Shuffle algebra and polylogarithms

Hoang Ngoc Minh
Université Lille II,
59024 Lille, France.
e-mail : hoang@lifl.fr

Michel Petitot
Université Lille I,
59655 Villeneuve d'Ascq, France.
e-mail : petitot@lifl.fr

Joris Van Der Hoeven
Université Lille I,
59655 Villeneuve d'Ascq, France.
e-mail : hoeven@lifl.fr

ABSTRACT – Generalized polylogarithms are defined as iterated integrals with respect to the two differential forms $\omega_0 = dz/z$ and $\omega_1 = dz/(1-z)$. We prove an algorithm which computes the *monodromy* of these special functions. This algorithm, implemented in AXIOM, is based on the Lyndon basis. The monodromy formulae involve special constants, called *multiple zeta values*. We prove that the algebra of polylogarithms is isomorphic to a *shuffle algebra*.

RÉSUMÉ – Les polylogarithmes généralisés sont des fonctions obtenues comme des intégrales itérées par rapport aux deux formes différentielles $\omega_0 = dz/z$ et $\omega_1 = dz/(1-z)$. On prouve un algorithme de calcul de la *monodromie* de ces fonctions spéciales. Cet algorithme, implanté dans le système de calcul formel AXIOM, repose sur la base de Lyndon. Les formules de monodromie comportent des constantes particulières appelées dans la littérature anglaise *multiple zeta values*. Nous démontrons que l'algèbre des polylogarithmes est isomorphe à une algèbre de mélange.

Keywords: polylogarithms, multiple zeta values, monodromy, Lyndon words.

1 Introduction

The Riemann ζ function and the polylogarithms arise in number theory, in physics (diagrams of Feynman), in the K–theory and in the knots theory. For each multi–index $s = (s_1, s_2, \ldots, s_k)$ of positive integers, one defines the generalized *polylogarithms*

$$\mathrm{Li}_s(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}. \tag{1}$$

This series in $z \in \mathbb{C}$ converges at the interior of the open unit disk. In $z = 1$, these polylogarithms yield the *Multiple Zeta Value* (MZV) – see([13]) :

$$\zeta(s) = \mathrm{Li}_s(1) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, \tag{2}$$

which converges for $s_1 \geq 2$. One gets thereby a generalization of classical polylogarithms $\mathrm{Li}_s(z) = \sum_{n>0} z^n/n^s$ and of the Riemann ζ function $\zeta(s) = \sum_{n>0} n^{-s}$.

One studies the $\mathbb{C}$–algebra LI generated by the constant function equal to 1, the log function and the Li_s functions defined by (1). These functions, which all are analytical over the real open segment $]0,1[$, form a commutative algebra with respect to the usual sum and product operations. By analytical prolongation one gets functions which are holomorphic on the simply connected domain formed by the complex plane minus the two real half lines $]-\infty,0[$ and $]1,\infty[$. From a more abstract standpoint, the LI elements are viewed as analytical functions on the universal Riemann surface $\mathcal{R}$ above $\mathbb{C}\backslash\{0,1\}$.

It is shown in [11] how to compute the monodromy of the *classical* polylogarithms $\mathrm{Li}_k(z)$ around the singularity $z = 1$:

$$\mathcal{M}_1 \, \mathrm{Li}_k(z) = \mathrm{Li}_k(z) - 2i\pi \frac{\log^{k-1}(z)}{(k-1)!}, \quad k > 0. \tag{3}$$

For generalizing this computation, one defines the LI functions as *iterated integrals* w.r.t. the two differential forms $\omega_0 = dz/z$ et $\omega_0 = dz/(1-z)$; this furnishes an epimorphism of $\mathbb{C}$–algebra $\alpha : \mathrm{Sh}_{\mathbb{C}}\langle X\rangle \to \mathrm{LI}$ where $X = \{x_0, x_1\}$ denotes is a two letters alphabet; $\mathrm{Sh}_{\mathbb{C}}\langle X\rangle$ denotes the ring of non commutative polynomials over X (with complex coefficients) endowed with the shuffle product. By using some combinatorial properties of Lie exponentials presented in [10, 5], one gets an algorithm (implemented in AXIOM [7]) to compute the monodromy of any function belonging to LI. One establishes the relationship with some ideas initiated by [3] (see also [2, 1, 12]).

The study of the monodromy of the polylogarithms permits to prove that the homomorphism α is injective. With other words, the $\mathbb{C}$–algebra LI is isomorphic to the shuffle algebra $\mathrm{Sh}_{\mathbb{C}}\langle X\rangle$; according to a theorem due to Radford [8], this latter is a free $\mathbb{C}$–algebra generated by the Lyndon words built over X. A detailed comparison with J. Ecalle [4] works must still be made.

2 Recalls from non commutative algebra

Let $R \supseteq \mathbb{Z}$ be a ring. Consider the alphabet $X = \{x_0, x_1\}$; let X^* be the set of words over X. Let $R\langle X\rangle$ and $R\langle\langle X\rangle\rangle$ be the algebras of non commutative polynomials resp. power series in x_0 and x_1 over a ring R. The coefficient of $w \in X^*$ in a series $S \in R\langle\langle X\rangle\rangle$ is denoted by $(S|w)$ or S_w. The set of Lie monomials is defined by induction: the letters in X are Lie monomials and the Lie bracket $[a,b] = ab - ba$ of two Lie monomials a and b is a Lie monomial. A Lie polynomial (resp. a Lie series) is a finite (resp. infinite) R-linear combination of Lie monomials. The set $\mathcal{L}ie_R\langle X\rangle \subset R\langle X\rangle$ of Lie polynomials is called the *free Lie algebra*.

2.1 The shuffle algebra

We recursively define a *shuffle product* $\sqcup\!\sqcup$ on $R\langle X\rangle$ as follows:

$$\begin{cases} \forall w \in X^*, \quad 1 \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, 1 = w, \\ \forall u, v \in X^*, xu \sqcup\!\sqcup\, yv = x(u \sqcup\!\sqcup\, yv) + y(xu \sqcup\!\sqcup\, v). \end{cases} \tag{4}$$

l	$P(l)$	$P^*(l)$
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0,\ x_1]$	x_0x_1
${x_0}^2x_1$	$[x_0,\ [x_0,\ x_1]]$	${x_0}^2x_1$
$x_0{x_1}^2$	$[[x_0,\ x_1],\ x_1]$	$x_0{x_1}^2$
${x_0}^3x_1$	$[x_0,\ [x_0,\ [x_0,\ x_1]]]$	${x_0}^3x_1$
...	...	...
${x_0}^3{x_1}^3$	$[x_0,\ [x_0,\ [[[x_0,\ x_1],\ x_1],\ x_1]]]$	${x_0}^3{x_1}^3$
${x_0}^2x_1x_0{x_1}^2$	$[x_0,\ [[x_0,\ x_1],\ [[x_0,\ x_1],\ x_1]]]$	$3{x_0}^3{x_1}^3 + {x_0}^2x_1x_0{x_1}^2$
${x_0}^2{x_1}^2x_0x_1$	$[[x_0,\ [[x_0,\ x_1],\ x_1]],\ [x_0,\ x_1]]$	$6{x_0}^3{x_1}^3 + 3{x_0}^2x_1x_0{x_1}^2 + {x_0}^2{x_1}^2x_0x_1$
${x_0}^2{x_1}^4$	$[x_0,\ [[[[x_0,\ x_1],\ x_1],\ x_1],\ x_1]]$	${x_0}^2{x_1}^4$
$x_0x_1x_0{x_1}^3$	$[[x_0,\ x_1],\ [[[x_0,\ x_1],\ x_1],\ x_1]]$	$4{x_0}^2{x_1}^4 + x_0x_1x_0{x_1}^3$
$x_0{x_1}^5$	$[[[[[x_0,\ x_1],\ x_1],\ x_1],\ x_1],\ x_1]$	$x_0{x_1}^5$

Table 1: Lyndon words, bracket forms and dual basis

Here 1 denotes the empty word.

EXAMPLE – $x_0x_1 \mathbin{ш} x_1 = x_1x_0x_1 + 2x_0x_1^2$

This product extends to $R\langle X\rangle$ by linearity. With this product, $R\langle X\rangle$ is a commutative and associative R-algebra, called the *shuffle algebra* $\mathrm{Sh}_R\langle X\rangle$.

2.2 Lyndon words and Radford's theorem

By definition, a *Lyndon word* is a non empty word $l \in X^*$, which is inferior to each of its strict right factors [10] (for the lexicographical ordering) ie. $\forall u, v \in X^+$, $l = uv$, $l < v$. The set of Lyndon words is denoted by $\mathcal{L}yndon(X)$.

EXAMPLE – For $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon words of length $\leqslant 6$ on X^* are the following (in lexicographical increasing order): see table 1.

Theorem 1 (Radford) *The* $\mathbb{Q}$*–algebra* $\mathrm{Sh}_{\mathbb{Q}}\langle X\rangle$ *is the algebra of polynomials generated by the Lyndon words.*

In [7, 6] efficient algorithms are given to rewrite a polynomial in $\mathbb{Q}\langle X\rangle$ as a linear combination of *shuffles* of Lyndon words.

2.3 Bracket forms and the dual basis

The *bracket form* $P(l) \in \mathcal{L}ie_R\langle X\rangle$ of a Lyndon word $l = uv$ with $l, u, v \in \mathcal{L}yndon(X)$ (v being as long as possible) is defined recursively by

$$\left\{ \begin{array}{rcl} P(l) & = & [P(u), P(v)] \\ P(x) & = & x \quad \text{for each letter } x \in X, \end{array} \right. \tag{5}$$

It is classical that the set $\mathcal{B}_1 = \{P(l);\ l \in \mathcal{L}yndon(X)\}$, ordered lexicographically, is a basis for the free Lie algebra. Moreover, each word $w \in X^*$ can be expressed uniquely as a decreasing product of Lyndon words:

$$w = l_1^{\alpha_1} l_2^{\alpha_2} \dots l_k^{\alpha_k}, \quad l_1 > l_2 > \dots > l_k, \quad k \geq 0. \tag{6}$$

The Poincaré–Birkhoff–Witt basis $\mathcal{B} = \{P(w);\ w \in X^*\}$ and its dual basis $\mathcal{B}^* = \{P^*(w);\ w \in X^*\}$ are obtained from (6) by setting [10]

$$\left\{ \begin{array}{rcl} P(w) & = & P(l_1)^{\alpha_1} P(l_2)^{\alpha_2} \dots P(l_k)^{\alpha_k}, \\ P^*(w) & = & C P^*(l_1)^{ш\,\alpha_1} \, ш \dots ш \, P^*(l_k)^{ш\,\alpha_k}, \\ & & \text{where } C = (\alpha_1! \alpha_2! \dots \alpha_k!)^{-1} \\ P^*(l) & = & x P^*(w), \quad \forall l \in \mathcal{L}yndon(X), \\ & & \text{where } l = xw,\ x \in X,\ w \in X^*. \end{array} \right. \tag{7}$$

In [10], it is proved that $\mathcal{B}$ and $\mathcal{B}^*$ are dual bases of $R\langle X \rangle : (P(u)|P^*(v)) = \delta_u^v$, for all words $u, v \in X^*$ with $\delta_u^v = 1$ if $u = v$ otherwise 0.

Lemma 1 *It holds $P^*(w) \in x_0 \mathbb{Z}\langle X \rangle x_1$ for all $w \in x_0 X^* x_1$.*

Proof – The Lyndon words involved in the decomposition (6) of a word $w \in X^* x_1$ (resp. $w \in x_0 X^* x_1$) all belong to $X^* x_1$ (resp. $x_0 X^* x_1$). □

2.4 Lie exponentials

A series $S \in R\langle\langle X \rangle\rangle$ is called a *Lie exponential* if there exists a *Lie series* $L \in \mathcal{L}ie_R\langle\langle X \rangle\rangle$ such that $S = e^L$. This is equivalent [9, 10] to $\forall u, v \in X^*$, $(S|u \, ш \, v) = (S|u)(S|v)$ or to $\Delta(S) = S \otimes S$, where Δ denotes the usual coproduct $\Delta : R\langle\langle X \rangle\rangle \to R\langle\langle X \rangle\rangle \otimes R\langle\langle X \rangle\rangle$, which is defined on letters $x \in X$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$. The product of two Lie exponentials is a Lie exponential.

Let $S \in R\langle\langle X \rangle\rangle$ be a Lie exponential. Then S can be factored as an infinite product of Lie exponentials [10]

$$S = \sum_{w \in X^*} (S|w)\ w = \prod_{l \in \mathcal{L}yndon(X) \searrow} e^{(S|P^*(l)) P(l)}. \tag{8}$$

The infinite product of the factors $S = e^{S_{x_1} x_1} \times \dots \times e^{S_{x_0} x_0}$ is ordered decreasingly by the ordering on Lyndon words.

3 Polylogarithms and Chen series

3.1 The polylogarithm generating series

Any multi–index $s = (s_1, s_2, \dots, s_k)$ can be encoded by a unique word $w \in X^* x_1$

$$w = x_0^{s_1 - 1} x_1 x_0^{s_2 - 1} x_1 \cdots x_0^{s_k - 1} x_1 \tag{9}$$

Now each function $\mathrm{Li}_s(z)$, which is also denoted by $L_w(z)$, can be obtained by an *iterated integral* as follows [13]:

$$L_{x_1}(z) = \int_0^z \frac{dt}{1-t} = -\log(1-z)$$

and

$$\left\{\begin{array}{rcl} L_{x_0 w}(z) &=& \displaystyle\int_0^z L_w(t)\frac{dt}{t}, \\ L_{x_1 w}(z) &=& \displaystyle\int_0^z L_w(t)\frac{dt}{1-t}, \end{array}\right. \tag{10}$$

for any $w \in X^* x_1$. These integrals are functions defined on the universal Riemann surface $\mathcal{R}$ above $\mathbb{C}\backslash\{0,1\}$. The real number $\zeta(s)$ is also denoted by $\zeta_w = L_w(1)$ for all $x \in x_0 X^* x_1$.

It is useful to extend the above definition of L_w to the case when $w \in X^*$. For each $n \geq 0$, we take

$$L_{x_0^n}(z) = \frac{1}{n!}\log^n(z), \tag{11}$$

and we extend the definition to $w \in X^*$ using (10).

The generating series L of polylogarithms is:

$$L(z) = \sum_{w \in X^*} L_w(z) w, \quad \forall z \in \mathcal{R}. \tag{12}$$

Proposition 1 *L satisfies the differential equation with border condition*

$$\begin{aligned} \frac{d}{dz}L(z) &= \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) L(z) && (13) \\ L(\varepsilon) &= e^{x_0 \log \varepsilon} + O(\sqrt{\varepsilon}) \quad \textit{where } \varepsilon \to 0^+. && (14) \end{aligned}$$

Proof – Observing that $L(z) = 1 + \sum_{u \in X^*} L_{x_0 u}(z) x_0 u + \sum_{v \in X^*} L_{x_1 v}(z) x_1 v$, this follows syntactically from the formulas (10). The exponential term $e^{(\log \varepsilon) x_0}$ comes from the definition (11). The coefficient of each other word w in $L(\varepsilon)$ is easily seen to be bounded by $O(\varepsilon^n \log^m \varepsilon)$, where $n > 0$ is the number of x_1's in w. □

Theorem 2 *$L(z)$ is a Lie exponential for all $z \in \mathcal{R}$. In particular, one has the* shuffle relations

$$L_{u ⧢ v}(z) = L_u(z) L_v(z), \quad \forall z \in \mathcal{R}, \forall u, v \in X^*. \tag{15}$$

Proof – Intuitively speaking, the theorem follows from the facts that the limit of $L(z)$ in 0 is a Lie exponential, because of (14), and that $L(z)$ is a Lie exponential for each z, if it is a Lie exponential for a particular z. We have to prove that $T(z) = \Delta L(z) - L(z) \otimes L(z)$ vanishes for all z. We claim that T satisfies

$$\left\{\begin{array}{rcl} \dfrac{d}{dz}T(z) &=& (\Delta V(z))\, T(z), \\ \lim_{\varepsilon \to 0^+} T(\varepsilon) &=& 0, \end{array}\right. \tag{16}$$

where $V(z) = (\frac{x_0}{z} + \frac{x_1}{1-z})$. Thus we have a recursive formula for the coefficients of $T(z)$ by means of differential equations with limit conditions in 0. Since these limits all vanish in 0, it follows by induction that the coefficients of T all vanish globally. □

3.2 Analytic continuation of polylogarithms

3.2.1 Chen series

For a differentiable path $\gamma : [0,1] \to \mathbb{C}\backslash\{0,1\}$ between a and b, let S_γ be the evaluation in $z = b$ of the solution to the differential equation

$$\frac{d}{dz}S(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right) S(z) \quad \text{for } z \in \gamma([0,1]) \tag{17}$$

with initial condition $S(a) = 1$. This series $S_\gamma \in \mathbb{C}\langle X\rangle$ is called [2] the *Chen series* along γ (and associated to the differential forms $\omega_0 = dz/z$ et $\omega_1 = dz/1-z$). It is classical [2] that S_γ is a Lie exponential, which only depends on the homotopy class of γ. Moreover, for the concatenation $\gamma_1\gamma_2$ of two paths γ_1 and γ_2, one has $S_{\gamma_1\gamma_2} = S_{\gamma_2}S_{\gamma_1}$. In particular, $S_{\gamma^{-1}} = S_\gamma^{-1}$.

Let z_0 be a point of $\mathcal{R}$, which we identify with its projection on $\mathbb{C}$ and let $z_0 \rightsquigarrow z$ be a differentiable path on $\mathbb{C}\backslash\{0,1\}$. Then L admits an analytic continuation along this path. The series $L(z)$ and $S_{z_0\rightsquigarrow z}L(z)$ both satisfy the differential equation (14) and take the same value in $z = z_0$. This proves that

$$L(z) = S_{z_0\rightsquigarrow z}L(z_0), \tag{18}$$

for all paths $z_0 \rightsquigarrow z$ in $\mathbb{C}\backslash\{0,1\}$.

3.2.2 Residues theorem (noncommutative version)

Let $R \in]0,1[$ and let $\gamma_0(R)$ (resp. $\gamma_1(R)$) be the circular path of radius R and turning around 0 (resp. 1) in the positive direction, starting in $z = R$ (resp. $z = 1-R$). By induction on the length of w one proves the bound

$$(S_{\gamma_0(R)}|w) \leqslant \frac{1}{|w|!}(2\pi)^{|w|}(2R)^{|w|_{x_1}}, \tag{19}$$

for the coefficients of the Chen series along $\gamma_0(R)$ (for $R < 1/2$), where $|w|$ denotes the length of the word w and $|w|_{x_1}$ the number of occurrences of x_1 in w. For $\varepsilon \to 0^+$, this estimate yields

$$\left\{ \begin{array}{rcl} S_{\gamma_0(\varepsilon)} &=& e^{2i\pi x_0} + O(\varepsilon), \\ S_{\gamma_1(\varepsilon)} &=& e^{-2i\pi x_1} + O(\varepsilon). \end{array} \right. \tag{20}$$

3.3 Asymptotic expansions of the polylogarithms at $z = 1$

By using the factorization (8)

$$L = e^{L_{x_1}x_1} \left(\prod_{l \neg\in \{x_0.x_1\}\searrow} e^{L_{P^*(l)}P(l)} \right) e^{L_{x_0}x_0} \tag{21}$$

at the point $z = 1-\varepsilon$, we obtain the asymptotic expansion

$$L(1-\varepsilon) \sim e^{-x_1\log\varepsilon} Z \quad \text{where } Z = \prod_{l \neg\in \{x_0.x_1\}\searrow} e^{\zeta_{P^*(l)}P(l)}. \tag{22}$$

From the lemma 1, $P^*(l) \in x_0\mathbb{Z}\langle X\rangle x_1$. Thus the quantities $\zeta_{P^*(l)}$ are all finite.

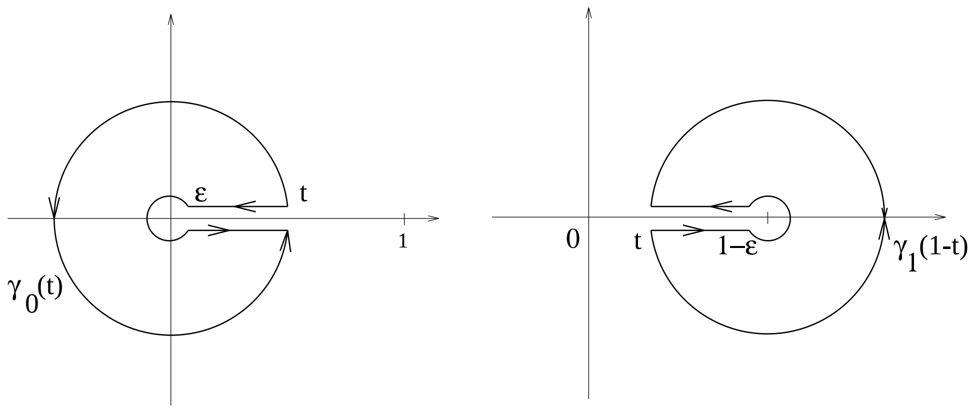

Figure 1: Paths of integration

3.3.1 Intrinsic definition of the series Z

Proposition 2 *The series Z is the unic Lie exponential verifying the two following properties :*

$$\left\{ \begin{array}{rcl} (Z \mid x_0) & = & (Z \mid x_1) = 0 \\ \forall w \in x_0 X^* x_1,\ (Z \mid w) & = & \zeta_w \end{array} \right. \tag{23}$$

4 Monodromy of L

For each $t \in]0,1[$, let $\mathcal{M}_0 L(t)$ (resp. $\mathcal{M}_1 L(t)$), be the analytic continuation of $L(t)$ along $\gamma_0(t)$, (resp. $\gamma_1(t)$). From (18), we get $\mathcal{M}_0 L(t) = S_{\gamma_0(t)} L(t)$ and $\mathcal{M}_1 L(t) = S_{\gamma_1(1-t)} L(t)$. We will now show how to compute two Lie exponentials $M_0, M_1 \in \mathbb{C}\langle\langle X \rangle\rangle$, which do not depend on t, such that $\forall t \in]0,1[$

$$\mathcal{M}_i L(t) = L(t) M_i, \quad \text{where } i = 0 \text{ or } 1. \tag{24}$$

4.1 Monodromy of L around 0 :

Since a Chen series only depends on the homotopy class of its underlying path, we deduce from (18) that – see figure 1

$$\begin{aligned} \mathcal{M}_0 L(t) &= S_{\varepsilon \rightsquigarrow t} S_{\gamma_0(\varepsilon)} S_{t \rightsquigarrow \varepsilon} L(t), \\ &= L(t)\ L^{-1}(\varepsilon) S_{\gamma_0(\varepsilon)} L(\varepsilon), \\ &= L(t) \lim_{\varepsilon \to 0^+} L^{-1}(\varepsilon) S_{\gamma_0(\varepsilon)} L(\varepsilon), \\ &= L(t) \lim_{\varepsilon \to 0^+} e^{-x_0 \log \varepsilon} e^{2i\pi x_0} e^{x_0 \log \varepsilon} \\ &= L(t)\ e^{2i\pi x_0}. \end{aligned}$$

4.2 Monodromy of L around 1 :

Similarly – see figure 1

$$\begin{aligned}
\mathcal{M}_1 L(t) &= S_{1-\varepsilon \rightsquigarrow t} S_{\gamma_1(\varepsilon)} S_{t \rightsquigarrow 1-\varepsilon} L(t), \\
&= L(t)\ L^{-1}(1-\varepsilon) S_{\gamma_1(\varepsilon)} L(1-\varepsilon), \\
&= L(t) \lim_{\varepsilon \to 0^+} Z^{-1} e^{x_1 \log \varepsilon} e^{-2i\pi x_1} e^{-x_1 \log \varepsilon} Z \\
&= L(t)\ Z^{-1} e^{-2i\pi x_1} Z.
\end{aligned}$$

Theorem 3 *The monodromy of the series $L(t)$ for $t \in]0,1[$ around $z = 0$ and $z = 1$ is given by*

$$\left\{ \begin{array}{ll} \mathcal{M}_0 L(t) &= L(t) e^{2i\pi \mathfrak{m}_0} \quad \text{with } \mathfrak{m}_0 = x_0, \\ \mathcal{M}_1 L(t) &= L(t) e^{2i\pi \mathfrak{m}_1}, \end{array} \right. \tag{25}$$

where $\mathfrak{m}_1$ is a Lie serie given by the formula

$$\mathfrak{m}_1 = \prod_{l \notin \{x_0, x_1\} \nearrow} e^{-\zeta_{P^*(l)} \operatorname{ad} P(l)} (-x_1). \tag{26}$$

The constants $\zeta_{P^*(l)}$ are finite, since $P^*(l) \in x_0 \mathbb{Z}\langle X \rangle x_1$, by lemma 1. Moreover, there exists an algorithm to compute $\mathcal{M}_1 L_w(t)$ for each word $w \in X^*$. The prof follows the computations of the factorization (22) on 4.1 and 4.2, and the classic properties of the adjoint transformation.

It is also possible to simplify the $\zeta(s)$ involved in the result using the algebraic relations for generalized zeta function given in [6].

4.3 Structure of the monodromy group

Corollary 1 *The monodromy of the polylogarithms L_w is given by*

$$\begin{aligned}
\forall w \in X^*, \quad \mathcal{M}_0 L_{w x_0} &= L_{w x_0} + 2i\pi L_w + \cdots \\
\mathcal{M}_1 L_{w x_1} &= L_{w x_1} - 2i\pi L_w + \cdots,
\end{aligned}$$

where the remaining terms are linear combinations of polylogarithms coded by words of length $< |w|$.

Proof – Consequence of (24), by remarking that thm. 3 implies that

$$\left\{ \begin{array}{lll} M_0 &= e^{2i\pi \mathfrak{m}_0} &= 1 + 2i\pi x_0 + \text{words of length} > 1 \\ M_1 &= e^{2i\pi \mathfrak{m}_1} &= 1 - 2i\pi x_1 + \text{words of length} > 1 \end{array} \right. \tag{27}$$

See also the results from appendix A. □

Corollary 2 *The monodromy group of the functions L_w for $|w| \leqslant n$ is nilpotent at order $n+1$.*

Proof – We have $M_0 = e^{2i\pi x_0}$ and $M_1 = e^{-2i\pi x_1 + \cdots}$. From $e^A e^B e^{-A} e^{-B} = e^{[A,B] + \cdots}$, it follows that the commutator $M_0 M_1 M_0^{-1} M_1^{-1}$ does not contain any Lie brackets of length 1. Iterating this computation, the brackets of lengths 2, next 3, etc. until n disappear. □

5 Linear independence of the polylogarithms

Theorem 4 *The functions L_w with $w \in X^*$ are $\mathbb{C}$-linearly independent.*

The generalizations of this theorem, when taking the rings of polynomials or entire functions as coefficients are easy.

PROOF – Given $n \geq 0$, assume that we have a $\mathbb{C}$-linear relation

$$\sum_{|w| \leqslant n} \lambda_w L_w = 0, \quad \lambda_w \in \mathbb{C} \tag{28}$$

between the L_w, where $w \in X^*$ and $|w|$ denotes the length of w. We prove by induction on n that $\lambda_w = 0$ for all w. This is trivial for $n = 0$. Assume therefore that we proved our assertion for all smaller n. Rewrite (28) as

$$\lambda_1 + \sum_{|u|<n} \lambda_{ux_0} L_{ux_0} + \sum_{|u|<n} \lambda_{ux_1} L_{ux_1} = 0. \tag{29}$$

Applying the operators $(\mathcal{M}_0 - Id)$ and $(Id - \mathcal{M}_1)$ on this expression, while using corollary 1 of theorem 3, we obtain two new linear relations

$$\left\{ \begin{array}{lcl} 2i\pi \displaystyle\sum_{|u|=n-1} \lambda_{ux_0} L_u + \sum_{|u|<n-1} \mu_u L_u & = & 0, \\ 2i\pi \displaystyle\sum_{|u|=n-1} \lambda_{ux_1} L_u + \sum_{|u|<n-1} \nu_u L_u & = & 0, \end{array} \right. \tag{30}$$

for certain coefficients μ_u and ν_u. By the induction hypothesis, the coefficients λ_{ux_0} and λ_{ux_1} with $|u| = n-1$ all vanish (as well as the coefficients μ_u and ν_u). Consequently,

$$\sum_{|w| \leqslant n-1} \lambda_w L_w = 0,$$

whence $\lambda_w = 0$ for all w, again by the induction hypothesis. □

We deduce that the $\mathbb{C}$-algebra LI generated by the L_w is isomorphic to $\mathrm{Sh}_{\mathbb{C}}\langle X \rangle$. By Radford's theorem, the polylogarithms coded by Lyndon words therefore form an infinite transcendence basis, but many other such bases are known, such as the dual basis $\mathcal{B}_1^* = \{P^*(l); l \in \mathcal{Lyndon}(X)\}$ defined by (7).

Corollary 3 *Each polylogarithm L_w can be expressed uniquely as a $\mathbb{Q}$-polynomial of polylogarithms coded by Lyndon words. The classical polylogarithms* Li_k *which are coded by Lyndon words $x_0^{k-1}x_1$ are algebraically independent.*

Acknowledgment Thanks to P. Cartier for useful discussions.

A Experimental results

A.1 The series $\mathfrak{m}_1$ up to order 6

$$
\begin{aligned}
\mathfrak{m}_1 &= \frac{1}{2i\pi}\log M_1\\
&= -[x_1] + \zeta_{x_0x_1}\left[x_0{x_1}^2\right] + \zeta_{{x_0}^2x_1}\left[{x_0}^2{x_1}^2\right] + \zeta_{x_0{x_1}^2}\left[x_0{x_1}^3\right] + \zeta_{{x_0}^3x_1}\left[{x_0}^3{x_1}^2\right]\\
&- \zeta_{{x_0}^3x_1}\left[{x_0}^2x_1x_0x_1\right] + \zeta_{{x_0}^2{x_1}^2}\left[{x_0}^2{x_1}^3\right] + \left(\zeta_{{x_0}^2{x_1}^2} - \frac{1}{2}{\zeta_{x_0x_1}}^2\right)\left[x_0x_1x_0{x_1}^2\right]\\
&+ \zeta_{x_0{x_1}^3}\left[x_0{x_1}^4\right] + \zeta_{{x_0}^4x_1}\left[{x_0}^4{x_1}^2\right] - 2\zeta_{{x_0}^4x_1}\left[{x_0}^3x_1x_0x_1\right]\\
&+ \zeta_{{x_0}^3{x_1}^2}\left[{x_0}^3{x_1}^3\right] + \left(3\zeta_{{x_0}^3{x_1}^2} + \zeta_{{x_0}^2x_1x_0x_1}\right)\left[{x_0}^2x_1x_0{x_1}^2\right]\\
&+ \left(3\zeta_{{x_0}^3{x_1}^2} + \zeta_{x_0x_1}\zeta_{{x_0}^2x_1} + 2\zeta_{{x_0}^2x_1x_0x_1}\right)\left[{x_0}^2{x_1}^2x_0x_1\right]\\
&+ \zeta_{{x_0}^2{x_1}^3}\left[{x_0}^2{x_1}^4\right] + \left(4\zeta_{{x_0}^2{x_1}^3} + \zeta_{x_0x_1x_0{x_1}^2}\right)\left[x_0x_1x_0{x_1}^3\right] + \zeta_{x_0{x_1}^4}\left[x_0{x_1}^5\right]
\end{aligned}
$$

A.2 Monodromy around $z = 1$

We abbreviated $2i\pi$ by p.

$$
\begin{aligned}
\mathcal{M}_1 L_{x_0} &= L_{x_0}\\
\mathcal{M}_1 L_{x_1} &= L_{x_1} - p\\
\mathcal{M}_1 L_{x_0x_1} &= L_{x_0x_1} - pL_{x_0}\\
\mathcal{M}_1 L_{x_0^2x_1} &= L_{x_0^2x_1} - \frac{1}{2}p{L_{x_0}}^2\\
\mathcal{M}_1 L_{x_0x_1^2} &= L_{x_0x_1^2} - pL_{x_0x_1} + \frac{1}{2}p^2L_{x_0} + p\zeta_{x_0x_1}\\
\mathcal{M}_1 L_{x_0^3x_1} &= L_{x_0^3x_1} - \frac{1}{6}p{L_{x_0}}^3\\
\mathcal{M}_1 L_{x_0^2x_1^2} &= L_{x_0^2x_1^2} - pL_{x_0^2x_1} + \frac{1}{4}p^2{L_{x_0}}^2 + p\zeta_{x_0x_1}L_{x_0} + p\zeta_{x_0^2x_1}\\
\mathcal{M}_1 L_{x_0x_1^3} &= L_{x_0x_1^3} - pL_{x_0x_1^2} + \frac{1}{2}p^2L_{x_0x_1} - \frac{1}{6}p^3L_{x_0} + p\zeta_{x_0x_1^2} - \frac{1}{2}p^2\zeta_{x_0x_1}\\
\mathcal{M}_1 L_{x_0^4x_1} &= L_{x_0^4x_1} - \frac{1}{24}p{L_{x_0}}^4\\
\mathcal{M}_1 L_{x_0^3x_1^2} &= L_{x_0^3x_1^2} - pL_{x_0^3x_1} + \frac{1}{12}p^2{L_{x_0}}^3 + \frac{1}{2}p\zeta_{x_0x_1}{L_{x_0}}^2 + p\zeta_{x_0^2x_1}L_{x_0} + p\zeta_{x_0^3x_1}\\
\mathcal{M}_1 L_{x_0^2x_1x_0x_1} &= L_{x_0^2x_1x_0x_1} + 3pL_{x_0^3x_1} - pL_{x_0}L_{x_0^2x_1} - p\zeta_{x_0x_1}{L_{x_0}}^2 - 2p\zeta_{x_0^2x_1}L_{x_0} - 3p\zeta_{x_0^3x_1}\\
\mathcal{M}_1 L_{x_0^2x_1^3} &= L_{x_0^2x_1^3} - pL_{x_0^2x_1^2} + \frac{1}{2}p^2L_{x_0^2x_1} - \frac{1}{12}p^3{L_{x_0}}^2 + \left(p\zeta_{x_0x_1^2} - \frac{1}{2}p^2\zeta_{x_0x_1}\right)L_{x_0}\\
&+ p\zeta_{x_0^2x_1^2} - \frac{1}{2}p^2\zeta_{x_0^2x_1}\\
\mathcal{M}_1 L_{x_0x_1x_0x_1^2} &= L_{x_0x_1x_0x_1^2} + 2pL_{x_0^2x_1^2} - p^2L_{x_0^2x_1} - \frac{1}{2}p{L_{x_0x_1}}^2 + \left(\frac{1}{2}p^2L_{x_0} + p\zeta_{x_0x_1}\right)L_{x_0x_1}\\
&+ \left(-3p\zeta_{x_0x_1^2} + \frac{1}{2}p^2\zeta_{x_0x_1}\right)L_{x_0} - 2p\zeta_{x_0^2x_1^2} + p^2\zeta_{x_0^2x_1} - \frac{1}{2}p{\zeta_{x_0x_1}}^2\\
\mathcal{M}_1 L_{x_0x_1^4} &= L_{x_0x_1^4} - pL_{x_0x_1^3} + \frac{1}{2}p^2L_{x_0x_1^2} - \frac{1}{6}p^3L_{x_0x_1} + \frac{1}{24}p^4L_{x_0} + p\zeta_{x_0x_1^3} - \frac{1}{2}p^2\zeta_{x_0x_1^2} + \frac{1}{6}p^3\zeta_{x_0x_1}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_1 L_{x_0^5x_1} &= L_{x_0^5x_1} - \frac{1}{120} p {L_{x_0}}^5 \\
\mathcal{M}_1 L_{x_0^4x_1^2} &= L_{x_0^4x_1^2} - pL_{x_0^4x_1} + \frac{1}{48}p^2 L_{x_0^4} + \frac{1}{6} p\zeta_{x_0x_1}L_{x_0^3} + \frac{1}{2} p\zeta_{x_0^2x_1}L_{x_0^2} + p\zeta_{x_0^3x_1}L_{x_0} + p\zeta_{x_0^4x_1} \\
\mathcal{M}_1 L_{x_0^3x_1x_0x_1} &= L_{x_0^3x_1x_0x_1} + 4pL_{x_0^4x_1} - pL_{x_0}L_{x_0^3x_1} - \frac{1}{3} p\zeta_{x_0x_1}{L_{x_0}}^3 - p\zeta_{x_0^2x_1}{L_{x_0}}^2 \\
&- 3p\zeta_{x_0^3x_1}L_{x_0} - 4p\zeta_{x_0^4x_1} \\
\mathcal{M}_1 L_{x_0^3x_1^3} &= L_{x_0^3x_1^3} - pL_{x_0^3x_1^2} + \frac{1}{2} p^2 L_{x_0^3x_1} - \frac{1}{36}p^3 {L_{x_0}}^3 + \left(\frac{1}{2} p\zeta_{x_0x_1^2} - \frac{1}{4} p^2\zeta_{x_0x_1}\right){L_{x_0}}^2 \\
&+ \left(p\zeta_{x_0^2x_1^2} - \frac{1}{2} p^2\zeta_{x_0^2x_1}\right)L_{x_0} + p\zeta_{x_0^3x_1^2} - \frac{1}{2} p^2\zeta_{x_0^3x_1} \\
\mathcal{M}_1 L_{x_0^2x_1x_0x_1^2} &= L_{x_0^2x_1x_0x_1^2} - pL_{x_0^2x_1x_0x_1} - \frac{3}{2} p^2 L_{x_0^3x_1} + \left(\frac{1}{2} p^2 L_{x_0} + p\zeta_{x_0x_1}\right)L_{x_0^2x_1} \\
&+ \left(-\frac{3}{2} p\zeta_{x_0x_1^2} + \frac{1}{4} p^2\zeta_{x_0x_1}\right){L_{x_0}}^2 + \left(-2p\zeta_{x_0^2x_1^2} + p^2\zeta_{x_0^2x_1} - \frac{1}{2} p{\zeta_{x_0x_1}}^2\right)L_{x_0} \\
&+ \frac{3}{2} p^2\zeta_{x_0^3x_1} + p\zeta_{x_0^2x_1x_0x_1} \\
\mathcal{M}_1 L_{x_0^2x_1^2x_0x_1} &= L_{x_0^2x_1^2x_0x_1} + pL_{x_0^2x_1x_0x_1} + 3pL_{x_0^3x_1^2} - pL_{x_0}L_{x_0^2x_1^2} - 2p\zeta_{x_0x_1}L_{x_0^2x_1} \\
&+ \left(\frac{3}{2} p\zeta_{x_0x_1^2} + \frac{1}{4} p^2\zeta_{x_0x_1}\right){L_{x_0}}^2 + \frac{3}{2} p{\zeta_{x_0x_1}}^2 L_{x_0} - 3p\zeta_{x_0^3x_1^2} + p\zeta_{x_0x_1}\zeta_{x_0^2x_1} - p\zeta_{x_0^2x_1x_0x_1} \\
\mathcal{M}_1 L_{x_0^2x_1^4} &= L_{x_0^2x_1^4} - pL_{x_0^2x_1^3} + \frac{1}{2} p^2L_{x_0^2x_1^2} - \frac{1}{6} p^3 L_{x_0^2x_1} + \frac{1}{48}p^4{L_{x_0}}^2 \\
&+ \left(p\zeta_{x_0x_1^3} - \frac{1}{2} p^2\zeta_{x_0x_1^2} + \frac{1}{6} p^3\zeta_{x_0x_1}\right)L_{x_0} + p\zeta_{x_0^2x_1^3} - \frac{1}{2} p^2\zeta_{x_0^2x_1^2} + \frac{1}{6} p^3\zeta_{x_0^2x_1} \\
\mathcal{M}_1 L_{x_0x_1x_0x_1^3} &= L_{x_0x_1x_0x_1^3} - pL_{x_0x_1x_0x_1^2} - p^2L_{x_0^2x_1^2} + \frac{1}{3} p^3 L_{x_0^2x_1} + \frac{1}{4} p^2 {L_{x_0x_1}}^2 \\
&+ \left(-\frac{1}{6} p^3 L_{x_0} + p\zeta_{x_0x_1^2} - \frac{1}{2} p^2\zeta_{x_0x_1}\right)L_{x_0x_1} + \left(-4p\zeta_{x_0x_1^3} + p^2\zeta_{x_0x_1^2} - \frac{1}{6} p^3\zeta_{x_0x_1}\right)L_{x_0} \\
&+ p^2\zeta_{x_0^2x_1^2} - \frac{1}{3} p^3\zeta_{x_0^2x_1} + p\zeta_{x_0x_1x_0x_1^2} + \frac{1}{4} p^2{\zeta_{x_0x_1}}^2 \\
\mathcal{M}_1 L_{x_0x_1^5} &= L_{x_0x_1^5} - pL_{x_0x_1^4} + \frac{1}{2} p^2L_{x_0x_1^3} - \frac{1}{6} p^3L_{x_0x_1^2} + \frac{1}{24}p^4L_{x_0x_1} \\
&- \frac{1}{120}p^5L_{x_0} + p\zeta_{x_0x_1^4} - \frac{1}{2} p^2\zeta_{x_0x_1^3} + \frac{1}{6} p^3\zeta_{x_0x_1^2} - \frac{1}{24}p^4\zeta_{x_0x_1}
\end{aligned}
$$

References

[1] Cartier, P. Jacobiennes généralisées, monodromie unipotente et intégrales itérées. *Séminaire Bourbaki*, 687 (1987), 31–52.

[2] Chen, K. Algebras of iterated path integrals and fondamental groups. *Trans. of the A.M.S. 156*, 3 (may 1971), 359–379.

[3] Drinfel'd, V. On quasitriangular quasi–hopf algebra and a group closely connected with gal($\bar{q}/q$). *Leningrad Math. J. 2*, 4 (1991), 829–860.

[4] Écalle, J. *Résurgence I et II.* Publications Univ. Orsay, Orsay, 1981.

[5] Hoang Ngoc, M. Fonction dirichlet d'ordre n et de paramètre t. *Discrete Mathematics 180* (1998).

[6] Hoang Ngoc, M., and Petitot, M. Mots de lyndon, polylogarithmes et fonction ζ de riemann. *submited to Discret Maths.*

[7] Petitot, M. *Algèbre non commutative en Scratchpad. Application au problème de la réalisation minimale analytique.* Thèse de Doctorat, Université Lille I, Janvier 1992.

[8] Radford, D. A natural ring basis for the shuffle algebra and an application to group schemes. *Journal of Algebra 58* (1979), 432–454.

[9] Ree, R. Lie elements and an algebra associated with shuffles. *Ann. of Math 68* (1958), 210–220.

[10] Reutenauer, C. *Free Lie Algebras*, vol. New Series-7 of *London Mathematical Society Monographs.* Oxford Science Publications, 1993.

[11] Wechsung, G. Functional equations of hyperlogarithms. In *Structural properties of polylogarithms*, L. Lewin, Ed., vol. 37 of *Mathematical surveys and monographs.* Amer.Math.Soc., 1992, ch. 8, pp. 171–184.

[12] Wojtkowiak, G. The basic structure of polylogarithmic functional equations. In *Structural properties of polylogarithms*, L. Lewin, Ed., vol. 37 of *Mathematical surveys and monographs.* Amer.Math.Soc., 1992, pp. 205–231.

[13] Zagier, D. Values of zeta functions and their applications. In *First European Congress of Mathematics* (1994), vol. 2, Birkhäuser, pp. 179–512.