A Three Parameters Hopf Deformation of the Algebra of Feynman-like Diagrams

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Abstract. We build here a three parameters deformation of the Hopf algebra **LDIAG**. This new Algebra is a true Hopf deformation, specializes to **LDIAG** on one side and to **MQSym** on the other, relating **LDIAG** to the other Hopf algebras of contemporary Physics. Moreover its product law covers the algebra of polyzeta functions.

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1. Introduction

`This first part is to be somehow altered`
The complete journey between a first appearance of a product formula by Bender et al. [3] and their related Feynman-like diagrams to the discovery of a Hopf algebra structure [15] (in fact, two distinct but isomorphic ones) on the diagram themselves compatible with their eveluations, goes roughly as follows.
First, C. M. Bender, D. C. Brody, and B. K. Meister [3] introduced a special field theory which proved to be particularly rich in combinatorial links and by-products.
Secondly, the Feynman-like diagrams produced by this theory label monomials and we had the surprise to see that they naturally combine in a way compatible with the monomial multiplication and co-addition. This is the Hopf algebra **DIAG**.
Thirdly, the natural noncommutative pull-back of this algebra, **LDIAG**, has a basis (the labelled diagrams) which is in one-to-one correspondence with that of the Matrix Quasi-Symmetric Fonctions (the packed matrices of **MQSym**), but their algebra and co-algebra structures are completely different. In particular, in this basis, the multiplication of **MQSym** implies a sort of shifted shuffle with overlappings reminiscent to Hoffmann's shuffle used in the theory of polyzeta functions [**?**]. The superpositions and overlappings involv ed there are not present in the (non-deformed) **LDIAG** and, moreover, the coproduct of **LDIAG** is co-commutative and the one of **MQSym** is not.

The aim of this paper is to introduce some "parametric algebra" between the two Hopf algebras **LDIAG** and **MQSym**. The striking result is that, introducing parameters to count, in the most natural way, the crossings and overlappings of the shifted shuffle one can witness that the resulting law is associative (graded with unit). We also show how to interpolate with a coproduct which makes, at each stage, our algebra a Hopf algebra. The result is thus a three-parameters Hopf algebra deformation wich specialises to **LDIAG** at $(0, 0, 0)$ and to **MQSym** at $(1, 1, 1)$. Moreover it appears that, for some set of parameters, the multiplication rule of **LDIAG** covers the one of Euler-Zagier sums.

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The structure of the contribution is the following ...

2. How and why these Feynman-like Diagrams arise

The beginning of the story was explained with full details in [13, 14, 17, 18, 19, 20], and the Hopf algebra structure is made precise in [15, 22]. Here, we will make the beginning shorter but focus on the last part, where the algebraic structure constructed on the diagrams themselves arise.

The very starting point is the formula (product formula) of Bender and al. [3],

which can be considered as an expression of the Hadamard product for exponential generating series. That is, with

$$F(z)=\sum_{n\geq 0}a_n\frac{z^n}{n!},\ G(z)=\sum_{n\geq 0}b_n\frac{z^n}{n!},\ \mathcal{H}(F,G):=\sum_{n\geq 0}a_nb_n\frac{z^n}{n!} \tag{1}$$

one can check that

$$\mathcal{H}(F,G)=\left.F\left(z\frac{d}{dx}\right)G(x)\right|_{x=0}. \tag{2}$$

This product being bilinear, in the case when $F(0)$ and $G(0)$ are not zero, one can renormalize the functions so that $F(0)=G(0)=1$ and one is interested to obtain compact and generic formulas. If we write the functions as

$$F(z)=\exp\left(\sum_{n=1}^{\infty}L_n\frac{z^n}{n!}\right),\qquad G(z)=\exp\left(\sum_{n=1}^{\infty}V_n\frac{z^n}{n!}\right). \tag{3}$$

that is as free exponentials, one gets, through the Bell polynomials in the sets of variables $\mathbb{L},\mathbb{V}$ (see [15] for details)

$$\mathcal{H}(F,G)=\sum_{n\geq 0}\frac{z^n}{n!}\sum_{P_1,P_2\in UP_n}\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)} \tag{4}$$

where UP_n is the set of unordered partitions of $[1\cdots n]$. An unordered partition P of a set X, is a subset of $P\subset\mathfrak{P}(X)-\{\emptyset\}$‡ (that is an unordered collection of blocks, i. e. non-empty subsets of X) such that

- the union $\bigcup_{Y\in P}Y=X$ (P is a covering)
- P consists of disjoint subsets, i. e.
  $Y_1,Y_2\in P$ *and* $Y_1\cap Y_2\neq\emptyset\Longrightarrow Y_1=Y_2$.

The type of $P\in UP_n$ (denoted above $Type(P)$) is the multiindex $(\alpha_i)_{i\in\mathbb{N}^+}$ such that α_k is the number of k-blocks, that is the number of members of P with cardinality k.

Here is the point where the formula entangles and the diagrams of the theory arise. The fundamental remarks are :

- the monomial $\mathbb{L}^{Type(P_1)}\mathbb{V}^{Type(P_2)}$ needs much less information than that which is contained in the individual partitions P_i (for example, one can relabel the elements without changing the monomial),
- two partitions have an incidence matrix *from which it is still possible to recover the types of the partitions*

‡ The set of subsets of X is denoted $\mathfrak{P}(X)$ (this denotation [5] owes to the former german school).

Now, the construction goes as follows.

(i) Take two unordered partitions of $[1 \cdots n]$, say P_1, P_2

(ii) Build their incidence matrix $(\mathrm{card}(Y \cap Z))_{(Y,Z)\in P_1\times P_2}$

(iii) Build the diagram representing the multiplicities of the incidence matrix : for each block of P_1 draw a black spot (resp. for each block of P_2 draw a white spot)

(iv) Draw lines between the white spot $Y \in P_1$ and the black spot $Z \in P_1$ as much as $\mathrm{card}(Y \cap Z)$

(v) Remove the information of the blocks $Y, Z, \cdots$.

So doing, one obtains a bipartite graph with p $(= \mathrm{card}(P_1))$ black spots, q $(= \mathrm{card}(P_2))$ white spots, no isolated vertex and integer multiplicities. Their set will be denoted **diag**.

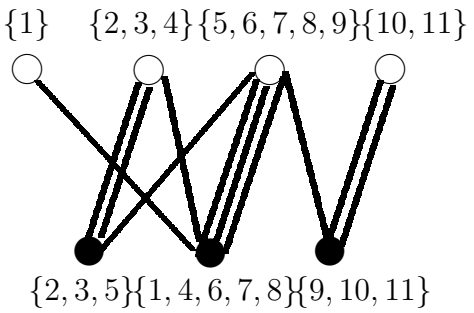

Fig 1. — *Diagram from* P_1, P_2 *(set partitions of* $[1 \cdots 11]$*).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for* P_1 *and white spots for* P_2*).*
The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$*. But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*

The product formula now reads

$$\mathcal{H}(F,G) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\substack{d\in diag \\ |d|=n}} mult(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{5}$$

where $\alpha(d)$ (resp. $\beta(d)$) is the "white spots type" (resp. the "black spots type") i.e. the multiindex $(\alpha_i)_{i\in\mathbb{N}^+}$ (resp. $(\beta_i)_{i\in\mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot) and $mult(d)$ is the number of pairs of unordered partitions of $[1 \cdots |d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of d) with associated diagram d.

Now the natural question arises :

Q1) "Is there a (graphically) natural multiplicative structure on **diag** *such that the arrow*

$$d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)} \tag{6}$$

be a morphism ?"

The answer is "yes". The desired product just consists in concatenating the diagrams (the result, for d_1, d_2, will be denoted $[d_1|d_2]_D$). One must check that this product is compatible with the permutation (of white and black spots between themselves) equivalence, which is rather straightforward (see [15]). We have

Proposition 2.1 *Let* **diag** *be the set of diagrams (including the void one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_D$ *endows* **diag** *with a structure of commutative monoid with the void diagram as neutral (which will, therefore, be denoted* $1_{\mathbf{diag}}$*).*
ii) The arrow $d \mapsto \mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$ *is a morphism of monoids, the codomain of this arrow being the monoid of (commutative) monomials in the alphabet* $\mathbb{L} \cup \mathbb{V}$ *i.e.*

$$\mathfrak{MON}(\mathbb{L} \cup \mathbb{V}) = \{\mathbb{L}^{\alpha}\mathbb{V}^{\beta}\}_{\alpha,\beta \in (\mathbb{N}^+)^{(\mathbb{N})}} = \bigcup_{n,m \geq 1} \{L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_n^{\alpha_n} V_1^{\beta_1} V_2^{\beta_2} \cdots V_m^{\beta_m}\}_{\alpha_i, \beta_j \in \mathbb{N}}.$$

iii) The monoid $(\mathbf{diag}, [-|-]_D, 1_{\mathbf{diag}})$ *is a free commutative monoid. Its letters are the connected (non void) diagrams.*

Remark 2.2 *The reader which is not familiar with the algebraic structure of* $\mathfrak{MON}(\mathbb{X})$ *can find rigourous definitions in paragraph (3.1) where this structure is needed to prepare the proofs relating to deformations.*

3. Non-commutative lifting (classical case)

The "classical" construction of the Hopf algebra **LDIAG** was annouced in [15]. We give below the proofs using a coding through "lists of monomials" which prepares the deformed (quantum) case.

3.1. Free monoids

We here recall the construction of the free and free commutative monoids generated by a given set of variables (i.e. an alphabet).
Let $\mathbb{X}$, be a set. We denote by $\mathbb{X}^*$ the set of lists of elements of $\mathbb{X}$, including the void one. In many works, and in the sequel, the list $[x_1, x_2, \cdots, x_n]$ will be denoted as a word $x_1 x_2 \cdots x_n$ so that the concatanation of two lists $[x_1, x_2, \cdots, x_n]$, $[y_1, y_2, \cdots, y_m]$ is just the word $x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m$. For this (associative) law, the void list [ ] is neutral and will therefore be denoted $1_{\mathbb{X}^*}$

Similarly, we denote $\mathbb{N}^{(\mathbb{X})}$ [4] the set of multisubsets of $\mathbb{X}$ (i.e. the set of - multiplicity - mappings with finite support $\mathbb{X} \mapsto \mathbb{N}$). Every element α of $\mathbb{N}^{(X)}$ can be written multiplicatively

$$\mathbb{X}^{\alpha} = \prod_{x \in \mathbb{X}} x^{\alpha(x)} \tag{7}$$

and the set $\mathfrak{MON}(X) = \{\mathbb{X}^{\alpha}\}_{\alpha \in \mathbb{N}^{(X)}}$ is exactly the set of (commutative) monomials with variables in $\mathbb{X}$. It is a monoid (indeed a - multiplicative - copy of $\mathbb{N}^{(X)}$). The

subset of its non-unit elements is a semigroup which will be denoted by $\mathfrak{MON}(X)^+$ $(= \mathfrak{MON}(X) - \{1\})$.

3.2. Labelling the nodes

There are (at least) two good reasons to look for non-commutative structures which may serve as a noncommutative pullback for **diag**.

- Rows and Columns of matrices are usually (linearly) ordered and we have seen that a diagram is not represented by a matrix but by a class of matrices
- The "expressive power" of **diag** and its algebra is not sufficient to connect it to other (non-commutative or non-cocommutative) algebras relevant in contemporary physics

The solution (of the non-deformed problem [15]) is simple and consists in labelling the nodes from left to right and from 1 to the desired amount as follows.

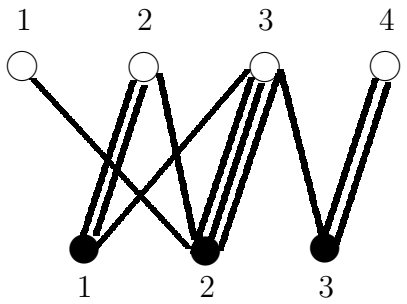

Fig 2. — *Labelled diagram of format* 3×4 *corresponding to the one of Fig 1.*

The set of these graphs (i.e. bipartite graphs on some product $[1..p] \times [1..q]$ with no isolated vertex) will be denoted **ldiag**. The composition law is, as previously, the concatenation in the obvious sense. Explicitely, if d_i, $i = 1, 2$ are two diagrams of dimensions $[1..p_i] \times [1..q_i]$, one relabels the black (resp. white) spots of d_2 from $p_1 + 1$ to $p_1 + p_2$ (resp. from $q_1 + 1$ to $q_1 + q_2$) the result will be noted $[d_1|d_2]_L$. One has

Proposition 3.1 *Let* **ldiag** *be the set of labelled diagrams (including the void one).*
i) The law $(d_1, d_2) \mapsto [d_1|d_2]_L$ *endows* **ldiag** *with a structure of noncommutative monoid with the void diagram* ($p = q = 0$) *as neutral (which will, therefore, be denoted* $1_{\mathbf{ldiag}}$*).*
ii) The arrow from **ldiag** *to* **diag**, *which consists in "forgetting the labels of the vertices" is a morphism of monoids.*
iii) The monoid $(\mathbf{ldiag}, [-|-]_L, 1_{\mathbf{ldiag}})$ *is a free (noncommutative) monoid. Its letters are the irreducible diagrams (denoted, from now on* $irr(\mathbf{ldiag})$*).*

Remark 3.2 *i) In a general monoid* $(M, \star, 1_M)$, *the irreducible elements are the elements* $x \neq 1_M$ *such that* $x = y \star z \Longrightarrow 1_M \in \{y, z\}$.
ii) It can happen that an irreducible of **ldiag** *has an image in* **diag** *which splits as shows the simple example of the cross defined by the incidence matrix* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3.3. Coding **ldiag** *with "lists of monomials"*

One can code every labelled diagram by a "list of (commutative) monomials" in the following way.

- Let $\mathbb{X} = \{x_i\}_{i\geq 1}$ an infinite set of indeterminates and $d \in \mathbf{ldiag}_{p\times q}$ a diagram ($\mathbf{ldiag}_{p\times q}$ is the set of diagrams with p black spots and q white spots).
- Associate to d the multiplicity function $[1..p] \times [1..q] \to \mathbb{N}$ such that $d(i,j)$ is the number of lines from the black spot i to the white spot j.
- The code associated to d is $\varphi_{lm}(d) = [m_1, m_2, \cdots, m_p$ such that $m_i = \prod_{j=1}^{q} x_j^{m(i,j)}$

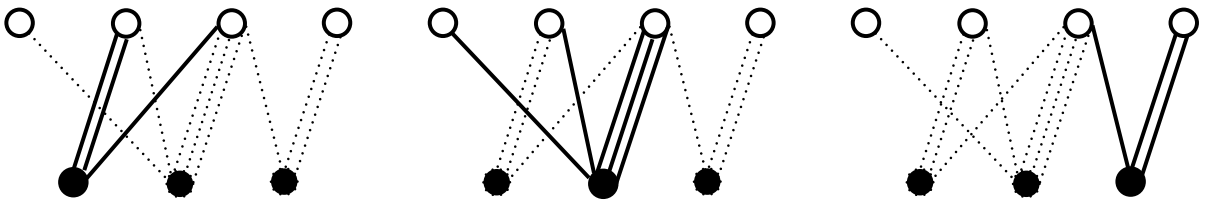

Fig 3. — *Coding the diagram of fig 2 by a word of monomials. The code here is* $[x_2^2x_3\ ,\ x_1x_2x_3^3\ ,\ x_3x_4^2]$

As a data structure, the lists of monomials are elements of $(\mathfrak{MON}(X)^+)^*$, the free monoid whose letters are $\mathfrak{MON}(X)^+ = \mathfrak{MON}(X) - \{1\}$, the semigroup of non-unit monomials over $\mathbb{X}$.

It is not difficult to see that, through this coding, the concatenation reflects according to the following formula

$$\varphi_{lm}([d_1|d_2]_L) = \varphi_{lm}(d_1) * T_{p_1}(\varphi_{lm}(d_2)) \tag{8}$$

where T_p is the translation operator which changes the variables according to $T_p(x_i) = x_{i+p}$ (corresponds to the relabelling of the white spots).

`TODO give an example`

3.3.1. Codes of diagrams

4. The (classical) Hopf algebra LDIAG

`The monoid` $(\mathfrak{MON}(X)^+)^*$`, evaluation, variables occupied, sub-monoid of diagrams, shifting and compacting, and` $Alph(w)$`,` w `beaing a list`.

5. The quantum case

5.1. Counting crossings (q_c) and superpositions (q_s)

The preceding coding is particularly well adapted to the the deformation we want to construct here. The philosophy of the deformed product is expressed by the descriptive

formula.

$$[d_1|d_2]_{L(q_c,q_s)} = \sum_{\substack{cs(?)\ all\ crossing\ and \\ superpositions\ of\ black\ spots}} q_c^{nc\times weight} q_s^{weight\times weight} cs([d_1|d_2]_L) \quad (9)$$

where

- $q_c, q_s \in \mathbb{C}$ or q_c, q_s formal. These and other cases being unified with considering the set of coefficients as taken in a ring K.
- the exponent of $q_c^{nc\times weight}$ is the number of crossings of "what crosses" times its weight
- the exponent of $q_s^{weight\times weight}$ is the product of the weights of "what is overlapped"
- $cs(?)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of d_2 to those of d_1, the order and distinguishability of the black spots of d_1 (i.e. d_2) being preserved.

What is striking is that this law is associative. This result will be established after the following paragraph.

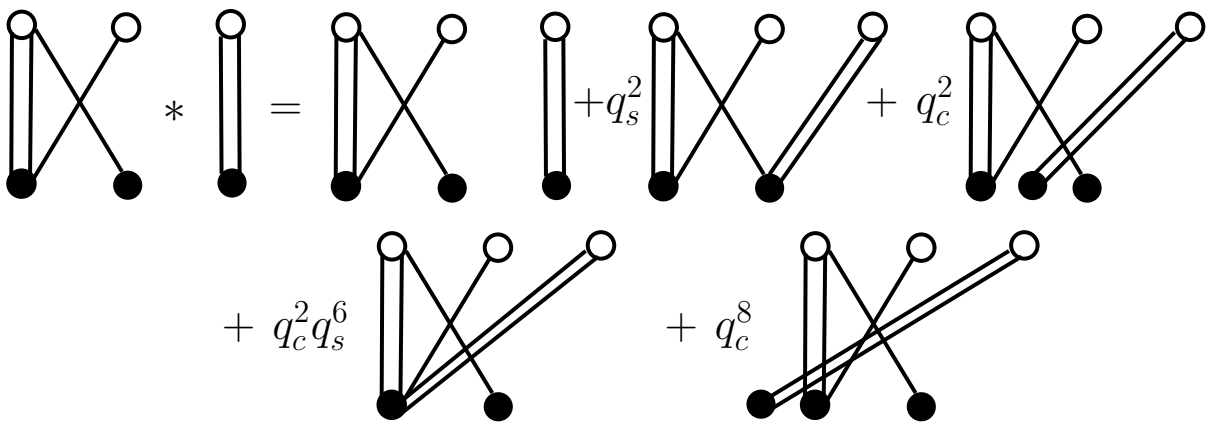

Fig 4. — *Counting crossings and superposings produces an associative law.*

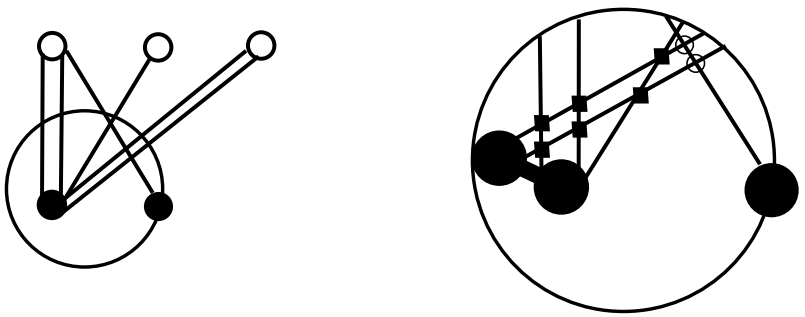

Fig 5. — *Detail of the fourth monomial (with coefficient $q_c^2 q_s^6$), crossings (circles) and superposings (black squares) are counted the same way but with a different variable.*

5.2. Modified laws

- *Twisting*

Introduction : Dual laws, shuffle, infiltration, Hoffmann.

`Maybe in an appendix or in concluding remarks`

Proposition 5.1 *Let $A = (A_n)_{n\in\mathbb{N}}$ a graded semigroup and A^* the set of lists (denoted $[a_1, a_2, \cdots a_k]$) with letters in A.*
For convenience we define the operator $*$ (left append) $A \times A^* \mapsto A^*$ by

$$a * [b_1, b_2, \cdots b_n] := [a, b_1, b_2, \cdots b_n] \tag{10}$$

Let $q_c, q_s \in k$ be two elements in a ring K. We define on $K < A >= K[A^*]$ a new law $\uparrow$ by

$$\begin{aligned} &w \uparrow 1_{A^*} = 1_{A^*} \uparrow w = w \\ &a * u \uparrow b * v = a * (u \uparrow b * v) + q_c^{|a*u||b|} b * (a * u \uparrow v) + q_c^{|u||b|} q_s^{|a||b|} ab * (u \uparrow v) \end{aligned} \tag{11}$$

where the weights ($|x| = n$ if $x \in A_n$) are extended additively to lists by

$$\Big|[a_1, a_2, \cdots, a_k]\Big| = \sum_{i=1}^{k} |a_i|$$

Then the new law $\uparrow$ is graded, associative with 1_{A^*} as unit.

Proof — Founding computation.
It suffices to prove the identity $x \uparrow (y \uparrow z) = (x \uparrow y) \uparrow z$ x, y, z being lists (as the two

members are trilinear). It is obviously true when one of the factors is the void list. Let's show it when the three factors are non-void.

$$
\begin{aligned}
&(a*u\uparrow b*v)\uparrow c*w=\\
&(a*(u\uparrow b*v)+q^{|u||b|}t^{|a||b|}(ab)(u\uparrow v)+q^{|a*u||b|}b(a*u\uparrow v))\uparrow c*w=\\
&\Big[a*((u\uparrow b*v)\uparrow c*w)+q^{(|u|+|b*v|)|c|}t^{|a||c|}(ac)((u\uparrow b*v)\uparrow w)\\
&+q^{(|a*u|+|b*v|)|c|}c*(a*(u\uparrow b*v)\uparrow w)\Big]+\\
&\Big[q^{|u||b|}t^{|a||b|}(ab)(u\uparrow v\uparrow c*w)+q^{|u||b|+(|u|+|v|)|c|}t^{|a||b|}t^{(|a|+|b|)|c|}(abc)(u\uparrow v\uparrow\ w)+\\
&q^{|u||b|+(|a*u|+|b*v|)|c|}t^{|a||b|}c(((ab)(u\uparrow v))\uparrow w)\Big]+\\
&\Big[q^{|a*u||b|}b((a*u\uparrow v)\uparrow c*w)+q^{|a*u||b|+(|a*u|+|v|)|c|}t^{|b||c|}(bc)(au\uparrow v\uparrow w)+\\
&q^{|a*u||b|+(|a*u|+|b*v|)|c|}c(b(a*u\uparrow v)\uparrow w)\Big]
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&a*u\uparrow(b*v\uparrow c*w)=\\
&a*u\uparrow(b*(v\uparrow c*w)+q^{|v||c|}t^{|b||c|}(bc)(v\uparrow w)+q^{|b*v||c|}c(b*v\uparrow w))=\\
&\Big[a*(u\uparrow b*(v\uparrow c*w))+q^{|u||b|}t^{|a||b|}(ab)(u\uparrow v\uparrow c*w)+q^{|a*u||b|}b(a*u\uparrow v\uparrow c*w)\Big]+\\
&\Big[q^{|v||c|}t^{|b||c|}a*(u\uparrow(bc)(v\uparrow w))+q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)}(abc)(u\uparrow v\uparrow w)+\\
&q^{|v||c|+|a*u|(|b|+|c|)}t^{|b||c|}(bc)(a*u\uparrow v\uparrow w)\Big]+\\
&\Big[q^{|b*v||c|}a*(u\uparrow c(b*v\uparrow w))+q^{(|u|+|b*v|)|c|}t^{|a||c|}(ac)(u\uparrow b*v\uparrow w)+\\
&q^{(|a*u|+|b*v|)|c|}c*(a*u\uparrow b*v\uparrow w)\Big]
\end{aligned}
\tag{13}
$$

in the second expression, one gathers the three terms which we find first in the square brackets and we get

$$
\begin{aligned}
&a*(u\uparrow b*(v\uparrow cw))+q^{|v||c|}t^{|b||c|}a*(u\uparrow(bc)*(v\uparrow w))+\\
&q^{|b*v||c|}a*(u\uparrow c*(b*v\uparrow w))=a*(u\uparrow b*v\uparrow c*w)
\end{aligned}
\tag{14}
$$

in the first expression, one gathers the three terms which we find last in the square brackets and we get

$$
\begin{aligned}
&q^{(|a*u|+|b*v|)|c|}c*(a*(u\uparrow b*v)\uparrow w)+\\
&q^{|u||b|+(|a*u|+|b*v|)|c|}t^{|a||b|}c*(((ab)*(u\uparrow v))\uparrow w)+\\
&q^{|a*u||b|+(|a*u|+|b*v|)|c|}c*(b*(a*u\uparrow v)\uparrow w)=\\
&q^{(|au|+|bv|)|c|}c*(a*u\uparrow b*v\uparrow w)
\end{aligned}
\tag{15}
$$

and one finds the 7 terms expression

$$
a*(u\uparrow b*v\uparrow c*w)+q^{|a*u|}b*(a*u\uparrow v\uparrow c*w)+
$$

$$q^{|a*u|+|b*v|}c*(a*u\uparrow b*v\uparrow w)+q^{|u||b|}t^{|a||b|}(ab)*(u\uparrow v\uparrow c*w)+$$
$$q^{(|u|+|b*v|)|c|}t^{|a||c|}(ac)*(u\uparrow b*v\uparrow w)+q^{|v||c|(|b|+|c|)|au|}t^{|b||c|}(bc)*(a*u\uparrow v\uparrow w)$$
$$+q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)}(abc)*(u\uparrow v\uparrow w) \quad (16)$$

□

The framework with diagrams will need another proposition on shifted laws.

• *Shifting*

We begin by the "shifting lemma" [24].

Lemma 5.2 *Let $\mathcal{A}$ be an associative algebra (which law will be denoted $\star$) and $\mathcal{A}=\oplus_{n\in\mathbb{N}}\mathcal{A}_n$ a decomposition of $\mathcal{A}$ in direct sum. Let $T\in \mathrm{End}(\mathcal{A})$ be an endomorphim of the algebra $\mathcal{A}$. We will denote $T^n=T\circ T\circ\cdots\circ T$ the n-th compositional power of T. We suppose that the shifted law*

$$a\ \bar{\star}\ b=a\star T^{\alpha}(b) \quad (17)$$

for $a\in\mathcal{A}_\alpha$ is graded for the decomposition $\mathcal{A}=\oplus_{n\in\mathbb{N}}\mathcal{A}_n$. Then, if the law $\star$ is associative so is the law $\bar{\star}$.

Remark 5.3 *The hypothesis that the shifted law given by eq.(17) be graded is automatically satisfied if $\mathcal{A}=\oplus_{n\in\mathbb{N}}\mathcal{A}_n$ is a graded algebra and if all the morphisms T_n are of degree 0.*

This lemma will be applied to the decomposition given by $n=sup(Alph(w))$ (the highest index of variables appearing in w) and the morphim given by $T(x_i)=x_{i+1}$.

What does these statements mean for us ?
Here the graded semigroup is $\mathfrak{MDN}(X)^+$ and we do not forget the coding arrow $\varphi_{lm}:\mathbf{ldiag}\to(\mathfrak{MDN}(X)^+)^*$. The image of φ_{lm} is exactly the set of lists of monomials $w=[m_1,m_2,\cdots,m_k]$ such that the set of variables involved $Alph(w)$ is of the form $x_1\cdots x_l$ (the labelling of the white spots is without hole). By abuse of language we will say that a list of monomials "is in **ldiag**" in this case. It is not difficult to see, from formulas (11,17) that if $w_i,\ i=1,2$ are in **ldiag** so are all the factors of $w_1\bar{\uparrow}w_2$, this defines a new law on $K[\mathbf{ldiag}]$ and this algebra will be called $\mathbf{LDIAG}(q_c,q_s)$. The properties of this algebra will be made precise in the following proposition.

Proposition 5.4 *Let $\mathcal{C}_{ldiag}$ be the subspace of $(K<\mathfrak{MDN}^+(\mathbb{X})>,\bar{\uparrow})$ generated by the codes of the diagrams (i.e. the lists $w\in\mathfrak{MDN}^+(\mathbb{X})$ such that $Alph(w)$ is without hole). Then*
i) $\mathcal{C}_{ldiag}$ is a unital subalgebra of $(K<\mathfrak{MDN}^+(\mathbb{X})>,\bar{\uparrow})$
ii) $\mathcal{C}_{ldiag}$ is a free algebra. More precisely, for any diagram decomposed in irreducibles $d=d_1.d_2\cdots d_k$ let

$$B(d):=\varphi_{lm}(d_1)\bar{\uparrow}\varphi_{lm}(d_2)\cdots\bar{\uparrow}\varphi_{lm}(d_k) \quad (18)$$

then
α) $(B(d))_{d\in ldiag}$ is a basis of $\mathcal{C}_{ldiag}$
β) $B(d_1.d_2) = B(d_1)\bar{\uparrow}B(d_2)$

As $K[\mathbf{ldiag}]$ is isomorphic to $\mathcal{C}_{ldiag}$ as a linear space, we denote $\mathbf{LDIAG}(q_c, q_s)$ the new algebra structure of $K[\mathbf{ldiag}]$ inherited from $\mathcal{C}_{ldiag}$. one has

$$\mathbf{LDIAG}(0,0) \simeq \mathbf{LDIAG};\ \mathbf{LDIAG}(1,1) \simeq \mathbf{MQSym} \tag{19}$$

6. Coproducts

Now, one has to define a parametrized (by, say, t) coproduct such that $(\mathbf{LDIAG}(q_c, q_s), \uparrow, 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon)$ be a graded bialgebra (the counity ε, the same as in the non-deformed Hopf algebra in [15] is just the "constant term" linear form).
We will take advantage of the freeness of $\mathbf{LDIAG}(q_c, q_s)$ through the following lemma.

Lemma 6.1 *Let $\mathbb{Y}$ be an alphabet, K a field of characteristic zero and $K<\mathbb{Y}>=K[\mathbb{Y}^*]$ be the free algebra constructed on $\mathbb{Y}$. For every mapping $\Delta : A \to K<\mathbb{Y}>\otimes K<\mathbb{Y}>$, we denote $\bar{\Delta} : K<\mathbb{Y}>\mapsto K<\mathbb{Y}>\otimes K<\mathbb{Y}>$ its extension as a morphism of algebras ($K<\mathbb{Y}>\otimes K<\mathbb{Y}>$ being endowed with its non-twisted structure of tensor product of algebras). Then, in order be coassociative, it is necessary and sufficient that*

$$(\bar{\Delta}\otimes I)\circ\Delta \text{ and } (I\otimes\bar{\Delta})\circ\Delta \tag{20}$$

coincide on $\mathbb{Y}$.

The preceding lemma expresses that, for a free algebra, the "phase space" of the possible coproducts is a linear subspace. This will be transparent in formula (23).

Now, we consider the structure constants of the coproduct of **MQSym** [23] expressed in the basis

$$\{\phi S_P\}_{P\in\mathcal{PM}^c}$$

where $\mathcal{PM}^c$ is the set of connex packed matrices.

$$\Delta_{\mathbf{MQSym}}(\phi S_P) = \sum_{Q,R\in\mathcal{PM}^c} \alpha_P^{Q,R}\ \phi S_Q\ \phi S_R \tag{21}$$

For d, irreducible diagram put

$$\Delta_1(d) = \sum_{d_1,d_2\in irr(\mathbf{ldiag})} \alpha_{\varphi_{pm}(d)}^{\varphi_{pm}(d_1),\varphi_{pm}(d_2)} d_1\otimes d_2 \tag{22}$$

and $\Delta_0(d) = \Delta_{WS}(d)$. Then proposition (6.1) proves that

$$\Delta_t = \overline{(1-t)\Delta_0 + t\Delta_1} \tag{23}$$

is a coproduct of graded bialgebra for $(\mathbf{LDIAG}(q_c, q_s), \uparrow, 1_{\mathbf{ldiag}})$.

Let us sum up the results

Proposition 6.2 *i) With the operations defined above*

$$\mathbf{LDIAG}(q_c, q_s, t) = (\mathbf{LDIAG}(q_c, q_s), \uparrow, 1_{\mathbf{ldiag}}, \Delta_t, \varepsilon)$$

is a Hopf algebra.
ii) At parameters $(0,0,0)$*, one has* $\mathbf{LDIAG}(0,0,0) \simeq \mathbf{LDIAG}$
iii) At parameters $(1,1,1)$*, one has* $\mathbf{LDIAG}(1,1,1) \simeq \mathbf{MQSym}$

Comment 6.3 *For a diagram* d *with* r *black spots, the code* $[m_1, m_2, \cdots, m_r]$ *can be temporarily seen as a "vector of coordinates" for the given diagram, but we prefered to stick to the structure of list as, firstly, the dimension of the vector varies with the diagram and secondly, we had to concatenate the codes. The coordinate functions of the diagram* d *are therefore the family* $(a_i)_{i>0}$ *defined by* $a_i(d) = m_i$ *for* $i \leq r$ *and* $a_i(d) = 0$ *for* $i > r$*. In this perspective the "*t*" of our three parameters deformation is a quantization in the sense of Moyal's star products [32] (see also the introduction of [7]), by the formula*

$$a_{i_1} * a_{i_1} \cdots * a_{i_k}(d) = \mu(\Delta_t^{[k]}(d)) \tag{24}$$

where μ *is the ordinary multiplication of monomials.*
The crossing parameter q_c *is also a quantization parameter as, for* $q_s = 0$*, one has*

$$code(d_1 * d_2) = code(d_1) \sqcup_{q_c} T(code(d_2)) \tag{25}$$

where T *is a suitable tranlation of the variables and* $\sqcup_{q_c}$ *is the quantum shuffle [31] for the braiding on* $V = \mathbb{C}[x_i;\ i \geq 1]$ *defined by*

$$B(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \otimes y_{j_1}^{\beta_1} y_{j_2}^{\beta_2} \cdots y_{j_l}^{\beta_l}) = q_c^{(\sum \alpha_i)(\sum \beta_j)}\ y_{j_1}^{\beta_1} y_{j_2}^{\beta_2} \cdots y_{j_l}^{\beta_l} \otimes x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \tag{26}$$

Let us add that q_s *and* q_c *are structurally different as* q_s *is the coefficient of a perturbation of the shuffle product (better seen on the coproduct). This kind of perturbation occurs in various domains as : computer science by means of the infiltration product introduced by Ochsenschläger [33] (see also [35] and [34]), algebra of the Euler-Zagier sums [36] and noncommutative symmetric functions [23]. The mathematics of this dual aspect is of geometrical nature and will be developped in [37].*

7. $\mathbf{LDIAG}(q_c, q_s, t)$ and its specializations

Graphic primitive elements, BELL, LBELL, Zagier, Foissy and the general picture.

8. Conclusion

Discuss the self-dual deformation.

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