A multipurpose Hopf deformation of the Algebra of Feynman-like Diagrams

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Abstract. We build here a three parameters deformation of the Hopf algebra **LDIAG**. This new Algebra is a true Hopf deformation, specializes to **LDIAG** on one side and to **MQSym** on the other, relating **LDIAG** to the other Hopf algebras of contemporary Physics. Moreover its product law covers the algebra of polyzeta functions.

Contents

1. Introduction

The complete journey between a first appearance of a product formula by Bender et al. [3] and their related Feynman-like diagrams to the discovery of a Hopf algebra structure [14] (in fact, two distinct but isomorphic ones) on the diagram themselves compatible with their eveluations, goes roughly as follows.
First, C. M. Bender, D. C. Brody, and B. K. Meister [3] introduced a special field theory which proved to be particularly rich in combinatorial links and by-products.
Secondly, the Feynman-like diagrams produced by this theory label monomials and we had the surprise to see that they naturally combine in a way compatible with the monomial multiplication and co-addition. This is the Hopf algebra **DIAG**.
Thirdly, the natural noncommutative pull-back of this algebra, **LDIAG**, has a basis (the labelled diagrams) which is in one-to-one correspondence with that of the Matrix Quasi-Symmetric Fonctions (the packed matrices of **MQSym**), but their algebra and co-algebra structures are completely different. In particular, in this basis, the multiplication of **MQSym** implies a sort of shifted shuffle with overlappings reminiscent to Hoffmann's shuffle used in the theory of polyzeta functions [**?**]. The superpositions and overlappings involv ed there are not present in the (non-deformed) **LDIAG** and, moreover, the coproduct of **LDIAG** is co-commutative and the one of **MQSym** is not.

The aim of this paper is to introduce some "dynamic algebra" between the two Hopf algebras **LDIAG** and **MQSym**. The striking result is that, introducing parameters to count, in the most natural way, the crossings and overlappings of the shifted shuffle one can witness that the resulting law is associative (graded with unit). We also show how to interpolate with a coproduct which makes, at each stage, our algebra a Hopf algebra. The result is thus a three-parameters Hopf algebra deformation wich specialises to **LDIAG** at $(0, 0, 0)$ and to **MQSym** at $(1, 1, 1)$.
The question of "graphic primitive elements" has a very elegant and simple solution : these elements are the diagrams with only one black spot. The diagrams obtained reflect the Bell numbers and the corresponding basis has the same multiplication rule as the polyzeta functions.

The structure of the contribution is the following ...

2. How and why these Feynman-like Diagrams arise

The beginning of the story was explained with full details in [12, 13, 16, 17, 18, 19], and the Hopf algebra structure was made precise in [14]. Here, we will make the developpement shorter but focus on the last part, where the algebraic structure constructed on the diagrams themselves arise.

The very starting point is Bender's and al. product formula [3], an expression of the Hadamard product for exponential generating series. That is, with

$$F(z)=\sum_{n\geq 0} a_n\frac{z^n}{n!},\ G(z)=\sum_{n\geq 0} b_n\frac{z^n}{n!},\ \mathcal{H}(F,G):=\sum_{n\geq 0} a_n b_n\frac{z^n}{n!} \tag{1}$$

one can check that

$$\mathcal{H}(F,G)=\ F\left(z\frac{d}{dx}\right)G(x)\bigg|_{x=0}. \tag{2}$$

The case when $F(0)=G(0)=1$ being of special interest, one is interested to obtain compact and generic formulas. If we write the functions as *free exponentials* that is with

$$F(z)=\exp\left(\sum_{n=1}^{\infty} L_n\frac{z^n}{n!}\right),\qquad G(z)=\exp\left(\sum_{n=1}^{\infty} V_n\frac{z^n}{n!}\right). \tag{3}$$

one gets, through the Bell polynomials in the alphabets $\mathbb{L},\mathbb{V}$ (see [14] for details)

$$\mathcal{H}(F,G)=\sum_{n\geq 0}\frac{z^n}{n!}\sum_{P_1,P_2\in UP_n}\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)} \tag{4}$$

where UP_n is the set of unordered partitions of $[1\cdots n]$. An unordered partition P of a set X, is a subset of $P\subset\mathfrak{P}(X)-\{\emptyset\}$ (that is an unordered collection of blocks, i. e. non-empty subsets, of X) such that

- the union $\bigcup_{Y\in P} Y=X$ (P is a covering)
- P consists of disjoint subsets, i. e.
  $Y_1,Y_2\in P\ and\ Y_1\cap Y_2\neq\emptyset\Longrightarrow Y_1=Y_2$.

The type of $P\in UP_n$ is the multiindex $(\alpha_i)_{i\in N^+}$ such that α_k is the number of k-blocks, that is the number of members of P with cardinality k.

Here is the point where the formula entangles and the diagrams of the theory arise. The fundamental remarks are :

- the monomial $\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)}$ needs much less information that is contained in the individual partitions P_i,
- two partitions have an incidence matrix *from which it is still possible to recover the types of the partitions*

Now the construction goes as follows.

(i) Take two unordered partitions of $[1\cdots n]$, say P_1,P_2
(ii) Build their incidence matrix $(\#(Y\cap Z))_{(Y,Z)\in P_1\times P_2}$
(iii) Build the diagram representing the multiplicities of the incidence matrix : for each block of P_1 (resp. P_2) draw a white spot (resp. a black spot)

(iv) Draw lines between the white spot $Y \in P_1$ and the black spot $Z \in P_1$ as much as $\#(Y \cap Z)$

(v) Remove the information of the blocks $Y, Z, \cdots$.

So doing, one obtains a bipartite graph with p $(= \#(P_1))$ white spots, q $(= \#(P_2)$ black spots, no isolated vertex and integer multiplicities. Their set will be denoted **diag**.

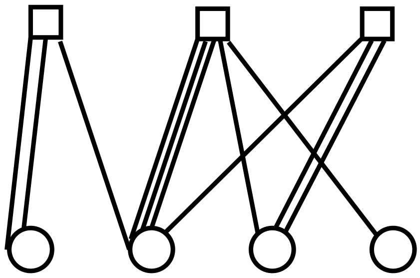

Fig 1. — *Diagram from the partitions of* $[1 \cdots 11]$.
$P_1 = \{\{1,3,5\},\{7,8,9,10,11\},\{2,4,6\}\}$ *(white spots (squares) above) and*
$P_2 = \{\{1,3\},\{2,5,7,8,9\},\{4,6,10\},\{11\}\}$ *(black spots underneath).*
The incidence matrix corresponding to this diagram (as drawn) is $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$. *But, due to the fact that the definig partitions are unordered, one can permute the spots (black and white) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*

The product formula now reads

$$\mathcal{H}(F,G) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{d \in diag \\ |d|=n}} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)} \tag{5}$$

where $\alpha(d)$ (resp. $\beta(d)$) is the "white spots type" (resp. the "black spots type") i.e. the multiindex $(\alpha_i)_{i \in \mathbb{N}^+}$ $((\beta_i)_{i \in \mathbb{N}^+})$ such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot) and $mult(d)$ is the number of pairs of unordered partitions of $[1 \cdots |d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of d) with associated diagram d.

Now the natural question arises :
"Is there a (graphically) natural multiplicative structure on **diag** *such that the arrow*

$$d \mapsto \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)} \tag{6}$$

be a morphism ?"

The answer is "yes". The desired product just consists in concatenating the diagrams (the result, for d_1, d_2, will be denoted $[d_1|d_2]_D$). One must check that this product is compatible with the permutation (of white and black spots) equivalence, which is rather straightforward (see [14]). We have

Proposition 2.1 *Let* **diag** *be the set of diagrams (including the void one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_D$ *endows* **diag** *with a structure of commutative monoid with the void diagram as neutral (which will, therefore, be denoted* $1_{\mathbf{diag}}$*).*
ii) The arrow $d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ *is a morphism of monoids, the codomain of this arrow being the monoid of (commutative) monomials in the alphabet* $\mathbb{L} + \mathbb{V}$ *i.e.*

$$\mathfrak{MON}(\mathbb{L} + \mathbb{V}) = \{\mathbb{L}^{\alpha}\mathbb{V}^{\beta}\}_{\alpha,\beta\in(\mathbb{N}^+)^{(\mathbb{N})}}.$$

iii) The monoid $(\mathbf{diag}, [?|?]_D, 1_{\mathbf{diag}})$ *is a free commutative monoid. Its letters are the connected (non void) diagrams.*

3. Non-commutative lifting

3.1. Labelling the nodes

There are (at least) two good reasons to look for non-commutative structures which may serve as a noncommutative pullback for **diag**.

- Rows and Columns of matrices are usually (linearly) ordered and we have seen that a diagram is not represented by a matrix but by a class of matrices
- The “expressive power” of **diag** and its algebra is not sufficient to connect it to other (non-commutative or non-cocommutative) algebras relevant in physics

The solution (of the non-deformed problem [14]) is simple and consists in labelling the nodes from left to right and from 1 to the desired amount as follows.

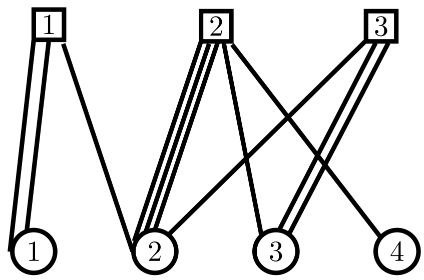

Fig 2. — *Labelled diagram of format* 3×4 *corresponding to the one of Fig 1.*

The set of these data structures (i.e. bipartite graphs on some product $[1..p] \times [1..q]$ with no isolated vertex) will be denoted **ldiag**. The composition law is, as previously, the concatenation in the obvious sense. Explicitely, if $d_i,\ i = 1, 2$ are two diagrams of dimensions $[1..p_i] \times [1..q_i]$, one relabels the white (resp. black) spots of d_2 from $p_1 + 1$ to $p_1 + p_2$ (resp. from $q_1 + 1$ to $q_1 + q_2$) the result will be noted $[d_1|d_2]_L$. One has

Proposition 3.1 *Let* **ldiag** *be the set of labelled diagrams (including the void one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_L$ *endows* **ldiag** *with a structure of noncommutative monoid with the void diagram* ($p = q = 0$) *as neutral (which will, theredfore, be denoted* $1_{\mathbf{ldiag}}$*).*
ii) The arrow from **ldiag** *to* **diag***, which consists in “forgetting the labels of the vertices”*

is a morphism of monoids.
iii) The monoid $(\mathbf{ldiag}, [?|?]_L, 1_{\mathbf{ldiag}})$ *is a free (noncommutative) monoid. Its letters are the irreducible diagrams (denoted, from now on* $irr(\mathbf{ldiag})$*).*

Note 3.2 *In a general monoid* $(M, \star, 1_M)$*, the irreducible elements are the elements* $x \neq 1_M$ *such that* $x = y \star z \Longrightarrow 1_M \in \{y, z\}$.

3.2. Coding **ldiag** *with "words of monomials"*

One can code every labelled diagram by a "word of (commutative) monomials" in the following way.

- Let $\mathbb{X} = \{x_i\}_{i\geq 1}$ an infinite set of indeterminates and $d \in \mathbf{ldiag}_{p\times q}$ a diagram ($\mathbf{ldiag}_{p\times q}$ is the set of diagrams with p white spots and q black spots).
- Associate to d the multiplicity function $[1..p] \times [1..q] \to \mathbb{N}$ such that $d(i, j)$ is the number of lines from the white spot i to the black spot j.
- The code associated to d is $\varphi_{wm}(d) = m_1 * m_2 * \cdots * m_q$ such that $m_j = \prod_{i=1}^{p} x_i^{m(i,j)}$

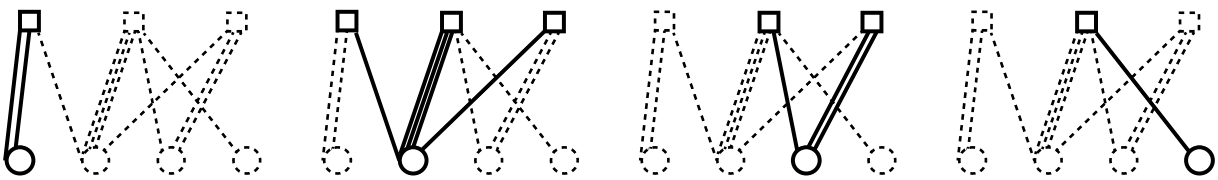

Fig 3. — *Coding a diagram with a word of monomials. The successive subgraphs from the blackspots correspond successively to the monomials* x_1^2 *(two lines to the first white spot),* $x_1x_2^3x^3$ *(one line to the first and third and three lines to the second), similarly the other black spots give* $x_2x_3^2$ *and* x_2*. The code is then* $x_1^2 * x_1x_2^3x^3 * x_2x_3^2 * x_2$.

As a data structure, the words of monomials are elements of $(\mathfrak{MON}(X)^+)^*$, the free monoid whose letters are $\mathfrak{MON}(X)^+ = \mathfrak{MON}(X) - \{1\}$, the semigroup of non-unit monomials over $\mathbb{X}$.

It is not difficult to see that, through this coding, the concatenation reflects according to the following formula

$$\varphi_{wm}([d_1|d_2]_L) = \varphi_{wm}(d_1) * T_{p_1}(\varphi_{wm}(d_2)) \quad (7)$$

where T_p is the translation operator which changes the variables according to $T_p(x_i) = x_{i+p}$ (corresponds to the relabelling of the white spots).

4. Counting crossings (q_c) and overlappings (q_o)

The preceding coding is particularly well adapted to the the deformation we want to construct here. The philosophy of the deformed product is expressed by the descriptive formula.

$$[d_1|d_2]_{L(q_c,q_o)} = \sum_{\substack{co(?)\ all\ crossing\ and\\ overlappings\ of\ black\ spots}} q_c^{nc\times weight} q_o^{weight\times weight} co([d_1|d_2]_L) \quad (8)$$

where

- $q_c, q_o \in \mathbb{C}$
- the exponent of $q_c^{nc\times weight}$ is the number of crossings of "what crosses" times its weight
- the exponent of $q_o^{weight\times weight}$ is the product of the weights of "what is overlapped"
- $co(?)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 to those of d_1, the order and distinguishability of the black spots of d_1 (i.e. d_2) being preserved.

What is striking is that this law is associative. This result will be established after the following paragraph, which is algebraic in nature and can be skipped.

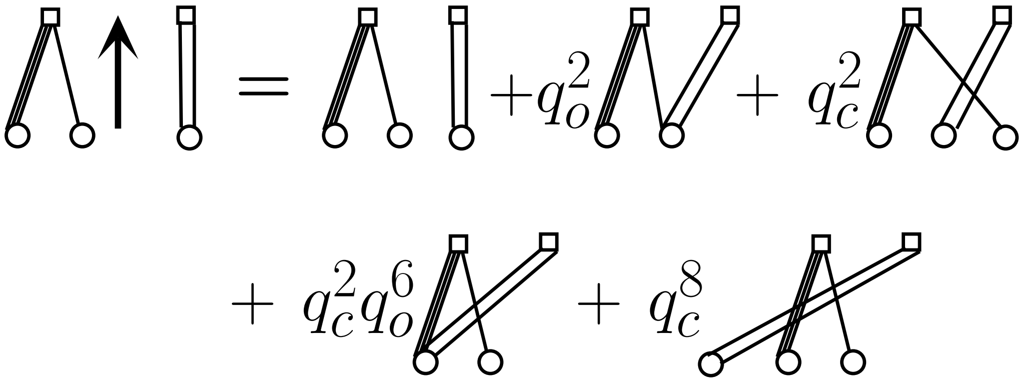

Fig 4. — *Counting crossings and overlappings produces an associative law.*

4.1. Twisted and shifted laws

Proposition 4.1 *Let $A = (A_n)_{n\in\mathbb{N}}$ a graded semigroup and A^* the set of words (denoted $a_1 * a_2 * \cdots * a_k$) with letters in A. Let $q_c, q_o \in k$ be two elements in a field K of characteristic zero. We define on $K < A >= K[A^*]$ a new law $\uparrow$ by*

$$w \uparrow 1_{A^*} = 1_{A^*} \uparrow w = w$$
$$a * u \uparrow b * v = a * (u \uparrow b * v) + q_c^{|a*u||b|} b * (a * u \uparrow v) + q_c^{|u||b|} q_o^{|a||b|} ab * (u \uparrow v) \quad (9)$$

where the weights ($|x| = n$ if $x \in A_n$) are extended additively

$$|a_1 * a_2 * \cdots * a_k| = \sum_{i=1}^{k} |a_i|$$

Then:
i) the law $\uparrow$ is graded, associative with 1_{A^*} as unit.

ii) moreover, if q_c is not a root of unity, the algebra $(k < A >, \uparrow, 1_{A^*})$ is free, generated by the elements of $(a_1 \uparrow a_2 \uparrow \cdots \uparrow a_k)_{\substack{a_i\in A\\ k\geq 0}}$.

The framework with diagrams will need another proposition on shifted laws.

Lemma 4.2 *Let $\mathcal{A}$ be an associative algebra (which law will be denoted $\star$) and $\mathcal{A} = \oplus_{n\in\mathbb{N}}\mathcal{A}_n$ a decomposition of $\mathcal{A}$ in direct sum. Let $n \mapsto T_n$ a morphim* $(\mathbb{N}, +) \to (\mathrm{End}(\mathcal{A}), \circ)$. *We suppose that the shifted law*

$$a \mathbin{\bar{\star}} b = a \star T_\alpha(b) \tag{10}$$

for $a \in \mathcal{A}_\alpha$ is graded for the decomposition $\mathcal{A} = \oplus_{n\in\mathbb{N}}\mathcal{A}_n$. Then, if the law $\star$ is associative so is the law $\bar{\star}$.

This lemma will be applied to the decomposition given by $n = sup(Alph(w))$ and the morphims given by $T_n(x_i) = x_{i+n}$.

What does these statements mean for us ?
Here the graded semigroup is $\mathfrak{MON}(X)^+$ and we do not forget the coding arrow φ_{wm} : **ldiag** $\to$ $(\mathfrak{MON}(X)^+)^*$. The image of φ_{wm} is exactly the set of words of monomials $w = m_1 * m_2 * \cdots * m_k$ such that the set of variables involved $Alph(w)$ is of the form $x_1 \cdots x_l$ (the labelling of the white spots is without hole). By abuse of language we will say that a word of monomials "is in **ldiag**" in this case. It is not difficult to see, from formulas (9,10) that if $w_i,\ i = 1, 2$ are in **ldiag** so are all the factors of $w_1 \bar{\uparrow} w_2$, this defines a new law on K[**ldiag**] and this algebra will be called **LDIAG**(q_c, q_o). The propoerties of this algebra will be made precise in the following proposition.

Proposition 4.3 *Let $\mathcal{C}_{ldiag}$ be the subspace of $(K < \mathfrak{MON}^+(\mathbb{X}) >, \bar{\uparrow})$ generated by the codes of the diagrams (i.e. the words $w \in \mathfrak{MON}^+(\mathbb{X})$ such that $Alph(w)$ is without hole). Then*
i) $\mathcal{C}_{ldiag}$ is a unital subalgebra of $(K < \mathfrak{MON}^+(\mathbb{X}) >, \bar{\uparrow})$
ii) $\mathcal{C}_{ldiag}$ is a free algebra. More precisely, for any diagram decomposed in irreducibles $d = d_1.d_2 \cdots d_k$ let

$$B(d) := \varphi_{wm}(d_1) \bar{\uparrow} \varphi_{wm}(d_2) \cdots \bar{\uparrow} \varphi_{wm}(d_k) \tag{11}$$

then
α) $(B(d))_{d\in ldiag}$ is a basis of $\mathcal{C}_{ldiag}$
β) $B(d_1.d_2) = B(d_1) \bar{\uparrow} B(d_2)$

As K[**ldiag**] is isomorphic to $\mathcal{C}_{ldiag}$ as a linear space, we denote **LDIAG**(q_c, q_o) the new algebra structure of K[**ldiag**] inherited from $\mathcal{C}_{ldiag}$. one has

$$\mathbf{LDIAG}(0, 0) \simeq \mathbf{LDIAG};\ \mathbf{LDIAG}(1, 1) \simeq \mathbf{MQSym} \tag{12}$$

5. Coproducts

Now, one has to define a parametrized (by, say, t) coproduct such that $(\mathbf{LDIAG}(q_c, q_o), \uparrow, 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon)$ be a graded bialgebra (the counity ε, the same as in the non-deformed Hopf algebra in [14] is just the "constant term" linear form).
We will take advantage of the freeness of $\mathbf{LDIAG}(q_c, q_o)$ through the following proposition.

Proposition 5.1 *Let $\mathbb{Y}$ be an alphabet, K a field of characteristic zero and $K < \mathbb{Y} >= K[\mathbb{Y}^*]$ be the free algebra constructed on $\mathbb{Y}$. For every mapping $\Delta : A \to K < \mathbb{Y} > \otimes K < \mathbb{Y} >$, we denote $\bar{\Delta} : K < \mathbb{Y} > \mapsto K < \mathbb{Y} > \otimes K < \mathbb{Y} >$ its extension as a morphism of algebras ($K < \mathbb{Y} > \otimes K < \mathbb{Y} >$ being endowed with its non-twisted structure of tensor product of algebras). Then, in order be coassociative, it is necessary and sufficient that*

$$(\bar{\Delta} \otimes I) \circ \Delta \text{ and } (I \otimes \bar{\Delta}) \circ \Delta \tag{13}$$

coincide on $\mathbb{Y}$.

The preceding proposition expresses that the possible coproducts for a free algebra "live" somehow in a linear subspace (to be precise, they are parametrized by a linear subspace). This will be transparent in formula (16).

Now, we consider the structure constants of the coproduct of **MQSym** [20] expressed in the basis

$$\{\phi S_P\}_{P \in \mathcal{PM}^c}$$

where $\mathcal{PM}^c$ is the set of connex packed matrices.

$$\Delta_{\mathbf{MQSym}}(\phi S_P) = \sum_{Q,R \in \mathcal{PM}^c} \alpha_P^{Q,R} \; \phi S_Q \; \phi S_R \tag{14}$$

For d, irreducible diagram put

$$\Delta_1(d) = \sum_{d_1, d_2 \in irr(\mathbf{ldiag})} \alpha_{\varphi_{pm}(d)}^{\varphi_{pm}(d_1), \varphi_{pm}(d_2)} d_1 \otimes d_2 \tag{15}$$

and $\Delta_0(d) = \Delta_{WS}(d)$. Then proposition (5.1) proves that

$$\Delta_t = \overline{(1-t)\Delta_0 + t\Delta_1} \tag{16}$$

is a coproduct of graded bialgebra for $(\mathbf{LDIAG}(q_c, q_o), \uparrow, 1_{\mathbf{ldiag}})$.

Let us sum up the results

Proposition 5.2 *i) With the operations defined above*

$$\mathbf{LDIAG}(q_c, q_o, t) = (\mathbf{LDIAG}(q_c, q_o), \uparrow, 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon)$$

is a Hopf algebra.
ii) At parameters $(0, 0, 0)$*, one has* $\mathbf{LDIAG}(0, 0, 0) \simeq \mathbf{LDIAG}$
iii) At parameters $(1, 1, 1)$*, one has* $\mathbf{LDIAG}(1, 1, 1) \simeq \mathbf{MQSym}$

6. LDIAG(q_c, q_o, t) and its specializations

Graphic primitive elements, BELL, LBELL, Zagier, Foissy and the general picture.

7. Conclusion

Discuss the self-dual deformation.

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