Faà di Bruno Hopf algebras

Faà di Bruno (Hopf, bi)algebras appear in several branches of mathematics and physics, and may be introduced in several ways. Here we start from exponential power series like

$$f(t) = \sum_{n=1}^{\infty} \frac{f_n}{n!}\, t^n,$$

with $f_1 > 0$. In view of Borel's theorem, one may regard them as local representatives of orientation-preserving diffeomorphisms of $\mathbb{R}$ leaving 0 fixed. On the group G of these diffeomorphisms we consider the coordinate functions

$$a_n(f) := f_n = f^{(n)}(0), \quad n \geq 1.$$

We wish to compute $h_n = a_n(h)$, where h is the composition $f \circ g$ of two such diffeomorphisms, in terms of the f_n and g_n. Now,

$$h(t) = \sum_{k=1}^{\infty} \frac{f_k}{k!} \biggl(\sum_{l=1}^{\infty} \frac{g_l}{l!}\, t^l \biggr)^{k}.$$

To compute the nth coefficient h_n we need only consider the sum up to $k = n$, since the remaining terms contain powers of t higher than n. From Cauchy's product formula,

$$h_n = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{l_i \geq 1,\, l_1 + \cdots + l_k = n} \frac{n!\, g_{l_1} \ldots g_{l_k}}{l_1! \ldots l_k!}.$$

If among the l_i there are λ_1 copies of 1, λ_2 copies of 2, and so on, then the sum $l_1 + \cdots + l_k = n$ can be rewritten as

$$\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n, \quad \text{with} \quad \lambda_1 + \cdots + \lambda_n = k. \tag{1}$$

Since there are $k!/\lambda_1! \ldots \lambda_n!$ contributions from g of this type, it follows that

$$h_n = \sum_{k=1}^{n} f_k \sum_{\lambda} \frac{n!}{\lambda_1! \ldots \lambda_n!} \frac{g_1^{\lambda_1} \ldots g_n^{\lambda_n}}{(1!)^{\lambda_1}\, (2!)^{\lambda_2} \ldots (n!)^{\lambda_n}} =: \sum_{k=1}^{n} f_k\, B_{n,k}(g_1, \ldots, g_{n+1-k}), \tag{2}$$

where the sum $\sum_\lambda$ runs over the sequences $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^{\mathbb{N}}$ satisfying (1), and the $B_{n,k}$ are called the (partial, exponential) *Bell polynomials*. Usually these are introduced by the expansion

$$\exp\biggl(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\biggr) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \biggl[\sum_{k=1}^{n} u^k B_{n,k}(x_1, \dots, x_{n+1-k})\biggr],$$

which is a particular case of (2). Each $B_{n,k}$ is a homogeneous polynomial of degree k. (This is a good moment to declare that the scalar field $\mathbb{R}$ may be replaced by any commutative field of characteristic zero.)

Formula (2) can be recast as

$$h^{(n)}(t) = \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1! \dots \lambda_n!} f^{(k)}(g(t)) \biggl(\frac{g^{(1)}(t)}{1!}\biggr)^{\lambda_1} \biggl(\frac{g^{(2)}(t)}{2!}\biggr)^{\lambda_2} \cdots \biggl(\frac{g^{(n)}(t)}{n!}\biggr)^{\lambda_n}. \tag{3}$$

Expression (3) is the famous formula attributed to Faà di Bruno (1855), who in fact followed previous authors; his original contribution was a determinant form of it. Apparently (3) goes back to Arbogast (1800); we refer the reader to [1] —and references therein— for these historical matters. This formula shows that the composition of two real-analytic functions is real-analytic [2]. Indeed, if there are constants A, B, C, D for which $|g^{(m)}(t)| \leq A\, m!/B^m$ and $|f^{(k)}(g(t))| \leq C\, k!/D^k$ for all k, m, then since $\sum_{k=1}^{n} \sum_\lambda \frac{k!}{\lambda_1! \dots \lambda_n!} x^k = x(1+x)^{n-1}$ by use of (2), it follows from (3) that

$$|h^{(n)}(t)| \leq \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1! \dots \lambda_n!} \frac{C k!}{D^k} \biggl(\frac{A}{B}\biggr)^{\lambda_1} \cdots \biggl(\frac{A}{B^n}\biggr)^{\lambda_n} = n!\, \frac{C}{B^n} \frac{A}{D} \biggl(1 + \frac{A}{D}\biggr)^{n-1} = \frac{E\, n!}{F^n},$$

with $E = AC/(A+D)$ and $F = BD/(A+D)$. Hence $f \circ g$ is analytic on the domain of g.

Introduce the notation $\binom{n}{\lambda;k} := n! \big/ \lambda_1! \lambda_2! \dots \lambda_n! (1!)^{\lambda_1} (2!)^{\lambda_2} \dots (n!)^{\lambda_n}$. A *Hopf algebra* dual to G is obtained when we define a coproduct Δ on the polynomial algebra $\mathbb{R}[a_1, a_2, \dots]$ by requiring that $\Delta a_n(g, f) = a_n(f \circ g)$, or equivalently, $a_n(f \circ g) = m(\Delta a_n(g \otimes f))$ where m means multiplication. This entails that

$$\Delta a_n = \sum_{k=1}^{n} \sum_{\lambda} \binom{n}{\lambda; k} a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n} \otimes a_k.$$

The unnecessary flip of f and g is traditional. This *Faà di Bruno bialgebra*, so called by Joni and Rota [3], is commutative but not cocommutative. Since a_1 is a grouplike element, it must be invertible to have a Hopf algebra. For that, one must either adjoin an inverse a_1^{-1}, or put $a_1 = 1$, as we do from now on, that is, we consider only the subgroup G_1 of diffeomorphisms tangent to the identity at 0. The first instances of the coproduct are, accordingly,

$$\begin{aligned}
\Delta a_2 &= a_2 \otimes 1 + 1 \otimes a_2, \\
\Delta a_3 &= a_3 \otimes 1 + 1 \otimes a_3 + 3a_2 \otimes a_2, \\
\Delta a_4 &= a_4 \otimes 1 + 1 \otimes a_4 + 6a_2 \otimes a_3 + (3a_2^2 + 4a_3) \otimes a_2, \\
\Delta a_5 &= a_5 \otimes 1 + 1 \otimes a_5 + 10a_2 \otimes a_4 + (10a_3 + 15a_2^2) \otimes a_3 + (5a_4 + 10a_2 a_3) \otimes a_2.
\end{aligned} \tag{4}$$

The resulting graded connected Hopf algebra $\mathcal{F}$ is called the *Faà di Bruno Hopf algebra*; the degree $\#$ is given by $\#a_n = n-1$.

Consider the graded dual Hopf algebra $\mathcal{F}'$. Its space of primitive elements has a basis $\{a'_n : n \geq 2\}$ defined by $\langle a'_n, a_m\rangle = \delta_{n,m}$ and $\langle a'_n, a_{m_1}a_{m_2}\dots a_{m_r}\rangle = 0$ for $r > 1$. Their product is given by the duality recipe $\langle b'c', a\rangle := \langle b' \otimes c', \Delta a\rangle$, leading to:

$$a'_n a'_m = \binom{m-1+n}{n} a'_{n+m-1} + (1+\delta_{n,m})(a_n a_m)'.$$

In particular, taking $b'_n := (n+1)!\, a'_{n+1}$ for $n \geq 1$, we are left with the commutator relations

$$[b'_n, b'_m] = (m-n)b'_{n+m}. \tag{5}$$

The Milnor–Moore theorem implies that $\mathcal{F}'$ is isomorphic to the enveloping algebra of the Lie algebra $\mathcal{A}$ spanned by the b'_n with these commutators. A curious consequence of (5) is that the space $P(\mathcal{F})$ of primitive elements of $\mathcal{F}$ just has dimension 2. Indeed, $P(\mathcal{F}) = (\mathbb{R}1 \oplus {\mathcal{F}'_+}^2)^\perp$, where $\mathcal{F}'_+$ is the augmentation ideal of $\mathcal{F}'$. But (5) entails that there is a dual basis of $\mathcal{F}'$ made of products, except for its first two elements: therefore, $\dim P(\mathcal{F}) = 2$. A basis of $P(\mathcal{F})$ is given by $\{a_2, a_3 - \frac{3}{2}a_2^2\}$. The second of these corresponds to the Schwarzian derivative, which is known [4] to be invariant under the projective group $PSL(2,\mathbb{R})$; inexistence of more primitive elements of $\mathcal{F}$ is related to the affine linear and Riccati equations being the only Lie–Scheffers systems [5] over the real line.

The Faà di Bruno algebra reappears as the maximal commutative Hopf subalgebra of the (noncommutative geometry) Hopf algebra of Connes and Moscovici [6]. Their description of $\mathcal{F}$ uses a different set of coordinates $\delta_n(f) := [\log f'(t)]^{(n)}(0)$, $n \geq 1$. Since

$$h(t) := \sum_{n\geq 1} \delta_n(f)\frac{t^n}{n!} = \log f'(t) = \log\biggl(1 + \sum_{n\geq 1} a_{n+1}(f)\,\frac{t^n}{n!}\biggr),$$

it follows from (2), for logarithm and exponential functions respectively, that

$$\delta_n = \sum_{k=1}^n (-1)^{k-1}(k-1)!\, B_{n,k}(a_2,\dots,a_{n+2-k}), \text{ inverted by } a_{n+1} = \sum_{k=1}^n B_{n,k}(\delta_1,\dots,\delta_{n+1-k}).$$

This way we get $\delta_1 = a_2$, $\delta_2 = a_3 - a_2^2$, $\delta_3 = a_4 - 3a_2a_3 + 2a_2^3$, and $\delta_4 = a_5 - 3a_3^2 - 4a_2a_4 + 12a_2^2a_3 - 6a_2^4$, and so on; since the coproduct is an algebra morphism, by use of (4) we may obtain the coproduct in the Connes–Moscovici coordinates. For instance,

$$\Delta\delta_4 = \delta_4 \otimes 1 + 1 \otimes \delta_4 + 6\delta_1 \otimes \delta_3 + (7\delta_1^2 + 4\delta_2) \otimes \delta_2 + (3\delta_1\delta_2 + \delta_1^3 + \delta_3) \otimes \delta_1.$$

It is not easy to find a closed formula for $\Delta(\delta_n)$ directly from (4). Fortunately, through $\mathcal{F}'$ another method is available. Using $B_{n,1}(a_2,\dots,a_{n+1}) = a_{n+1}$, one finds that $\langle b'_n, \delta_m\rangle = (n+1)!\,\delta_{n,m}$. Let A be the graded free Lie algebra generated by primitive elements X_n, $n \geq 1$. Its enveloping algebra $\mathcal{U}(A)$ is the *concatenation* Hopf algebra. A linear basis for

$\mathcal{U}(A)$, indexed by all vectors with positive integer components $\bar{n} = (n_1, \dots, n_r)$, is made of products $X_{\bar{n}} := X_{n_1} X_{n_2} \dots X_{n_r}$, together with the unit $X_\emptyset = 1$. The coproduct is

$$\Delta(X_{\bar{n}}) = \sum_{\bar{n}^1, \bar{n}^2} \mathrm{sh}_{\bar{n}}^{\bar{n}^1, \bar{n}^2} \, X_{\bar{n}^1} \otimes X_{\bar{n}^2},$$

with $\mathrm{sh}_{\bar{n}}^{\bar{n}^1, \bar{n}^2}$ denoting the number of shuffles of the vectors $\bar{n}^1, \bar{n}^2$ that produce $\bar{n}$. Let $u^{\bar{n}}$ denote a dual basis to $X_{\bar{n}}$; the graded dual of $\mathcal{U}(A)$ is the *shuffle* Hopf algebra H with product and coproduct, respectively

$$u^{\bar{n}^1} u^{\bar{n}^2} = \sum_{\bar{n}} \mathrm{sh}_{\bar{n}}^{\bar{n}^1, \bar{n}^2} \, u^{\bar{n}}, \qquad \Delta(u^{\bar{n}}) = \sum_{\bar{n}^1 \bar{n}^2 = \bar{n}} u^{\bar{n}^1} \otimes u^{\bar{n}^2},$$

where $\bar{n}^1 \bar{n}^2$ is the concatenation of the vectors $\bar{n}^1$, $\bar{n}^2$. The surjective morphism $\rho \colon A \to \mathcal{A}$ defined by $\rho(X_n) := b'_n$ extends, by the universal property of enveloping algebras, to a surjective morphism $\rho \colon \mathcal{U}(A) \to \mathcal{F}'$, whose transpose is the injective Hopf map $\rho^t \colon \mathcal{F} \to H$ given by $\delta_n \mapsto \Gamma_n := \delta_n \circ \rho$. We may thus regard $\mathcal{F}$ as a Hopf subalgebra of H, and thereby compute the coproduct of $\mathcal{F}$ from that of H. The argument may look circular, since we seem to need an expression for Γ_n, which in turn requires computing $\Delta(\delta_n)$. But we can write

$$\langle \Gamma_m, X_{\bar{n}} \rangle = \langle \delta_m, \rho(X_{\bar{n}}) \rangle = \langle \delta_m, b'_{n_1} \dots b'_{n_r} \rangle = \langle \Delta(\delta_m), b'_{n_1} \otimes b'_{n_2} \dots b'_{n_r} \rangle. \tag{6}$$

Thus, to compute Γ_n, the only terms we need in the expansion of Δ_n are $\delta_n \otimes 1 + 1 \otimes \delta_n$ and the bilinear terms, namely multiples of $\delta_i \otimes \delta_j$; the remaining terms are of the form $\mathrm{const}\, \delta_{i_1}^{r_1} \dots \delta_{i_k}^{r_k} \otimes \delta_j$, where $r_1 i_1 + \dots + r_k i_k + j = n$. The bilinear part $B(\delta_n)$ may be computed by induction [6] to be

$$B(\delta_n) = \sum_{i=1}^{n-1} \binom{n}{i-1} \delta_{n-i} \otimes \delta_i. \tag{7}$$

Substituting (7) repeatedly in (6), one obtains $\Gamma_n = n! \sum_{\bar{n} \in N_n} C^{\bar{n}} u^{\bar{n}}$ with $N_n = \{\, \bar{n} : n_1 + \dots + n_r = n \,\}$, whose coefficients are given by $C^{\bar{n}} = (n_r + 1) \prod_{i=2}^{r} (n_i + \dots + n_r)$. For instance, $\Gamma_1 = 2u^1$ and $\Gamma_3 = 12(2u^3 + u^{(2,1)} + 3u^{(1,2)} + 2u^{(1,1,1)})$. Another calculation of Γ_n is sketched in [7]; it eventually allows to improve (7) to

$$\Delta(\delta_n) = \delta_n \otimes 1 + 1 \otimes \delta_n + \sum_{\bar{n} \in N'_n} \frac{n!}{n_1! \dots n_r!} \, K_{n_r}^{n_1, \dots, n_{r-1}} \, \delta_{n_1} \dots \delta_{n_{r-1}} \otimes \delta_{n_r},$$

where $N'_n = \{\, \bar{n} \in N_n : r > 1 \,\}$ and, mindful that $\binom{n_r}{k} = 0$ when $n_r < k$,

$$K_{n_r}^{n_1, \dots, n_{r-1}} = \sum_{k=1}^{r-1} \binom{n_r}{k} \sum_{\bar{n}^1 \dots \bar{n}^k = (n_1, \dots, n_{r-1})} \frac{1}{r^1! \dots r^k!} \prod_{i=1}^{k} \frac{1}{1 + n_1^i + \dots + n_{r^i}^i}.$$

For $r = 2$, this becomes $K_i^{n-i} = \frac{i}{1+n-i}$, thus the coefficient of $\delta_{n-i} \otimes \delta_i$ is $\binom{n}{i} \frac{i}{1+n-i} = \binom{n}{i-1}$, as in (7).

The Faà di Bruno Hopf algebra is from the combinatorial viewpoint the *incidence Hopf algebra* corresponding to intervals formed by partitions of finite sets. This is no surprise, since the coefficients of a Bell polynomial $B_{n,k}$ just count the number of partitions of $\{1, \dots, n\}$ into k blocks. A *partition* $\pi \in \Pi(S)$, of a finite set S with n elements, is a collection $\{B_1, B_2, \dots, B_k\}$ of nonempty disjoint subsets, called *blocks*, such that $\bigcup_{i=1}^k B_i = S$. We say that π is of *type* $(\alpha_1, \dots, \alpha_n)$ if exactly α_i of these B_j have i elements; thus $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ and $\alpha_1 + \cdots + \alpha_n = k$ [8]. We write $\{A_1, \dots, A_n\} = \pi \leq \tau = \{B_1, \dots, B_m\}$, and say π *refines* τ, if each A_i is contained in some B_j. A subinterval $[\pi, \tau] = \{\sigma : \pi \leq \sigma \leq \tau\}$ of the lattice $\mathcal{P}$ of partitions of finite sets is isomorphic to the poset $\Pi_1^{\lambda_1} \times \cdots \times \Pi_n^{\lambda_n}$, where $\Pi_j := \Pi(\{1, \dots, j\})$ and λ_i blocks of τ are unions of exactly i blocks of π. One assigns to each interval the sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ and declares two intervals in $\mathcal{P}$ to be equivalent when their vectors λ are equal. From the matching $[\pi, \tau]^{\sim} \leftrightarrow \lambda \leftrightarrow \widetilde{\Pi}_1^{\lambda_1} \widetilde{\Pi}_2^{\lambda_2} \dots \widetilde{\Pi}_n^{\lambda_n}$ of equivalence classes, one may regard the family $\widetilde{\mathcal{P}}$ of equivalence classes as the algebra of polynomials of infinitely many variables $\mathbb{R}[\widetilde{\Pi}_1, \widetilde{\Pi}_2, \dots]$. By means of the general theory of coproducts for incidence bialgebras [9] one then recovers the Faà di Bruno algebra under the identifications $a_n \leftrightarrow \widetilde{\Pi}_n$. The cardinality in the sense of category theory [10] of the groupoid of finite sets equipped with a partition is given by

$$\sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{1}{n!} B_{n,k}(1, \dots, 1) = e^{e-1}.$$

The *characters* of $\mathcal{F}$ form a group $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ under the convolution operation of Hopf algebra theory. The action of a character f is determined by its values on the a_n. The map $f \mapsto f(t) = \sum_{n=1}^{\infty} f_n t^n / n!$, where $f_n := \langle f, a_n \rangle$, matches characters with exponential power series over $\mathbb{R}$ such that $f_1 = 1$. This correspondence is an *anti-isomorphism* of groups: indeed, the convolution $f * g$ of $f, g \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ is given by

$$\langle f * g, a_n \rangle := m(f \otimes g) \Delta a_n = \langle g \circ f, a_n \rangle.$$

This is just the nth coefficient of $h(t) = g(f(t))$. Also, the algebra endomorphisms $\mathrm{End}_{\mathrm{alg}}(\mathcal{F})$ form a group under the convolution of the unital algebra $\mathrm{End}(\mathcal{F})$ of linear endomorphisms. The inverse under functional composition of an exponential series is given by the reversion formula of Lagrange [11], one of whose forms [12] states that if f and g are two of such series and if $f_1 = 1$, $f \circ g(t) = g \circ f(t) = t$, then

$$g_n = \sum_{k=1}^{n-1} (-1)^k B_{n-1+k,k}(0, f_2, f_3, \dots). \tag{8}$$

Now, the inverse under convolution of $f \in \mathrm{End}_{\mathrm{alg}}(\mathcal{F})$ is $g = f \circ S$, with S the antipode map of $\mathcal{F}$. The multiplicativity of f forces $S(a_n) = \sum_{k=1}^{n-1} (-1)^k B_{n-1+k,k}(0, a_2, a_3, \dots)$. One may reverse the roles and prove the combinatorial identity (8) from Hopf algebra theory [13]. Use of partitions with special properties may lead to other incidence algebras: for instance, if we restrict to noncrossing partitions, we obtain a cocommutative Hopf algebra, with the commutative group operation on characters essentially corresponding to Lagrange reversion of the Cauchy product of reverted series [14].

To go to higher Faà di Bruno algebras, we consider exponential N-series in N variables of the form

$$f(t) = f(t_1, \dots, t_N) = \Bigl(t_1 + \sum_{|\bar m|>1} f^1_{\bar m} \frac{t^{\bar m}}{\bar m!}, t_2 + \sum_{|\bar n|>1} f^2_{\bar n} \frac{t^{\bar n}}{\bar n!}, \dots, t_N + \sum_{|\bar p|>1} f^N_{\bar p} \frac{t^{\bar p}}{\bar p!} \Bigr), \qquad (9)$$

where $\bar m, \bar n, \dots, \bar p \in \mathbb{N}^N$. If g is of the same form as f, then $f \circ g(t) = \sum_{|\bar k|\geq 1} (f_{\bar k}/\bar k!)\, g(t)^{\bar k}$ is also of the same form. We look for the coefficient of $t^{\bar n}/\bar n!$ in the r-th term of this expansion, which is

$$\sum_{(\lambda_{i,\bar m_i})} \Bigl\{ \frac{f^r_{\bar k}\, \bar n! \prod_{i,\bar m_i} (g^i_{\bar m_i})^{\lambda_{i,\bar m_i}}}{\lambda! \prod_{\bar m_i} (\bar m_i!)^{\lambda_{i,\bar m_i}}} : \sum_{\bar m_i} \lambda_{i,\bar m_i} = k_i,\ \sum_{i,\bar m_i} \lambda_{i,\bar m_i}\, \bar m_i = \bar n \Bigr\}. \qquad (10)$$

Here it is understood that $i \in \{1, \dots, N\}$ and for each i, $\bar m_i$ lies in $\mathbb{N}^N$. In this form, the comparison with (2) is immediate. The expansion (10) holds [15] without the requirement that the components of f have leading coefficient 1; but this condition is natural both from the Hopf algebra viewpoint and for application of the N-series to renormalization analysis in quantum field theory. These series (9) can be regarded as characters of certain "coloured" Faà di Bruno Hopf algebras $\mathcal{F}(N)$ [13]. For any finite set X gifted with a colouring map $\theta\colon X \to \{1, \dots, N\}$, we consider partitions π whose sets of blocks are also coloured, provided $\theta(\{x\}) = \theta(x)$ for singletons. Such coloured partitions form a poset, with $\pi \leq \rho$ if π refines ρ as partitions, and if $\theta_\pi(B) = \theta_\rho(B)$ for each block B of π which is also a block of ρ; this condition entails that ρ induces a coloured partition $\rho|\pi$ of the set of blocks of π. Coloured partitions π of X with $\theta(X) = r$ form a poset $\Pi^r_{\bar n}$, where $|\pi| = \bar n \in \mathbb{N}^N$ counts the colours of its elements; their types $\widetilde\Pi^r_{\bar n}$ generate the Hopf algebra $\mathcal{F}(N)$, with coproduct given by

$$\Delta \widetilde\Pi^r_{\bar n} = \sum_{\pi \in \Pi^r_{\bar n}} \Bigl(\prod_{B \in \pi} \widetilde\Pi^{\theta(B)}_{|B|} \Bigr) \otimes \widetilde\Pi^r_{|\pi|}.$$

A character f of $\mathcal{F}(N)$ is specified by its values on algebra generators $f^r_{\bar n} = f(\widetilde\Pi^r_{\bar n})$, which yield coefficients of the N-series (9). The convolution of two such characters g, f has coefficients

$$g * f(\widetilde\Pi^r_{\bar n}) = \sum_{\pi \in \Pi^r_{\bar n}} f^r_{|\pi|} \prod_{B \in \pi} g^{\theta(B)}_{|B|} = \sum_{|\bar k| \leq |\bar n|} \frac{f^r_{\bar k}}{\bar k!} \prod_{(B_1, \dots, B_{|\bar k|})} g^1_{|B_1|} \cdots g^N_{|B_{|\bar k|}|}$$

where the second product ranges over ordered coloured partitions of a set with $|\bar n|$ elements; since there are $\bar n! \big/ \prod_i (\bar m_i!)^{\lambda_{i,\bar m_i}}$ of these with prescribed colours, rearrangement of the right hand side yields (10). Thus, the character group of $\mathcal{F}(N)$ is anti-isomorphic to the group of N-series like (9) under composition. Also, the antipode on $\mathcal{F}(N)$ provides Lagrange reversion in several variables [13].

As mentioned, the Faà di Bruno algebras (perhaps involving functional derivatives) have applications in quantum field theory. Some elementary ones are described in [9]. Deeper ones related to renormalization theory are broached in [16, 17]; much remains to be explored.

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