Matrix coefficients and the structure of LDIAG

Gérard H. E. Duchamp, Christophe Tollu

LIPN, ***Université de Paris XIII, France***

and

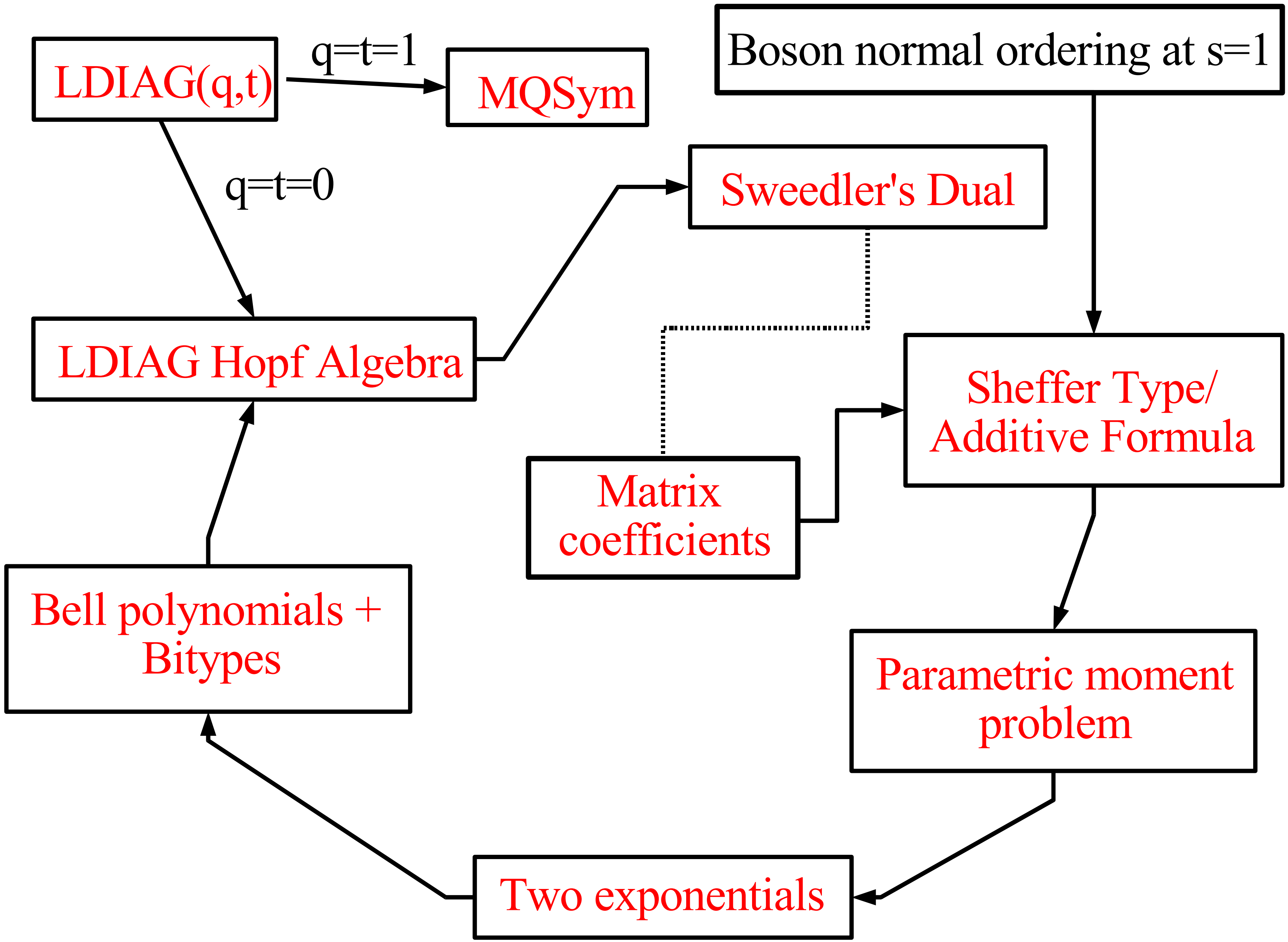
Karol A. Penson, LPTMC, ***Université de Paris VI, France***

Collaborators :

Allan I. Solomon, ***The Open University, United Kingdom***

Pawel Blasiak, ***Instit. of Nucl. Phys., Krakow, Pologne***

Andrzej Horzela, ***Instit. of Nucl. Phys., Krakow, Pologne***


LDIAG(q,t)
q=t=1
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Boson normal ordering at s=1
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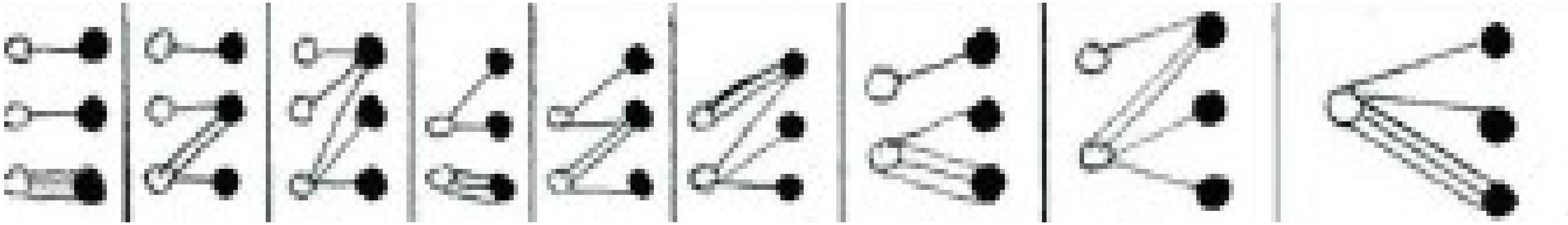
Two exponentials

About the LDIAG Hopf algebra

In a relatively recent paper Bender, Brody and Meister (*) introduce a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs (see third Part of this talk).

These graphs label monomials and are obtained in the case of special interest when the two EGF have a constant term equal to unity.

Bender, C.M, Brody, D.C. and Meister,
Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)



Some 5-line diagrams

If we write these functions as exponentials, we are led to witness a surprising interplay between the following aspects: **algebra** (of normal forms or of the exponential formula), **geometry** (of one-parameter groups of transformations and their conjugates) and **analysis** (parametric Stieltjes moment problem and convolution of kernels).

Classical normal ordering problem for bosons

The normal ordering problem goes as follows.

- Weyl (two-dimensional) algebra defined as

$$< a^+, a \; ; [a , a^+]=1 > \mid aa^+ \text{--->} a^+a +1$$

- Known to have no (faithful) representation by bounded operators in a Banach space.

There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

$$a \text{--->} d/dx \; ; \; a+ \text{--->} x$$

where a has degree -1 and a^+ has degree 1.

A typical element in the Weyl algebra is of the form (normal form).

$$\Omega = \sum_{k,l \geq 0} c(k,l)(a^{+})^{k} a^{l}$$

When Ω is a single monomial, a word i.e. a product of generators a^+, a, there is solution to the normal ordering problem (and thus, by linearity to the general problem) using rook numbers.

Today, we will be interested with the use of matrix coefficients in two instances :

normal ordering --> infinite matrices --> moments

finite representations --> Sweedler's dual and automata

A word (boson string) and more generally an homogeneous operator (for the grading where a has degree -1 and a^+ has degree 1) of degree e reads

$$\Omega = \sum_{\substack{k,l \geq 0 \\ k-l=e}} c(k,l)(a^+)^k a^l$$

Due to the symmetry of the Weyl algebra, we can suppose, with no loss of generality that $e \geq 0$. For homogeneous operators one can define generalized Stirling numbers (GSN) by

$$\Omega^n = (a^+)^{ne} \sum_{k \geq 0} S_\Omega(n,k)(a^+)^k a^k \quad \text{(Eq1)}$$

The case of a pure string is of special interest for physics and can be solved combinatorally. The recipe, for a string W is the following:

- associate a path with north east steps for every a^+ and a south east step for every a.
- construct the Ferrers diagram B over this path

The normal form of W is

$$W = \sum_{k \geq 0} R(B,k)(a^+)^{r-k} a^{s-k}$$

where R(B,k) is the k-th rook number of the board B.

In particular, the boson string $w=[(a^+)^r a^s]$ was considered by Penson, Solomon, Blasiak and al. In this case the GSN will be denoted by $S_{r,s}(n,k)$ and, due to the particular form of W, one has

$$\left[(a^+)^r a^s\right]^n = (a^+)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^+)^k a^k$$

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3)$$

For $w = a^+ aa^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+ aaa^+ a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

Setting

$$B_{r,s}(n,y)=\sum_{k=0}^{\infty} S_{r,s}(n,k)y^k$$

for the generating polynomials of the lines of the generalized Stirling matrix, one has the formulas

$$B_{r,s}(n,y)=\sum_{k=s}^{ns} S_{r,s}(n,k)y^k$$

$$=e^{-y}\sum_{k=s}^{\infty}\frac{1}{k!}\prod_{j=1}^{n}[(k+(j-1)(r-s))(k+(j-1)(r-s)-1)$$

$$\cdots(k+(j-1)(r-s)-s+1)]y^k.$$

... and, when s=1, the EGF of these polynomials is an exponential which gives an additive formula in the variable y (see the paper One-parameter Groups)

$$e^{y(e^x-1)} = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{1,1}(n,k)y^k\right)\frac{x^n}{n!}$$

and

$$\exp\left[y\left(\frac{1}{\sqrt[r-1]{1-(r-1)x^{r-1}}}-1\right)\right] = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{r,1}(n,k)y^k\right)\frac{x^n}{n!} \qquad r=2,3,\ldots.$$

For which, we have Dobi’nski-type relations

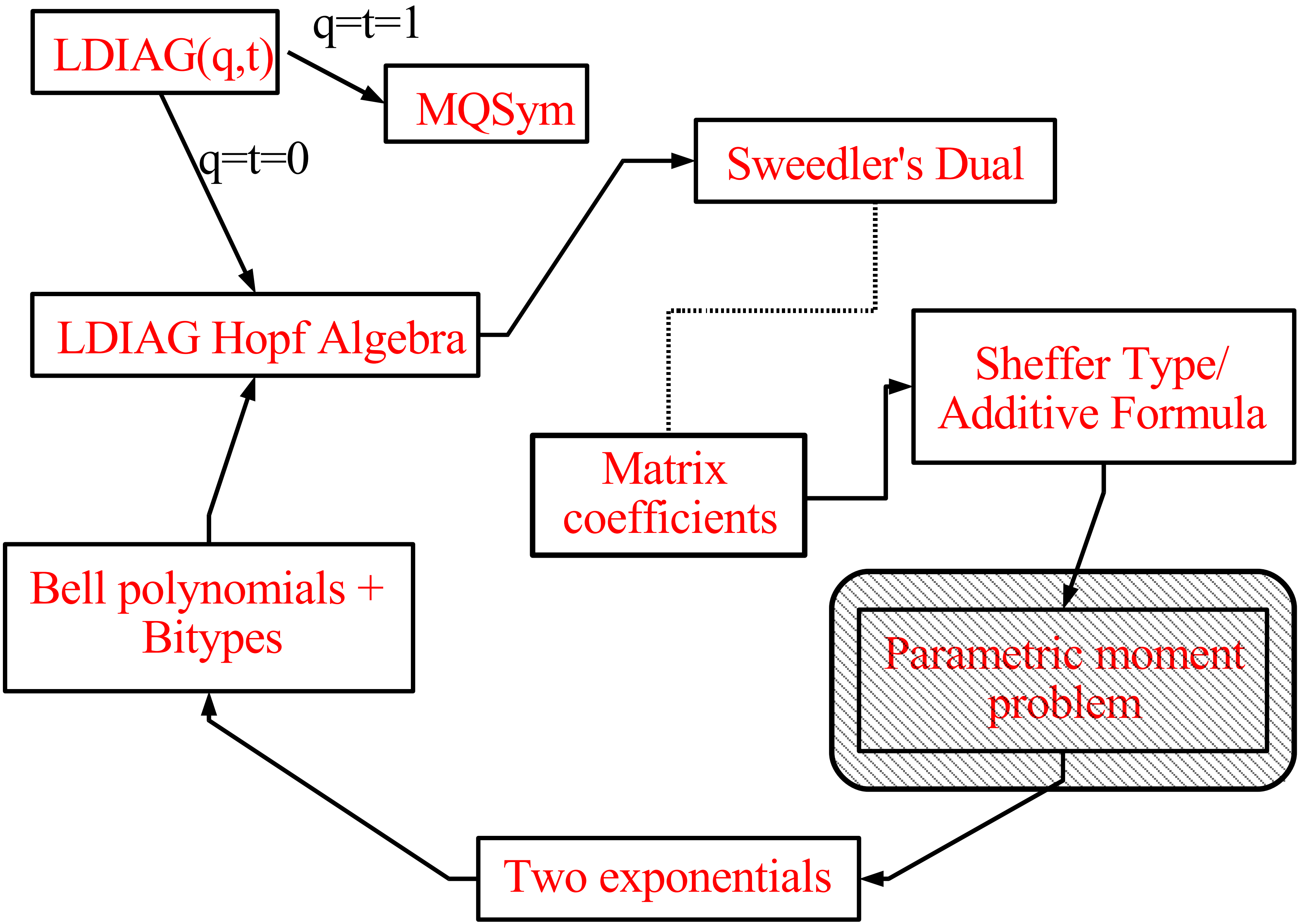
$$\frac{1}{e^y}\sum_{k=1}^{\infty}\frac{k^n}{k!}y^k = \sum_{k=1}^{n} S_{1,1}(n,k)y^k \qquad n=0,1,\ldots$$

$$\frac{(r-1)^n}{e^y}\sum_{k=1}^{\infty}\frac{\Gamma\left(n+\frac{k}{r-1}\right)}{k!\,\Gamma\left(\frac{k}{r-1}\right)}y^k = \sum_{k=1}^{n} S_{r,1}(n,k)y^k \qquad n=1,2,\ldots.$$

Classical Stieltjes moment problem

Consider a sequence of real numbers *B(n)*. The classical Stieltjes moment problem consists in finding a positive measure *W(x)dx* on the half-line]0,+∞[such that

$$B(n) = \int_0^{+\infty} x^n W(x) dx$$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Parametric Stieltjes moment problem

Consider a sequence of real functions *B(n,y)*. The parametric Stieltjes moment problem consists in finding a family of positive measures *W(x,y)dx* on the half-line]0,+∞[such that

$$B(n,y) = \int_0^{+\infty} x^n W(x,y)dx$$

Using the first Dobinski relation of slide (10), one can solve the parametric Stieltjes moment problem for the classical Stirling numbers as

$$S_{1,1}(n,y) = \int_0^{+\infty} x^n W_1(x,y) dx$$

with

$$W_1(x,y) = e^{-y} \sum_{k=1}^{\infty} \frac{y^k \delta(x-k)}{k!}$$

which is a Poisson distribution on the half-line]0,+∞[.

Using an inverse Mellin transform, one can solve the second parameric moment problem, which gives, this time, a continuous measure

$$S_{2,1}(n,y) = \int_0^{+\infty} x^n W_2(x,y) dx$$

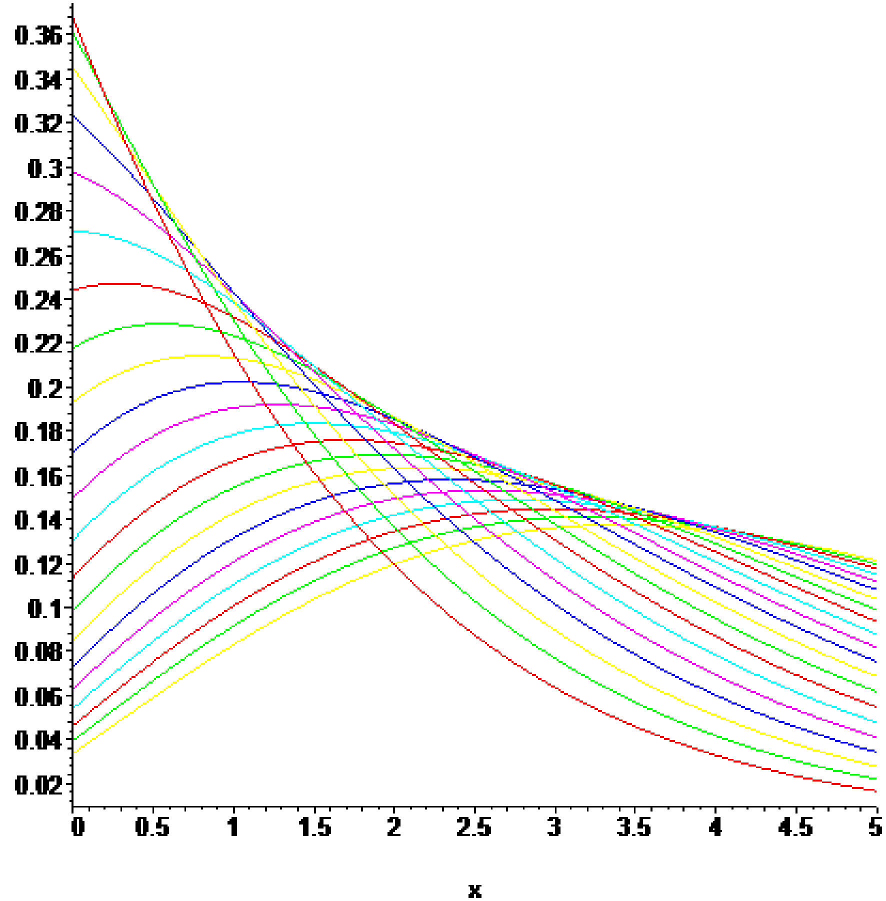
with

$$W_2(x,y) = y\, e^{-(x+y)} \frac{I_1(2\sqrt{xy})}{\sqrt{xy}}$$

```
> f1:=(x,y)->y*exp(-(x+y))*BesselI(1,2*sqrt(x*y))/sqrt(x*y);
```

$$f1 := (x, y) \to \frac{y\, e^{(-x-y)}\, \mathrm{BesselI}(1, 2\sqrt{x\,y})}{\sqrt{x\,y}}$$

```
> plot([seq(f1(x,1+0.2*k),k=0..20)],x=0..5,tickmarks=[8,10]);
```




```
> seq(evalf(f1(0.001,1+0.2*k),3),k=0..20);
```

0.367, 0.361, 0.345, 0.322, 0.297, 0.271, 0.245, 0.218, 0.193, 0.171, 0.150, 0.131, 0.115, 0.0985, 0.0851, 0.0731, 0.0633, 0.0543, 0.0468, 0.0397, 0.0337

Ongoing work

Realizations of the product for some types of infinite matrices

Convolution of kernels: We first suppose given two infinite matrices F(n,k), G(n,k) (n,k integers) admitting solutions for the parametric moment problem (PMP) which means that there are two (parametric) measures W_F, W_G such that

$$B_F(n,y) = \int_0^{+\infty} x^n W_F(x,y)dx$$

$$B_G(n,y) = \int_0^{+\infty} x^n W_G(x,y)dx$$

Then one can check easily that, if the two kernels W_F *and* W_G are convolable, then FG admits a solution for the PMP and

$$W_{FG}(x,y) = \int_0^{+\infty} W_F(x,z) W_G(z,y) dz$$

Questions: Q1) What are the types of matrices for which there is a PMP solution ?

Q2) Which are the ones for which the kernel is discrete ? Continuous ?

Q3) Are there general laws for convolution of these types of kernels.

Link with grafting: Certain classes of graphs (i.e. closed by relabelling and extraction of connected components) provide lower triangular matrices via

$M(n,k)$=number of graphs with labels $\{1, 2,..n\}$ and k connected components

the product of the matrices associated with two classes corresponds to the grafting obtained by considering the connected components of a graph of the first kind as vertices of a graph of the second kind.

Question: What are the legal types of grafting when we change denominators ? Link with renormalisation ?

Substitutions: An infinite matrice F(n,k) with finite rows can be seen as defining a transformation between EGF. The transformation is of the form f --> u(x)f(v(x)) with u(x)=1+... and v(x)=λx+... if the sequence of polynomials $B_F(n,y)$ is of Extended Sheffer Type (EST).

There is a « calculus » using vector fields on the half-line and their conjugates. (see SLC Viennot - Lucelle - and Myczcowce talks)

Questions: Q1) Combinatorial fields ? What is the «Stirling field » for instance ?

Q2) Make precise the dictionnaries (formal or analytic)
vector fields ↔ combinatorial matrices

Q3) What are the matrices coming from classes of graphs

For these one-parameter groups and conjugates of vector fields

G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,

One-parameter groups and combinatorial physics,

Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003. arXiv : quant-ph/0401126.

For the Sheffer-type sequences and coherent states

P Blasiak, A Horzela , K A Penson, G H E Duchamp and A I Solomon,

Boson Normal Ordering via Substitutions and Sheffer-type Polynomials,

(Published in Physics Letters A)

A word on the construction of LDIAG

Hadamard product of two sequences

$$(a_n)_{n\geq 0} \quad (b_n)_{n\geq 0}$$

can at once be transferred to EGFs by

$$F=\sum_{n\geq 0} a_n \frac{y^n}{n!};\; G=\sum_{m\geq 0} b_m \frac{y^m}{m!};\; \mathcal{H}(F,G):=\sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

which can be written

$$\mathcal{H}(F,G)=F(y\frac{d}{dx})G(x)|_{x=0}$$

and, in case *F(0)=G(0)=1* can be expressed in terms of (set) partitions.

We will adopt the notation

$$\alpha = 1^{a_1} 2^{a_2} \cdots r^{a_r}$$

for the *type* of a (set) partition which means that there are a_1 singletons a_2 pairs a_3 3-blocks a_4 4-blocks and so on.

The number of set partitions of type α as above is well known (see Comtet for example)

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Then, with

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

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one has

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

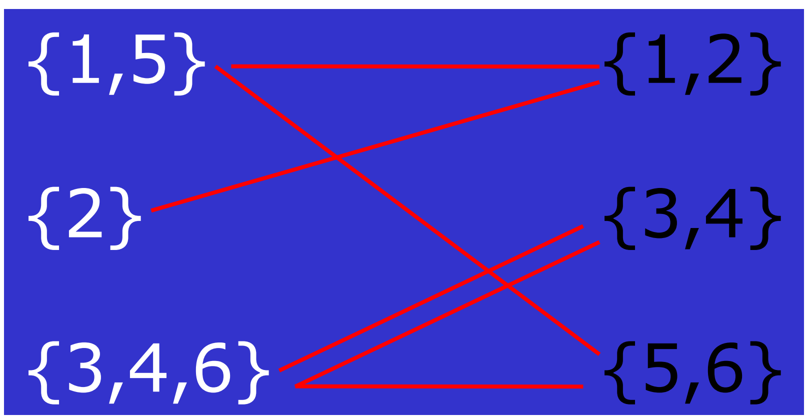
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

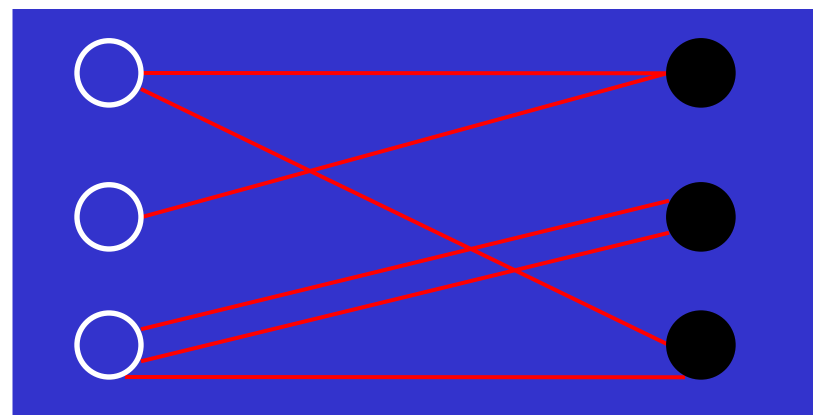
Now, one can count in another way the term *numpart(α)numpart(β)*. Remarking that this is the number of pairs of set partitions (P1,P2) with type(P1)=α, type(P2)=β. But every pair of partitions (P1,P2) has an intersection matrix ...

	{1,5}	{2}	{3,4,6}
{1,2}	1	1	0
{3,4}	0	0	2
{5,6}	1	0	1

Classes of packed matrices see NCSF VI (GD, Hivert, and Thibon)

Feynman-type diagram (Bender & al.)





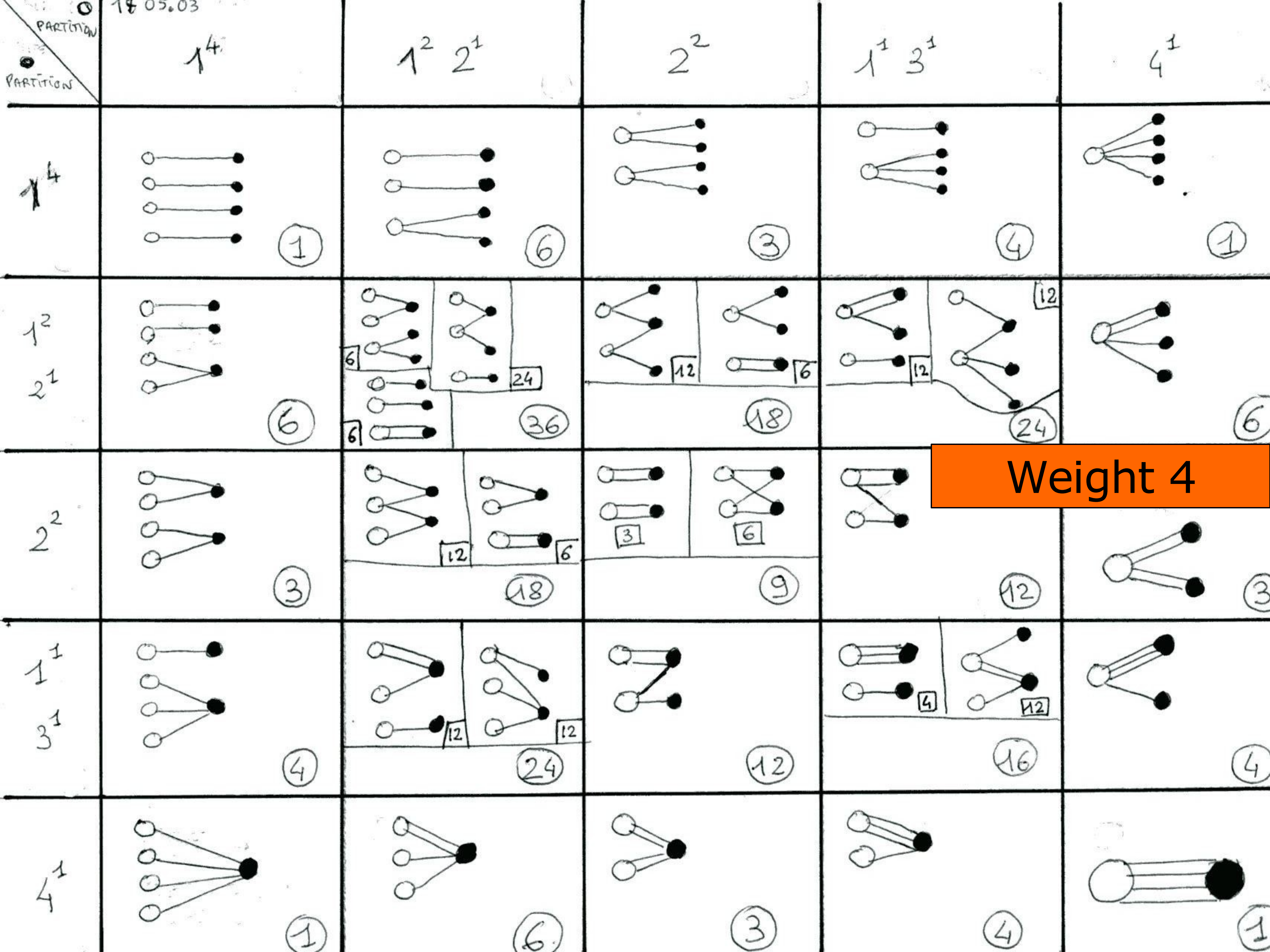
Now the product formula for EGFs reads

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{d \text{ diagram}} mult(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}\frac{y^{|d|}}{|d|!}$$

The main interest of this new form is that we can impose rules on the counted graphs.

Weight 4



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PARTITION
PARTITION
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2^2
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1^4
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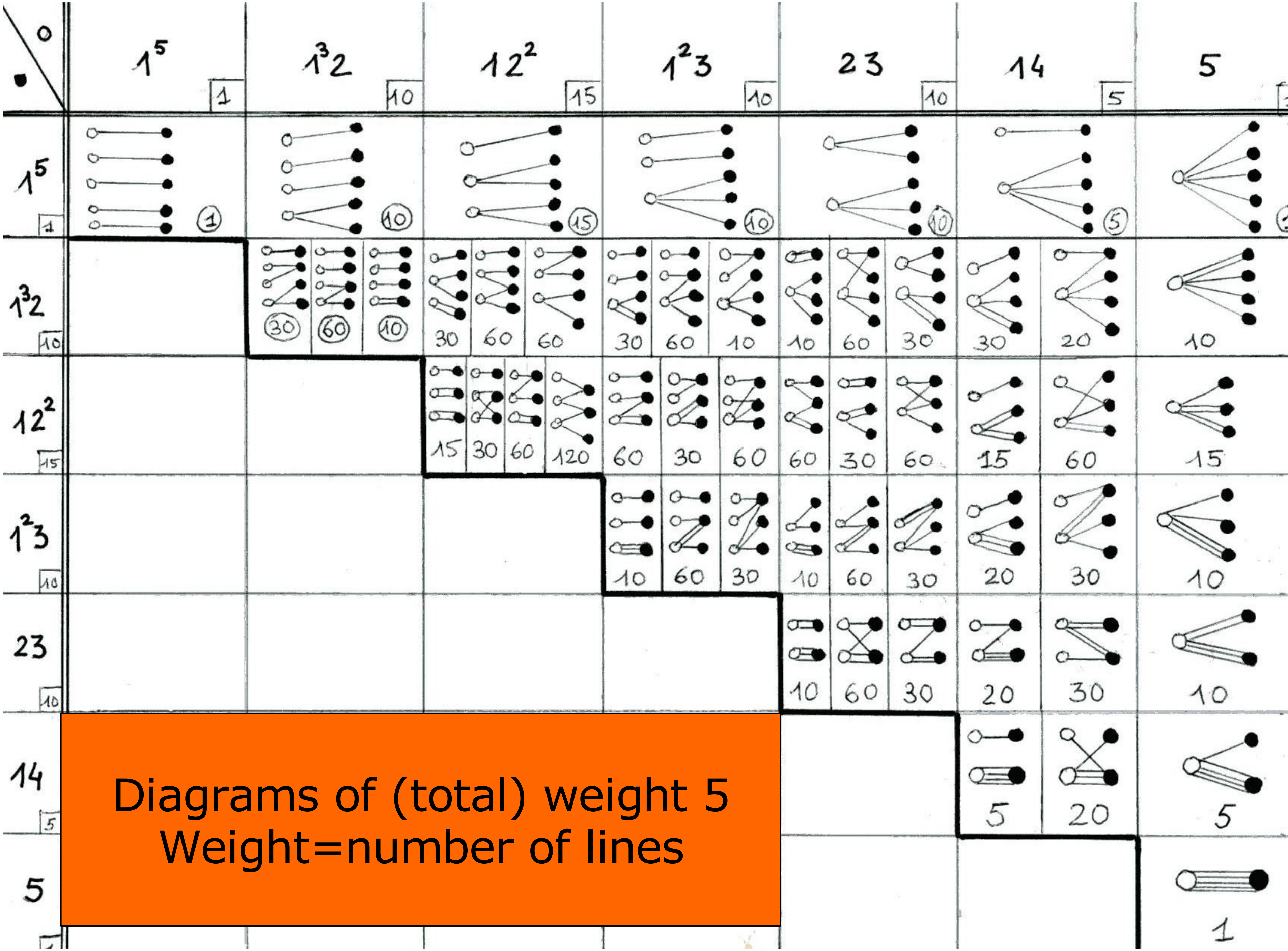


Diagrams of (total) weight 5
Weight=number of lines

Hopf algebra structure

$$(H,\mu,\Delta,1_H,\varepsilon,\alpha)$$

Satisfying the following axioms

- $(H,\mu,1_H)$ is an associative k-algebra with unit (here k will be a – commutative - field)
- (H,Δ,ε) is a coassociative k-coalgebra with counit
- $\Delta : H \to H\otimes H$ is a morphism of algebras
- $\alpha : H \to H$ is an anti-automorphism (the antipode) which is the inverse of Id for convolution.

Convolution is defined on End(H) by

$$\varphi \bullet \psi = \mu\,(\varphi \otimes \psi)\,\Delta$$

with this law End(H) is endowed with a structure of associative algebra with unit $1_H\varepsilon$.

First step: Defining the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d$ $\qquad$ $LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

(functions with finite supports on the set of diagrams). At this stage, we have a natural arrow *LDiag* → *Diag*.

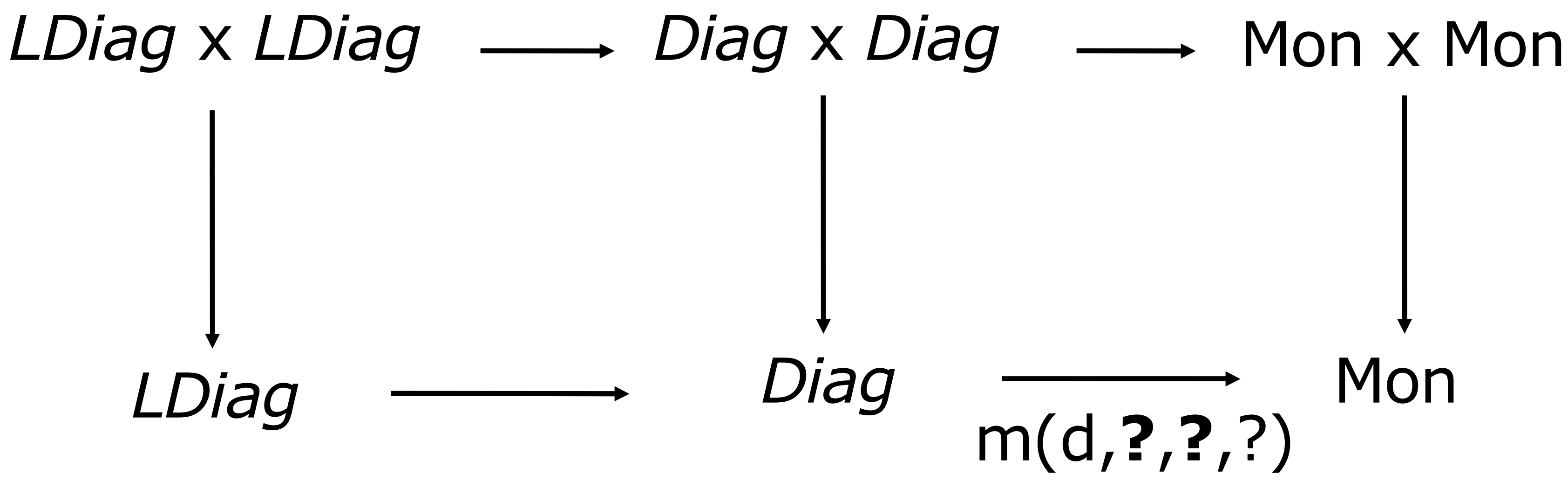
Second step: The product on *Ldiag* is just the superposing of diagrams

$$d_1 \star d_2 = \boxed{\begin{matrix} d_1 \\ d_2 \end{matrix}}$$

And, setting $m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\mathbf{V}^{\beta(d)}\, z^{|d|}$ one gets

$$m(d_1 * d_2,\mathbf{L},\mathbf{V},z) = m(d_1,\mathbf{L},\mathbf{V},z)\, m(d_2,\mathbf{L},\mathbf{V},z)$$

This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which, in turn, is compatible with the product of monomials.

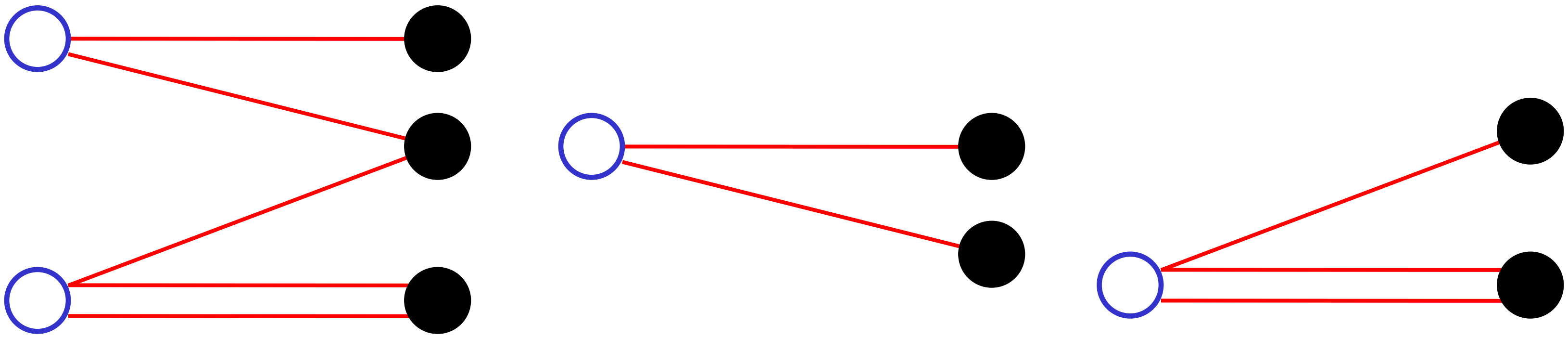


The coproduct needs to be compatible with m(d,**?**,**?**,?). One has two symmetric possibilities. The « white spots coproduct » reads

$$\Delta_{ws}(d)=\sum d_I \otimes d_J$$

the sum being taken over all the decompositions, (I,J) of the White Spots of d into two subsets.

For example, with the following diagrams d, d_1,d_2 ,

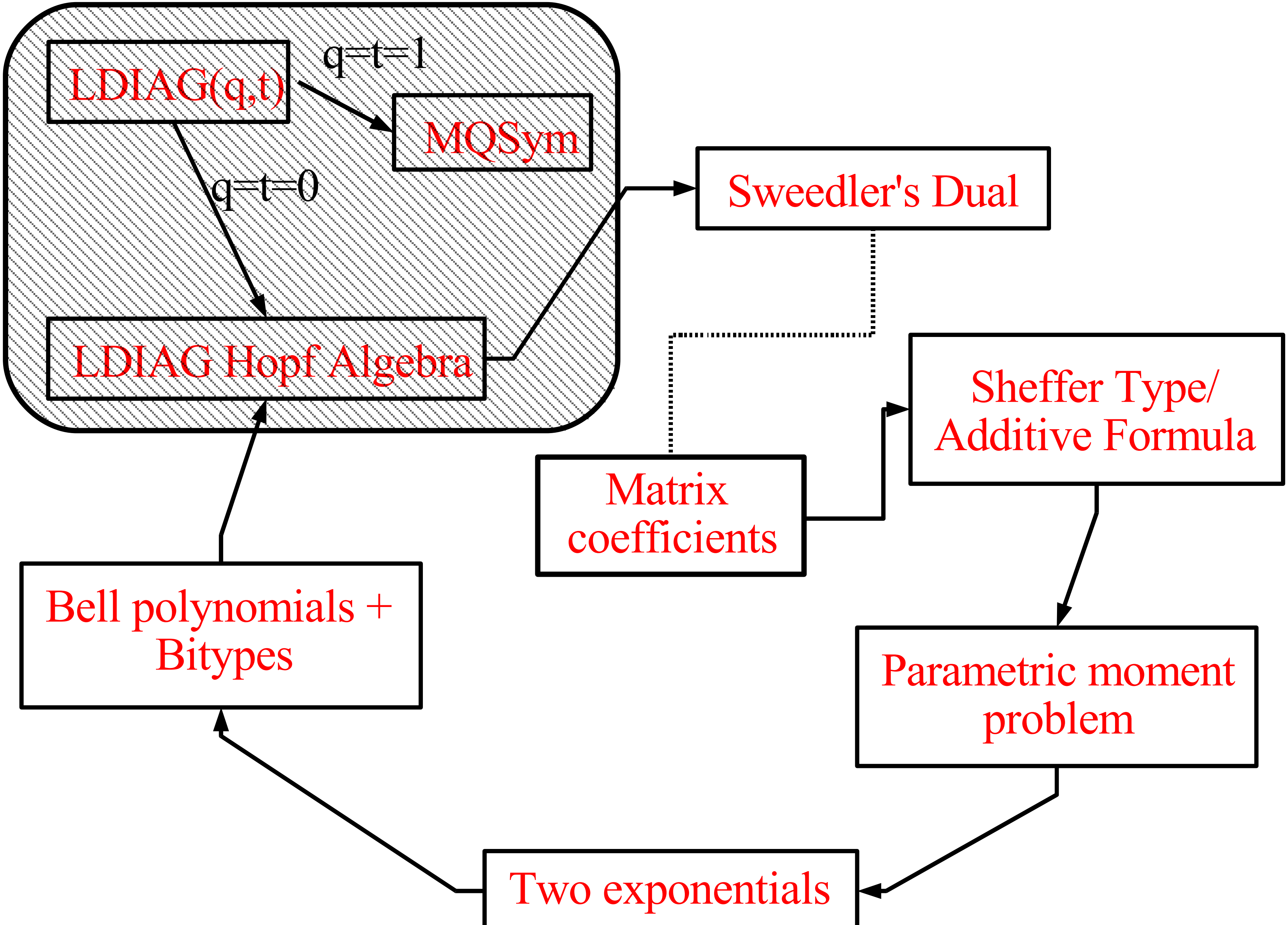


one has $\Delta_{ws}(d)=d\otimes\varnothing + \varnothing\otimes d + d_1\otimes d_2 + d_2\otimes d_1$

Today, we focus on the multiplicative structure of *Ldiag* remarking that the objects are in one-to-one correspondence with the so-called packed matrices of NCSFVI (Hopf algebra MQSym), but the product of MQSym is given (on a certain basis **MS**) according to the following example

$$\mathbf{MS}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{MS}_{[3\ 1]} =$$

$$\mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$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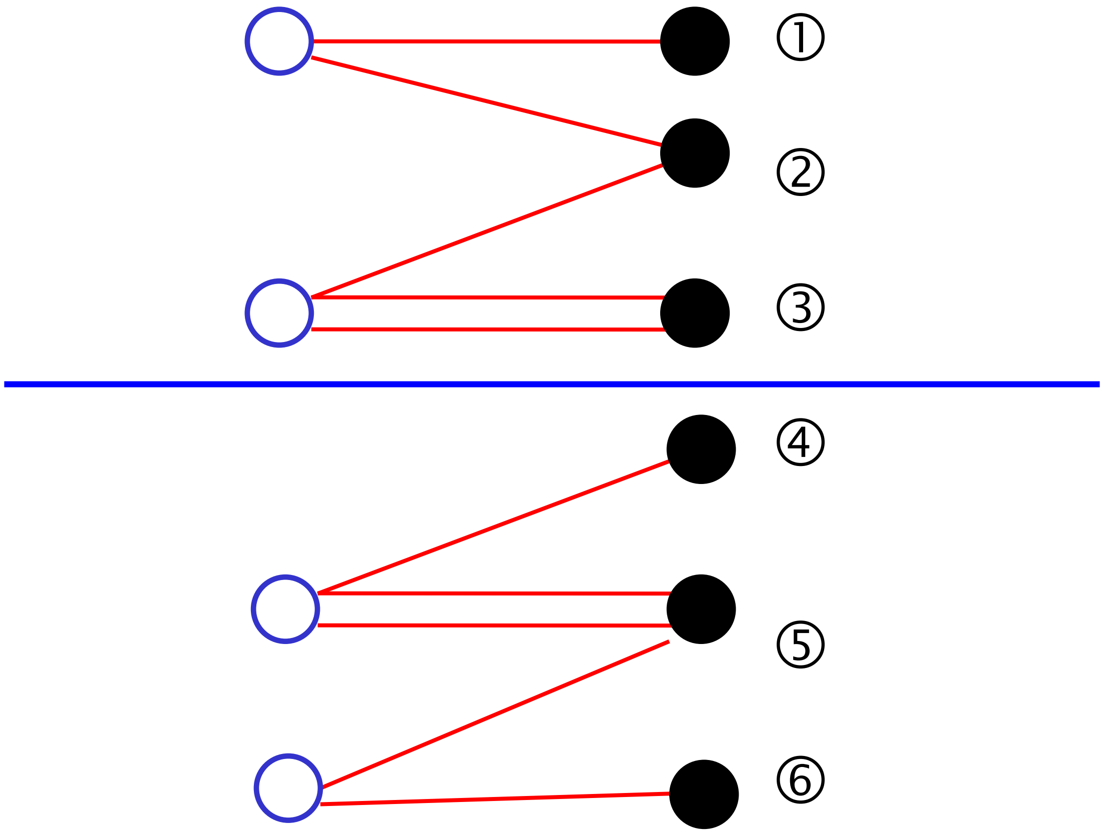
The double deformation provides an identity in the algebra of the symmetric semigroup. It goes as follows

- Stack the diagrams

- Develop according to the rules :
    - Every crossing “pays” a q
    - Every node-stacking “pays” a t

One has to show associativity (the remaining properties are straightforward)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Associativity can be shown by direct computation

$$(au \uparrow bv) \uparrow cw = \big(a(u \uparrow bv) + q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v) + q^{|au||b|} b(au \uparrow v)\big) \uparrow cw$$

$$\Big[a((u \uparrow bv) \uparrow cw) + q^{(|u|+|bv|)|c|} t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} ((u \uparrow bv) \uparrow w) + q^{(|au|+|bv|)|c|} c(a(u \uparrow bv) \uparrow w)\Big]$$

$$\Big[q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|u||b|+(|u|+|v|)|c|} t^{|a||b|} t^{(|a|+|b|)|c|} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow \; w)$$

$$q^{|u||b|+(|au|+|bv|)|c|} t^{|a||b|} c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w)\Big]$$

$$\Big[q^{|au||b|} b((au \uparrow v) \uparrow cw) + q^{|au||b|+(|au|+|v|)|c|} t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w) + q^{|au||b|+(|au|+|bv|)|c|} c(b(au \uparrow v) \uparrow w$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \left(b(v \uparrow cw) + q^{|v||c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|}c(bv \uparrow w)\right) =$$

$$\left[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|}b(au \uparrow v \uparrow cw)\right] +$$

$$\left[q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) +\right.$$

$$\left. q^{|v||c|+|au|(|b|+|c|)}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\right] +$$

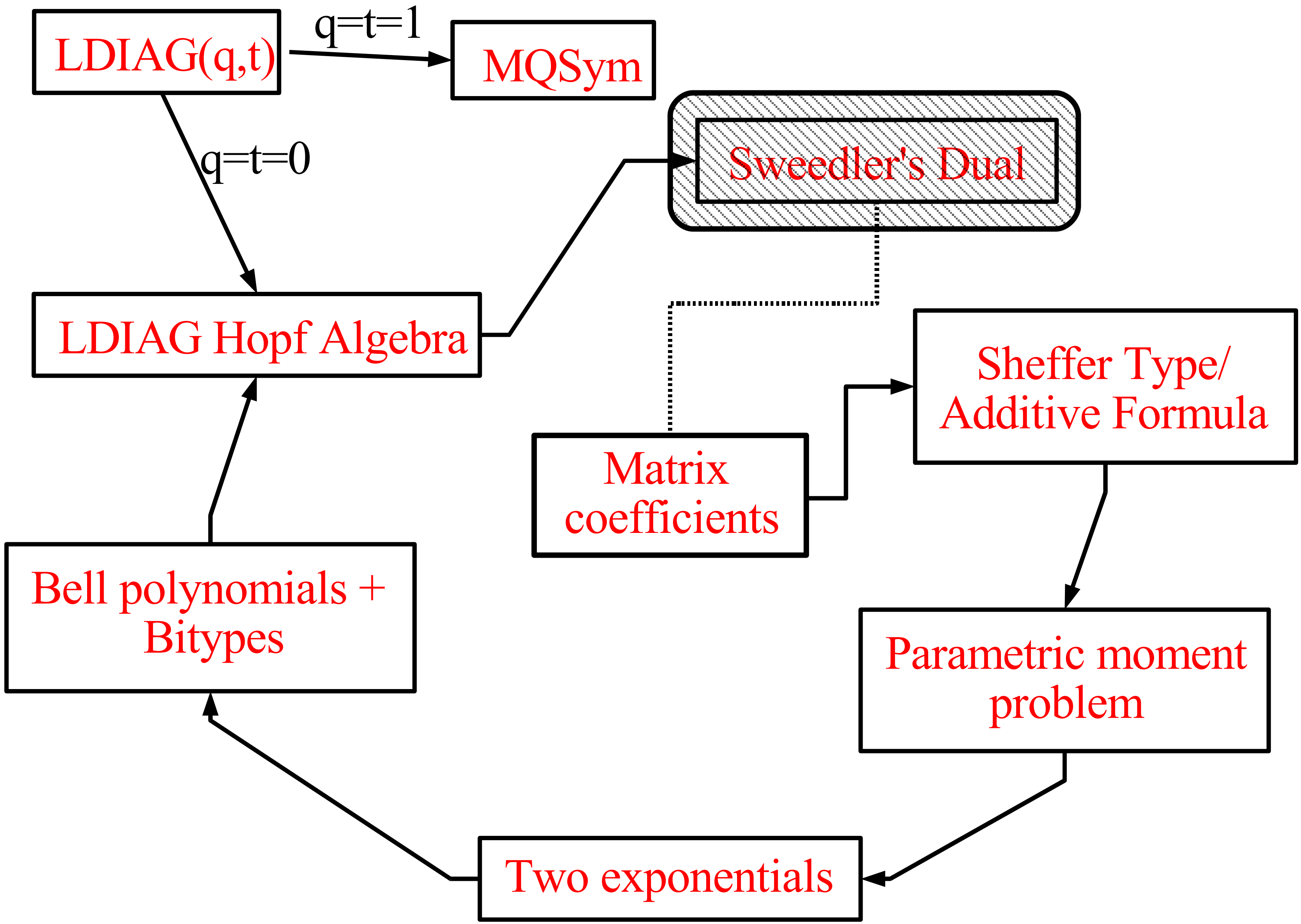
$$\left[q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w)\right] \quad (3)$$

dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$
$$q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w) \quad (5)$$


LDIAG(q,t)
q=t=1
MQSym
q=t=0
Sweedler's Dual
LDIAG Hopf Algebra
Sheffer Type/
Additive Formula
Matrix
coefficients
Bell polynomials +
Bitypes
Parametric moment
problem
Two exponentials

Sweedler's duals and automata theory

i) The drama of Sweedler's dual is the following

Let $\mathcal{A}$ be an associative algebra with unit. The product is a linear mapping

$$\mathcal{A}\otimes\mathcal{A} \xrightarrow{\mu} \mathcal{A}$$

ii) By dualization one gets

$$(\mathcal{A})^* \xrightarrow{{}^t\mu} (\mathcal{A}\otimes\mathcal{A})^*$$

but not a "stable calculus" as

$$(\mathcal{A})^*\otimes(\mathcal{A})^* \subseteq (\mathcal{A}\otimes\mathcal{A})^*$$

(strict in general). We ask for elements $x\in\mathcal{A}$ such that

$${}^t\mu(x)\in(\mathcal{A})^*\otimes(\mathcal{A})^*$$

Here, we will be concerned with the case $\mathcal{A}$= k$\langle A \rangle$ (non-commutative polynomials with coefficients in a field k).

Indeed, we have the following theorem (the beginning can be found in [ABE : Hopf algebras]) and the end is the starting point of Schützenberger's school of automata and language theory.

Theorem A: TFAE (the notations being as above)

i) ${}^{t}\mu(c) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$

ii) There are functions f_i, g_i $i=1,2..n$ such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in A^* (the free monoid of alphabet A).

iii) There is a morphism of monoids $\mu: A^* \to k^{n \times n}$ (square matrices of size n x n), a line λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If A is finite) c lies in the rational closure of A within the algebra k<<A>>.

We can safely apply the first three conditions of **Theorem A** to *Ldiag.* The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more “basic” functions. The modification reads

iv) (A is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in A} p(a)\, a$$

within the algebra k<<A>>.

Concluding remarks

i) We have much information on the structures of *Ldiag* and *Diag* (multiplicative and Hopf structures). (by Cartier-Milnor-Moore theorem).

ii) One can change the constants $V_k=1$ to a condition with level (i.e. $V_k=1$ for $k\leq N$ and $V_k= 0$ for $k>N$). We obtain then sub-Hopf algebras of the one constructed above. These can apply to the manipulation of partition functions of physical models including Free Boson Gas, Kerr model and Superfluidity.

iii) Schützenberger's theorem could be rephrased in saying that functions in Sweedler's dual are behaviours of finite (state and alphabet) automata.

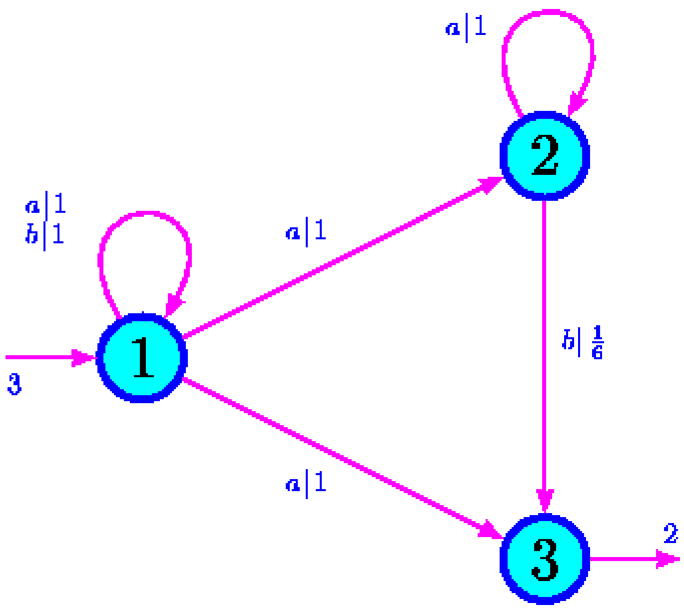


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6 + a^*b).$$

In our case, we are obliged to allow infinitely many edges.