for the new coproduct Δ_χ is still an algebra map:

$$\Delta_\chi(ab) = \chi\Delta(ab)\chi^{-1} = \chi\Delta(a)\Delta(b)\chi^{-1} = \chi\Delta(a)\chi^{-1}\chi\Delta(b)\chi^{-1} = \Delta_\chi(a)\Delta_\chi(b).$$

Let us check coassociativity of Δ_χ:

$$\begin{aligned}(\Delta_\chi \otimes \mathrm{id})\Delta_\chi(a) &= \chi_{12}(\Delta \otimes \mathrm{id})(\chi\Delta(a)\chi^{-1})\chi_{12}^{-1} \\ &= \chi_{12}((\Delta \otimes \mathrm{id})\chi)((\Delta \otimes \mathrm{id})\Delta a)((\Delta \otimes \mathrm{id})\chi^{-1})\chi_{12}^{-1} \\ &= \chi_{23}((\mathrm{id} \otimes \Delta)\chi)((\mathrm{id} \otimes \Delta)\Delta a)((\mathrm{id} \otimes \Delta)\chi^{-1})\chi_{23}^{-1} = (\mathrm{id} \otimes \Delta_\chi)\Delta_\chi(a).\end{aligned}$$

Here χ_{12} of course means $\chi \otimes 1 \in H \otimes H \otimes H$, and so on. We have used (1.1). The resulting Hopf algebra is denoted H_χ. Naturally this twisting procedure to create new Hopf algebras, when used with cohomologous cocycles, gives Hopf algebras that are isomorphic via inner automorphisms; but often an appropriate twist gives a novel construction.

Now, for H any Hopf algebra, not necessarily commutative or cocommutative, a left (Hopf) H-module algebra $(A, \times)$ is a not necessarily commutative algebra which is a representation space for (the algebra structure of) H, and moreover

$$h \cdot (a \times b) = h_{(1)} \cdot a \times h_{(2)} \cdot b \quad \text{whenever} \quad h \in H,\ a, b \in A.$$

The formula $h \cdot 1_A = \eta(h)1_A$ usually added here is redundant [5]. Write also $\times(a \otimes b) = a \times b$. The consequence of the twist is that the product

$$a \star_\chi b := \times\big(\chi^{-1}(a \otimes b)\big), \tag{1.2}$$

for $a, b \in A$, defines a new associative algebra A_χ, covariant under H_χ. In effect, associativity of $\star_\chi$ follows from the 2-cocycle condition. One trivially checks covariance: for $h \in H$,

$$\begin{aligned}h \cdot (a \star_\chi b) &:= h \cdot \times\big(\chi^{-1}(a \otimes b)\big) = \times\big(\Delta(h) \cdot \chi^{-1}(a \otimes b)\big) \\ &= \times\big(\chi^{-1}\Delta_\chi(h) \cdot (a \otimes b)\big) =: \star_\chi\big(\Delta_\chi(h) \cdot (a \otimes b)\big).\end{aligned}$$

We ask forgiveness from the reader for the heavy notation; it will be needed later.

The mystery is this: in principle there is *no more* information in the Hopf algebra $\mathcal{U}(\mathfrak{g})$ than in the Lie algebra $\mathfrak{g}$. So, in terms of symmetry, what may the twisting procedure mean?

For $\mathfrak{P}$ with its usual generators $T_\nu, M_{\alpha\beta}$, taking $\chi_\Theta := \exp(-\frac{i}{2}\theta^{\mu\nu}\, T_\mu \otimes T_\nu)$, where $\Theta := (\theta^{\mu\nu})$ is a skew-symmetric matrix, a little calculation with a glance at (3.2) gives:

$$\Delta_\chi(M_{\alpha\beta}) = M_{\alpha\beta} \otimes 1 + 1 \otimes M_{\alpha\beta} + \tfrac{i}{2}\theta^{\rho\sigma}\big((g_{\alpha\rho}T_\beta - g_{\beta\rho}T_\alpha) \otimes T_\sigma + T_\rho \otimes (g_{\alpha\sigma}T_\beta - g_{\beta\sigma}T_\alpha)\big);$$

while the coproduct for the T_ν is not modified. This is, in cohomological terms, what was done in [6]. As in that reference, to check the cocycle condition (1.1) is left to the reader. Moreover, $\star_{\chi_\Theta}$ is a Moyal product [7], and the apparently unlikely deed of having the Poincaré Lie algebra act on Moyal algebra has been done. Deforming the coproduct of an enveloping algebra is much less drastic than deforming the product, and, very gratifyingly, the Casimirs and the whole paraphernalia of relativistic fields remain unaffected. Still, the manner in which the action of $\mathfrak{P}$ on its representation spaces propagates to their tensor products has been modified.

2 Introduction II: a physical discussion

The motivation for [6] was that the question of relativistic symmetry on noncommutative Minkowski (or Euclidean, as the case may be) spacetime is apparently a vexing one. Indeed, periodically, and as recently as [8], there are complaints about the calamitous state of the study of covariance in noncommutative field theory (NCFT). Often, authors just look at the Moyal commutators $[x^\mu, x^\nu]_{\star_\Theta} = i\theta^{\mu\nu}$ and conclude that Poincaré invariance is broken down to a subgroup. To wit, with $\mathfrak{P}_n, \mathfrak{E}_n$ respectively denoting the $\frac{1}{2}(n^2+n)$ dimensional Poincaré (respectively, Euclidean) Lie algebras of Lorentz transformations (respectively, rotations) and translations on $\mathbb{R}^n$, and assuming Θ has maximal rank, $\mathfrak{P} \equiv \mathfrak{P}_4$ reduces to a six-dimensional Lie algebra isomorphic to $\mathfrak{P}_2 \oplus \mathfrak{E}_2$. See [9]. Respectively, $\mathfrak{E}_4$ would break down to $\mathfrak{E}_2 \oplus \mathfrak{E}_2$. One easily finds in the NCFT literature statements like: "the physics depends on the frame of reference" [10]; picturesquely adding that it must be so, because the speed of light in a noncommutative geometry depends on the direction of motion. Also [11] espouses the viewpoint of [10]. On the face of it, this is a defensible one.

But if so, would it be justified to use the Wigner particle structure of scalar, vector, spinor fields and so on, as done as a matter of course in almost every paper in NCFT? This is the second mystery.

This is why the ideas in paper [6] —see also [12]— were welcome. These authors apparently establish that a form of Poincaré covariance is relevant in NCFT. In particular, Poincaré group representations and their tensor products are totally pertinent. Later, it has been claimed that the analysis of [6] extends to the conformal group [13]; also twisted conformal symmetry in NCFT in two dimensions has been examined [14].

There is, however, a touch of obscurantism in [6]. For a start, their treatment is couched in the abstract language of quantum groups, and no physical interpretation of their reconstruction of Moyal algebra from a twisting of the coproduct on the Poincaré enveloping algebra was attempted. Also, the twisting is a general geometrical fact, not specifically linked to the the Poincaré group.

Actually, some of the early treatments of relativistic symmetry in NCFT are more forthcoming: we refer to the lucid remarks in [15] and the analysis in the deep paper [16]. One may rephrase their argument as follows. Assume that in a region of the space there is a background field. Its presence modifies the vacuum, breaking Poincaré invariance, in the sense that active Lorentz transformations are no longer symmetries of the physical system. But this does not stop an electron under the influence of that background from being a relativistic electron. It remains possible for observers to describe the system in a Poincaré covariant way, by suitable changes in the description of the background (the so-called observer or passive Lorentz transformations). In doing so, one stays within the same theory; were we to modify charges, masses or other internal variables of the system it would be otherwise. The phenomenon of variation of the speed of light is expected with any background field [17]; a result that does not contradict relativity: rather, relativity is used to derive it.

Now, one may contend that the situation is analogous in NCFT, where one has the skew-symmetric tensor Θ describing the background. Noncommutativity would brook no ether, even we ignore as yet what its dynamical equations —and boundary conditions— are. The origin of the theory in string dynamics [18] does not appear to contradict this view. Our analysis has points of contact with the recent papers [19], that use the Hopf dual H_χ° of H_χ, and with [20]. One should rather not speak alternatively of unbroken/broken symmetry, but of *manifest/hidden* invariance. Relativistic symmetry is simply hidden in NCFT.

The solution of the second conundrum holds a key to the first. Symmetries in the noncommutative regime (no less than in the commutative one) are *always* described by automorphisms —that is, derivations at the infinitesimal level— of the algebra of observables. When the symmetry is hidden, those derivations involve the parameters of the vacuum state. We discover that, in some cases at least, twists or deformations of Hopf algebras are related to hidden symmetries.

3 Conventions

The form of the Moyal product used in this paper is that of Rieffel [21]; this is good for any rank of Θ, and is moreover an exact (nonperturbative in Θ) deformation of the ordinary product. Due to the singular nature of the $\Theta \downarrow 0$ limits, all kind of pitfalls await the unwary user of perturbative forms. For the precise relation between both kinds we refer to [22], in the analogous framework of phase space Quantum Mechanics. Of course, at some points we need to fall back on perturbative forms for comparison purposes. Given the 4×4 skewsymmetric matrix Θ, the Moyal star product on $\mathbb{R}^4$ is:

$$f \star_\Theta h(x) = \frac{1}{(2\pi)^4} \int d^4y\, d^4u\, f\big(x + \tfrac{1}{2}\Theta y\big)\, h(x+u)\, e^{iy\cdot u}. \tag{3.1}$$

The group $A(4;\mathbb{R})$ of affine transformations acts on four-vectors by $x \mapsto Lx + a$, where $a \in \mathbb{R}^4$ and L denotes a matrix with $\det L \neq 0$. We have $(L,a)(L',a') = (LL', La' + a)$. Thus the inverse transformation of (L,a) is $(L^{-1}, -L^{-1}a)$. Often we write just g for (L,a) and $g \cdot x$ for $Lx + a$. An action on functions on $\mathbb{R}^4$ ensues, of the form:

$$[(L,a) \triangleright f](x) := f\big(L^{-1}(x-a)\big).$$

This definition leads to the natural $g_1 \triangleright [g_2 \triangleright f] = (g_1 g_2) \triangleright f$.

The 11-dimensional Weyl group W of *rigid* conformal transformations (translations and dilations plus Lorentz transformations) generated by $\{\, T_\tau, D, M_{\alpha\beta} \,\}$, with commutation relations:

$$\begin{aligned} [T_\tau, T_\sigma] = 0; \qquad [T_\tau, D] = T_\tau; \qquad [T_\tau, M_{\alpha\beta}] = g_{\tau\alpha} T_\beta - g_{\tau\beta} T_\alpha; \\ [D, M_{\alpha\beta}] = 0; \quad [M_{\alpha\beta}, M_{\gamma\delta}] = g_{\beta\gamma} M_{\alpha\delta} + g_{\alpha\delta} M_{\beta\gamma} - g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\delta} M_{\alpha\gamma}, \end{aligned} \tag{3.2}$$

will be envisaged. This subgroup of $A(4;\mathbb{R})$ is singled out in relation with dynamical —as opposed to merely geometrical— aspects: for definiteness we consider now $\star$-gauge (noncommutative Yang–Mills) theories, whose action is invariant under W. This is as in [16]. The prototype is the Maxwell-$\star$ theory on $\mathbb{R}^4$, with gauge potential A_μ. Unfortunately, lack of space prevents us from going into the particulars of gauge theory: almost solely its vector aspect is important here. Throughout, we consider $\mathbb{R}^4_\Theta$ with constant (position-independent) noncommutativity. Let us note, however, that the interplay between coordinate, gauge and Θ-variables characteristic of NCFT is even more patent in non-constant noncommutativity spaces [23], of whose the one considered here must be regarded as a limit case.

4 Twisted affine transformations

The question is to compute $[g \triangleright f] \star_\Theta [g \triangleright h]$. Denote by L^{-t} the contragredient matrix of L. By a simple change of variables in the integral (3.1) one obtains:

$$\begin{aligned} &[g \triangleright f] \star_\Theta [g \triangleright h](g \cdot x) = f \star_{L^{-1}\Theta L^{-t}} h(x); \quad \text{that is to say,} \\ &[g \triangleright f] \star_\Theta [g \triangleright h] = g \triangleright (f \star_{L\Theta L^t} h). \end{aligned} \tag{4.1}$$

In the noncommutative world, i.e., for $\Theta \neq 0$, spacetime and parameter transformations are intimately linked; we see in (4.1) emerging an action, trivial for translations, of the affine group on the linear space of skewsymmetric matrices, given by

$$(L, a) \cdot \Theta = L \cdot \Theta := L\Theta L^t.$$

There is neither novelty nor mystery about this action: it is just classical congruence, studied by Lagrange and Sylvester centuries ago. Its only invariant is the rank, so the orbits are constituted respectively by the generic set of invertible skewsymmetric matrices, the set of non-invertible, nonvanishing skewsymmetric matrices, and the zero matrix. Given Θ, the matrices $L \in A(4;\mathbb{R})$ such that $L \cdot \Theta = \Theta$ form a "little group" A_Θ, of dimension 10 for the generic orbit (then and only then does Θ define a symplectic form). There is of course an enormous difference between merely regarding A_Θ —or $A_\Theta \cap W$— as 'the' symmetry group, and regarding it as the result of a symmetry breaking $A(4;\mathbb{R}) \downarrow A_\Theta$ of a larger group.

In summary, on the variables (x, Θ) the affine transformations act by

$$(L, a) \cdot (x, \Theta) = (Lx + a, L\Theta L^t). \tag{4.2}$$

For the induced action on the sections of the field of $\star$-algebras over the space of all Θ's, regarded as functions of (x, Θ), from (4.1) we conclude that

$$[g \triangleright f] \star_\Theta [g \triangleright h] = g \triangleright (f \star_\Theta h). \tag{4.3}$$

Such an automorphism equation is the trademark of covariance. The paper is but a corollary of this fundamental formula. Incidentally, the oldest avatar of these formulae we know of was found in [24]. Also, recently (4.3) has been rederived from a different viewpoint in [25].

If $g \in A_\Theta$, its action is vertical on that field, and then we may replace (4.3) by:

$$[g \triangleright f] \star_\Theta [g \triangleright h](x) = f \star_\Theta h(g^{-1} \cdot x).$$

Moreover this equivariance can be realized by global gauge transformations, that is, by conjugation with $\star_\Theta$-unitary elements. Properties of those unitaries were reported in [26].

Next we descend to the infinitesimal level. The action (4.2) possesses infinitesimal generators, which are vector fields in the (x, Θ) spaces. As convenient coordinates on the noncommutativity parameter sector we may take the six nonvanishing components of Θ. In some sense, this is whole point: the variable is Θ, the coordinates do not have intrinsic physical meaning. Writing $L = 1 + B$ in (4.2), for small B we have

$$(L, a) \cdot (x, \Theta) \sim (x + Bx + a, \Theta + B\Theta + \Theta B^t) = \big(x + Bx + a, \Theta + B\Theta - (B\Theta)^t\big).$$

This means that suitable generators are

$$\Gamma_{B,a} := \big(a^\alpha + b^\alpha_\beta x^\beta\big)\frac{\partial}{\partial x^\alpha} + \big(b^\rho_\beta\theta^{\beta\sigma} + \theta^{\rho\beta}b^\sigma_\beta\big)\frac{\partial}{\partial\theta^{\rho\sigma}} = \big(a^\alpha + b^\alpha_\beta x^\beta\big)\frac{\partial}{\partial x^\alpha} + (B\Theta)^{[\rho\sigma]}\frac{\partial}{\partial\theta^{\rho\sigma}},$$

where we have put $(b^\alpha_\beta) = B$. We write $\partial_\alpha \equiv \partial/\partial x^\alpha$ and for a while omit from our considerations the $a^\alpha\partial_\alpha$ part: it is well known that the Leibniz rule for these operators with the Moyal product holds. The remaining vector fields Γ_B have components linear in the respective coordinates. We rewrite

$$\Gamma_B = b^\alpha_\beta x^\beta\partial_\alpha + \big(b^\rho_\beta\theta^{\beta\sigma} + \theta^{\rho\beta}b^\sigma_\beta\big)\frac{\partial}{\partial\theta^{\rho\sigma}} =: \varepsilon^\alpha_B(x)\partial_\alpha - \delta_{\varepsilon_B}\theta^{\rho\sigma}\frac{\partial}{\partial\theta^{\rho\sigma}}. \tag{4.4}$$

The last form of the second part of Γ_B points to its geometrical meaning: it is (minus) the *Lie derivative* with respect to the vector field ε_B of the contravariant components of the matrix Θ, regarded as a tensor:

$$\delta_\varepsilon\theta^{\rho\sigma} = \varepsilon^\alpha_B(x)\partial_\alpha\theta^{\rho\sigma} - \theta^{\beta\sigma}\partial_\beta\varepsilon^\rho_B - \theta^{\rho\beta}\partial_\beta\varepsilon^\sigma_B = -b^\rho_\beta\theta^{\beta\sigma} - \theta^{\rho\beta}b^\sigma_\beta = -\delta_{\varepsilon_B}\theta^{\sigma\rho}.$$

This is an indication that we are on the right track. It is obviously important —in physics in relation with application of Noether's theorem, for instance— to record the 4×4 matrices B such that $B\Theta + \Theta B^t = 0$ or $\delta_{\varepsilon_B}\theta^{\mu\nu} = 0$. We identify the Lie algebra $\mathfrak{a}_\Theta$ of matrices B such that $B\Theta$ is symmetric. Now, from (4.3) we quote its infinitesimal version

$$\Gamma_B(f \star_\Theta h) = \Gamma_B f \star_\Theta h + f \star_\Theta \Gamma_B h. \tag{4.5}$$

The simplicity of (4.5) *and* of the path leading to it is remarkable. For $B \in \mathfrak{a}_\Theta$, in view of our remark at the end of the previous section, Γ_B is an inner derivation of the $\star$-algebra (precisely, it is equivalent to a $\star_\Theta$-commutator in a multiplier $\star$-algebra) for $\det\Theta \neq 0$; otherwise Γ_B is outer. This kind of derivations were not considered in the previous analysis [27]. In the simpler case of $\mathbb{R}^2$, with $b^\alpha_\beta = \delta^\alpha_\beta$ and $\theta^{\alpha\beta} = \varepsilon^{\alpha\beta}\theta$, we get only the derivation $x^\mu\frac{\partial}{\partial x^\mu} + 2\theta\frac{\partial}{\partial\theta}$. This had been noticed by some mathematicians [28].

The reader is encouraged to check (4.5) by brute-force calculations: compute $\frac{\partial}{\partial\theta^{\rho\sigma}}(f\star_\Theta h)$ and $\varepsilon_B(f\star_\Theta h)$, directly from (3.1) in both cases, using integration by parts.

Summarizing: for $\Theta = 0$ (the commutative world), automorphisms of the algebra of observables are diffeomorphisms. These are locally generated by vector fields, with components which are arbitrary in principle. In the noncommutative world, vector fields no longer represent infinitesimal symmetries. However, vector fields with components up to degree one in the coordinates can still be interpreted as —manifest or hidden— symmetries of Moyal algebra.

5 Coming back to [16]

The comparison with [16] is very instructive. All the generators in (3.2) are affine. In [16] their action is written down only on the (unquantized) gauge potentials and the gauge field strengths $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_{\star_\Theta}$, in terms of functional derivatives with respect to the former. That method is unnecessarily complicated. It is enough to treat the

$A_\nu, F_{\mu\nu}$ as covariant vectors and 2-tensors, respectively, and substitute the Lie derivative for the action (4.4) of Γ_B on scalar functions, for the corresponding matrix B. For the gauge potentials, as the $\partial/\partial\theta^{\rho\sigma}$ do not intervene, this gives

$$\Gamma_B(A_\mu) = b^\rho_\tau x^\tau \partial_\rho A_\mu + A_\rho \partial_\mu (b^\rho_\tau x^\tau) = b^\rho_\tau x^\tau \partial_\rho A_\mu + b^\rho_\mu A_\rho.$$

In particular, if $B = M_{\alpha\beta}$ then $b^\rho_\tau = \delta^\rho_\beta g_{\alpha\tau} - \delta^\rho_\alpha g_{\beta\tau}$, and if $B = D$ then $b^\rho_\tau = \delta^\rho_\tau$, so we get

$$\Gamma_{M_{\alpha\beta}}(A_\mu) = x_\alpha \partial_\beta A_\mu - x_\beta \partial_\alpha A_\mu + g_{\mu\alpha} A_\beta - g_{\mu\beta} A_\alpha; \quad \Gamma_D(A_\mu) = x^\rho \partial_\rho A_\mu + A_\mu;$$

together with $\Gamma_{T_\tau}(A_\mu) = \partial_\tau A_\mu$, of course. For the field strengths, one has to take into account the Θ-dependence in their definition. Still the corresponding terms cancel and one concludes

$$\begin{aligned} \Gamma_{M_{\alpha\beta}}(F_{\mu\nu}) &= x_\alpha \partial_\beta F_{\mu\nu} - x_\beta \partial_\alpha F_{\mu\nu} + g_{\mu\alpha} F_{\beta\nu} - g_{\mu\beta} F_{\alpha\nu} + g_{\nu\alpha} F_{\beta\mu} - g_{\nu\beta} F_{\alpha\mu}; \\ \Gamma_D(F_{\mu\nu}) &= x^\alpha \partial_\alpha F_{\mu\nu} + 2F_{\mu\nu}; \qquad \Gamma_{T_\tau}(F_{\mu\nu}) = \partial_\tau F_{\mu\nu}. \end{aligned}$$

We have recovered in all simplicity the results of [16], with the proviso that the widespread use of $\star$-anticommutators in that reference is another unnecessary complication, because $\frac{1}{2}(x_\alpha \star_\Theta \partial_\beta F_{\mu\nu} + \partial_\beta F_{\mu\nu} \star_\Theta x_\alpha)$ is the same as $x_\alpha \partial_\beta F_{\mu\nu}$ for any Θ. All looks like in the commutative world, and invariance of the noncommutative Yang–Mills action ensues.

6 Coming back to [6] and [13]

In this last discussion, our *point de départ* is (1.2) for $\chi = \chi_\Theta$. We write $\star_\Theta$ for $\star_{\chi_\Theta}$, giving the asymptotic version of the Moyal product [18], and Δ_Θ for Δ_{χ_Θ}. Now, let X be *any* derivation of the commutative product $\times$, i.e., any vector field. It has the property that

$$X \cdot \times(a \otimes b) = \times\big(\Delta_0(X) \cdot (a \otimes b)\big).$$

Then

$$\begin{aligned} X \cdot (a \star_\Theta b) &= X \cdot \times\big(\chi_\Theta^{-1}(a \otimes b)\big) = \times\big(\Delta_0(X) \cdot \chi_\Theta^{-1}(a \otimes b)\big) \\ &= \times\big(\chi_\Theta^{-1} \Delta_\Theta(X) \cdot (a \otimes b)\big) = \star_\Theta\big(\Delta_\Theta(X) \cdot (a \otimes b)\big). \end{aligned}$$

This is a general geometrical fact, independent of whether X is the generator of a Poincaré transformation or not. It is then scarcely surprising that Matlock [13] has found it to be valid for local conformal transformations. For similar reasons, sections 3 of [6] and 4 of [13] are tautological.

Next we need a more explicit name, say ρ, for the representation of X as a Moyal algebra operator. What we have been able to prove in the above is that, for X an *affine* transformation, if $\rho(\Delta_\Theta(X)) = \rho(\Delta_0(X)) + R(X)$, then there is another linear operation $\tilde\rho$ of X on the Moyal algebra, not a derivation either, such that

$$\tilde\rho(X) \cdot (a \star_\Theta b) = \tilde\rho(X) \cdot a \star_\Theta b + a \star_\Theta \tilde\rho(X) b - \star_\Theta\big(R(X)(a \otimes b)\big);$$

and so $\rho + \tilde\rho$ is a $\star$-derivation. Thus hidden and twist covariance boil respectively down to

$$X^\Theta m_\Theta = m_\Theta \Delta_0(X) \qquad \text{and} \quad X m_\Theta = m_\Theta \Delta_\Theta(X),$$

where we have written X^Θ for the realization of X as a derivation in (x, Θ)-space. This does not seem to work for special conformal transformations, as noted in [16, 19, 29].

7 Conclusion

We have examined in parallel references [6] and [16]. This in particular amounts to a (partial) physical interpretation of the manipulation in [6], in terms of an action of the Poincaré group by observer transformations, involving the space of parameters describing a noncommutativity background field. For Euclidean symmetry everything would work out essentially the same.

Our results show by the way that the seminal ‘quantum spacetime’ formalism by Doplicher, Fredenhagen and Roberts [30] and NCFT as currently practiced essentially coincide. Mathematically, the space of states in [30] is a particular orbit of congruence (4.2) when L is in the Lorentz group. This should have been clear at least since reference [31]. In the quantum spacetime formalism questions of relativistic symmetry breaking can be adjourned for a while by use of the Heisenberg picture for fields depending on the position variables; however, to perform physical evaluations, one is forced to choose a state, that is, a finite measure on the Θ-space; and in so doing Lorentz symmetry becomes hidden.

The moral of our story is that sometimes concrete group actions are able to complement what ‘twisted symmetry’ teaches us. It would be good to know under which general conditions cocycles for cocommutative Hopf algebras relate to hidden symmetry.

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References

[1] B. Kostant, unpublished.

[2] M. E. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[3] H. Figueroa and J. M. Gracia-Bondía, Reviews in Math. Phys. **17** (2005) 881.

[4] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.

[5] M. Cohen, Contemp. Math. **134** (1992) 1.

[6] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, Phys. Lett. B **604** (2004) 98.

[7] J. E. Moyal, Proc. Camb. Philos. Soc. **45** (1949) 99.

[8] B. Schroer, Ann. Phys. **319** (2005) 92.

[9] L. Alvarez-Gaumé and M. A. Vázquez-Mozo, Nucl. Phys. B **668** (2003) 293.

[10] A. Iorio and T. Sýkora, Int. J. Modern Phys. A **17** (2002) 2369.

[11] R. Jackiw and S.-Y. Pi, Phys. Rev. Lett. **88** (2002) 111603.

[12] J. Wess, "Deformed coordinate spaces derivatives", hep-th/0408080.

[13] P. Matlock, Phys. Rev. D **71** (2005) 126007.

[14] F. Lizzi, S. Vaidya and P. Vitale, "Infinite conformal symmetry in noncommutative two-dimensional quantum field theory", hep-th/0601056.

[15] S. M. Carroll, J. A. Harvey, V. A. Kostelecký, C. D. Lane and T. Okamoto, Phys. Rev. Lett. **87** (2001) 141601.

[16] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, Eur. Phys. J. C **24** (2002) 165.

[17] J. I. Latorre, P. Pascual and R. Tarrasch, Nucl. Phys. B **437** (1995) 60.

[18] N. Seiberg and E. Witten, J. High Energy Phys. **9909** (1999) 032.

[19] C. Gonera, P. Kosiński, P. Maślanka and S. Giller, Phys. Lett. B **622** (2005) 192; Phys. Rev. D **72** (2005) 067702.

[20] R. Banerjee, B. Chakraborty and K. Kumar, Phys. Rev. D **70** (2004) 125004.

[21] M. A. Rieffel, *Deformation Quantization for Actions of* $\mathbb{R}^d$, Memoirs Amer. Math. Soc. **506**, Providence, RI, 1993.

[22] R. Estrada, J. M. Gracia-Bondía and J. C. Várilly, J. Math. Phys. **30** (1989) 2789.

[23] V. Gayral, J. M. Gracia-Bondía and F. Ruiz Ruiz, Nucl. Phys. B **727** (2005) 513.

[24] J. C. Várilly, E. de Faría and J. M. Gracia-Bondía, Cienc. Tec. (Costa Rica) **10** (1986) 81.

[25] L. Alvarez-Gaumé, F. Meyer and M. A. Vázquez-Mozo, "Comments on noncommutative gravity", hep-th/0605113.

[26] J. M. Gracia-Bondía, F. Lizzi, G. Marmo and P. Vitale, J. High Energy Phys. **0204** (2002) 026.

[27] F. Lizzi, R. G. Szabo and A. Zampini, J. High Energy Phys. **0108** (2001) 032.

[28] S. Gutt and J. Rawnsley, J. Geom. Phys. **29** (1999) 347.

[29] J. M. Gracia-Bondía, F. Lizzi, F. Ruiz Ruiz and P. Vitale, "Noncommutative spacetime symmetries: twist versus covariance", hep-th/0604206.

[30] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. **172** (1995) 187.

[31] M. A. Rieffel, in *Operator algebras and quantum field theory*, International Press, Cambridge, MA, 1997; pp. 374–382.