Hopf Algebras of Diagrams and Deformations

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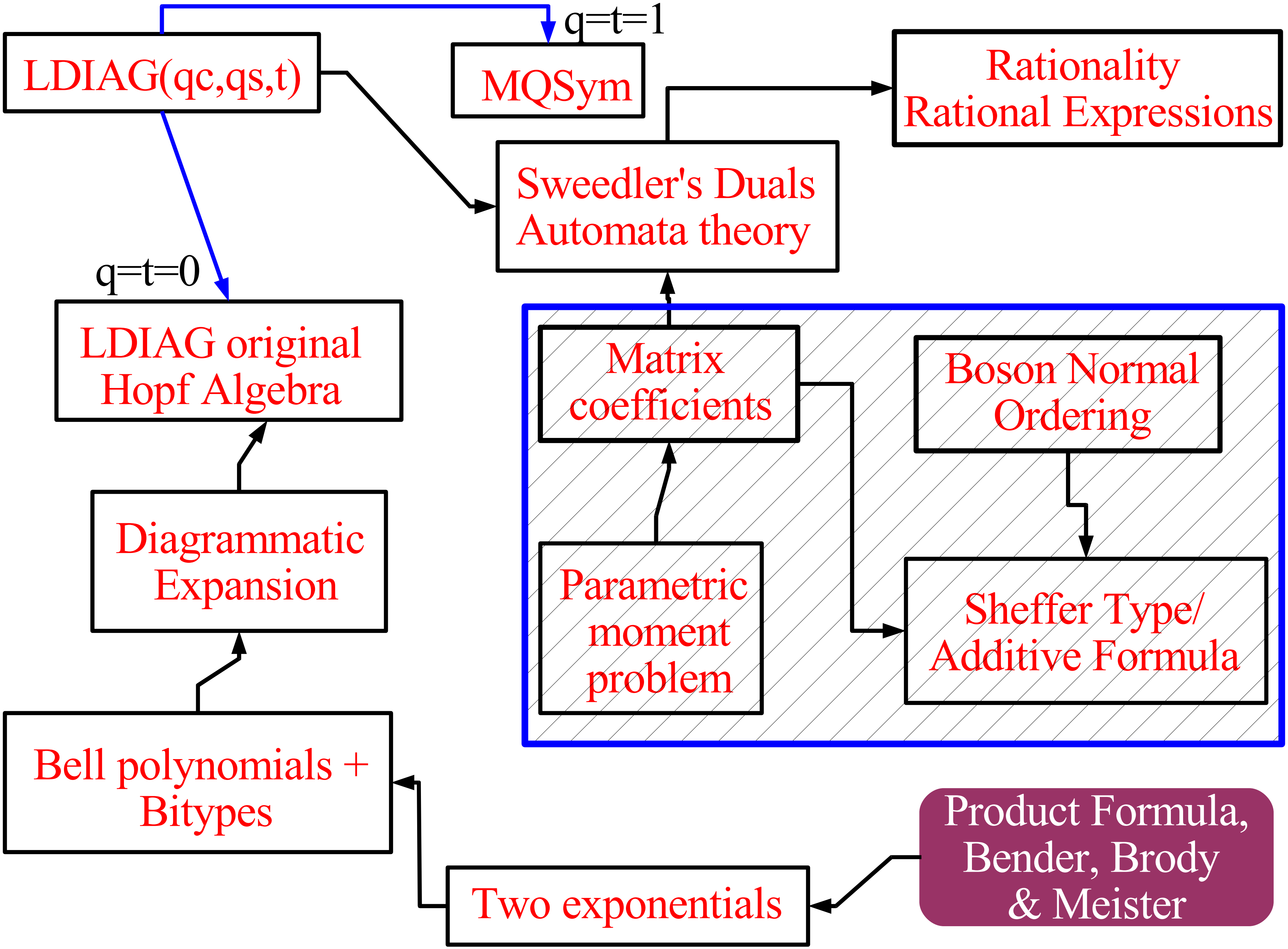


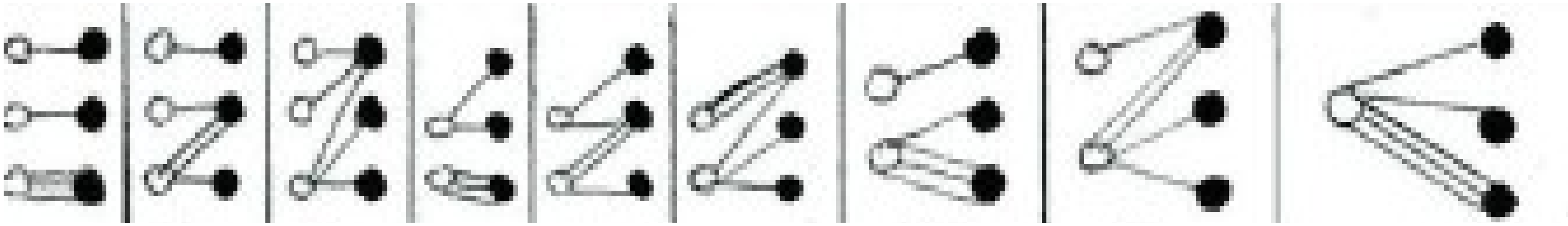
LDIAG(qc,qs,t)
q=t=1
MQSym
Rationality
Rational Expressions
Sweedler's Duals
Automata theory
q=t=0
LDIAG original
Hopf Algebra
Matrix
coefficients
Boson Normal
Ordering
Diagrammatic
Expansion
Parametric
moment
problem
Sheffer Type/
Additive Formula
Bell polynomials +
Bitypes
Product Formula,
Bender, Brody
& Meister
Two exponentials

About the LDIAG Hopf algebra

In a relatively recent paper Bender, Brody and Meister (*) introduce a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs (see third Part of this talk).

These graphs label monomials and are obtained in the case of special interest when the two EGF have a constant term equal to unity.

Bender, C.M, Brody, D.C. and Meister,
Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)



Some 5-line diagrams

If we write these functions as exponentials, we are led to witness a surprising interplay between the following aspects: **algebra** (of normal forms or of the exponential formula, Hopf structure), **geometry** (of one-parameter groups of transformations and their conjugates) and **analysis** (parametric Stieltjes moment problem and convolution of kernels). Today, we will first focus on the algebra. If time permits, we will touch on the other aspects.

Construction of the Hopf algebra LDIAG

How these diagrams arise and which data structures are around them

Let F, G be two EGFs.

$$F = \sum_{n\geq 0} a_n \frac{y^n}{n!};\ G = \sum_{m\geq 0} b_m \frac{y^m}{m!};\ \mathcal{H}(F,G) := \sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0}$$

Called « product formula » in the QFTP of Bender, Brody and Meister.

In case *F(0)=G(0)=1*, one can set

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

and then,

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

with $\alpha, \beta \in \mathbb{N}^{(\mathbb{N}^*)}$ multiindices

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

We will adopt the notation

$$\alpha = 1^{a_1} 2^{a_2} \cdots r^{a_r}$$

for the *type* of a (set) partition which means that there are a_1 singletons a_2 pairs a_3 3-blocks a_4 4-blocks and so on.

The number of set partitions of type α as above is well known (see Comtet for example)

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Then, with

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

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one has

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

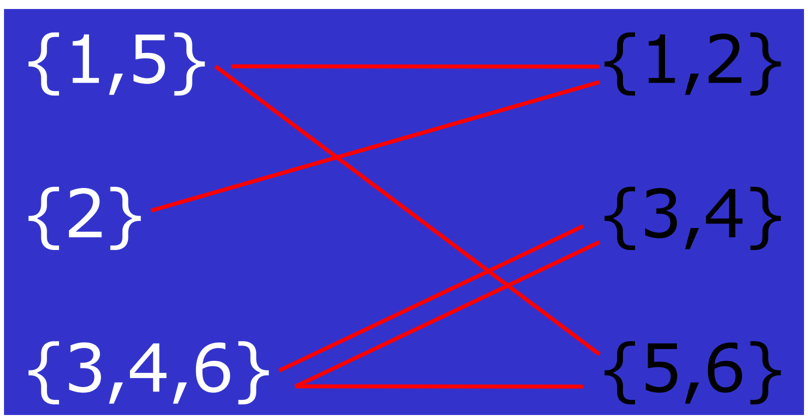
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

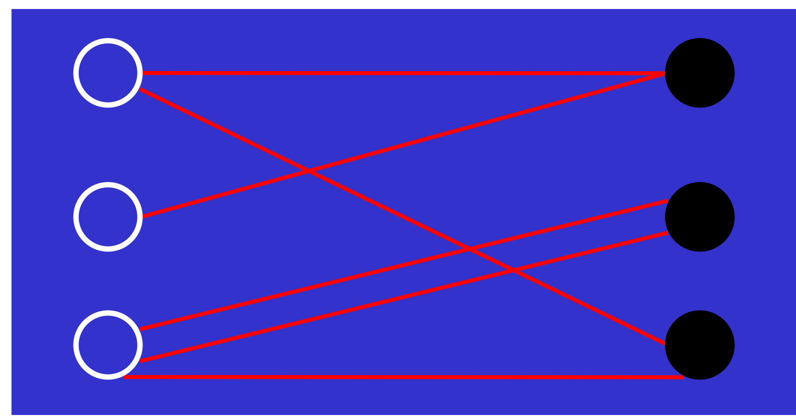
Now, one can count in another way the term *numpart(α)numpart(β)*. Remarking that this is the number of pairs of set partitions (P1,P2) with type(P1)=α, type(P2)=β. But every pair of partitions (P1,P2) has an intersection matrix ...

	{1,5}	{2}	{3,4,6}
{1,2}	1	1	0
{3,4}	0	0	2
{5,6}	1	0	1

Classes of packed matrices see NCSF VI (GD, Hivert, and Thibon)

Feynman-type diagram (Bender & al.)





Now the product formula for EGFs reads

$$\mathcal{H}(F,G) = \sum_{d\ FB-diagram} \frac{y^{|d|}}{|d|!} \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

$$\mathcal{H}(F,G) = \sum_{d \in \mathbf{diag}} \frac{y^{|d|}}{|d|!} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

The main interest of these new forms is that we can impose rules on the counted graphs and we can call these (and their relatives) graphs : Feynman-Bender Diagrams of this theory (here, the simplified model of Quantum Field Theory of Partitions).

One has now 3 types of diagrams :

- the diagrams with labelled edges (from 1 to |d|). Their set is denoted (see above) FB-diagrams.

- the unlabelled diagrams (where permutations of black and white spots are allowed). Their set is denoted (see above) **diag.**

- the diagrams, as drawn, with black (resp. white) spots ordered i.e. labelled. Their set is denoted **ldiag.**

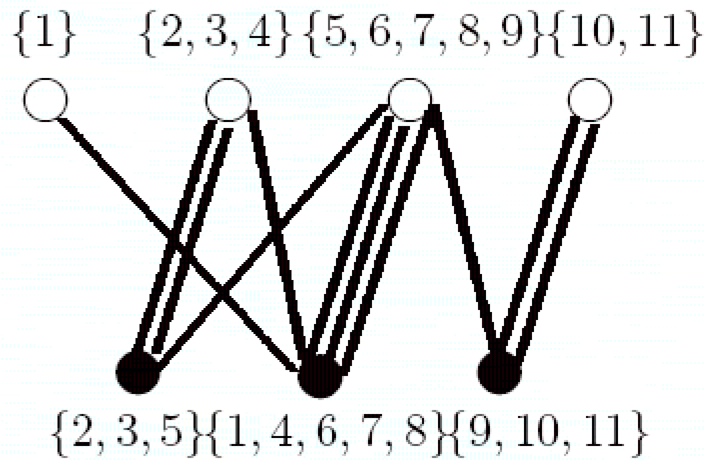
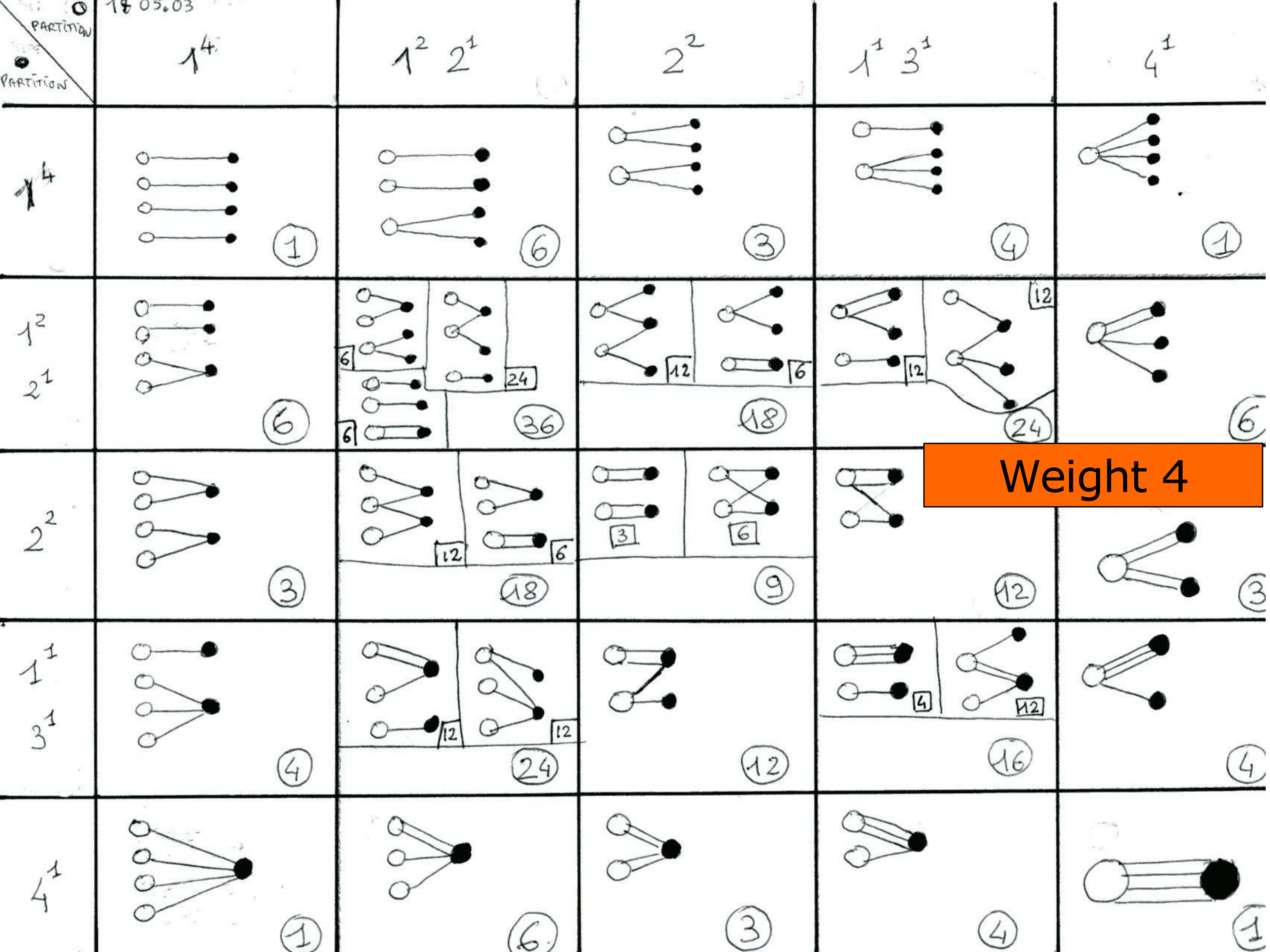
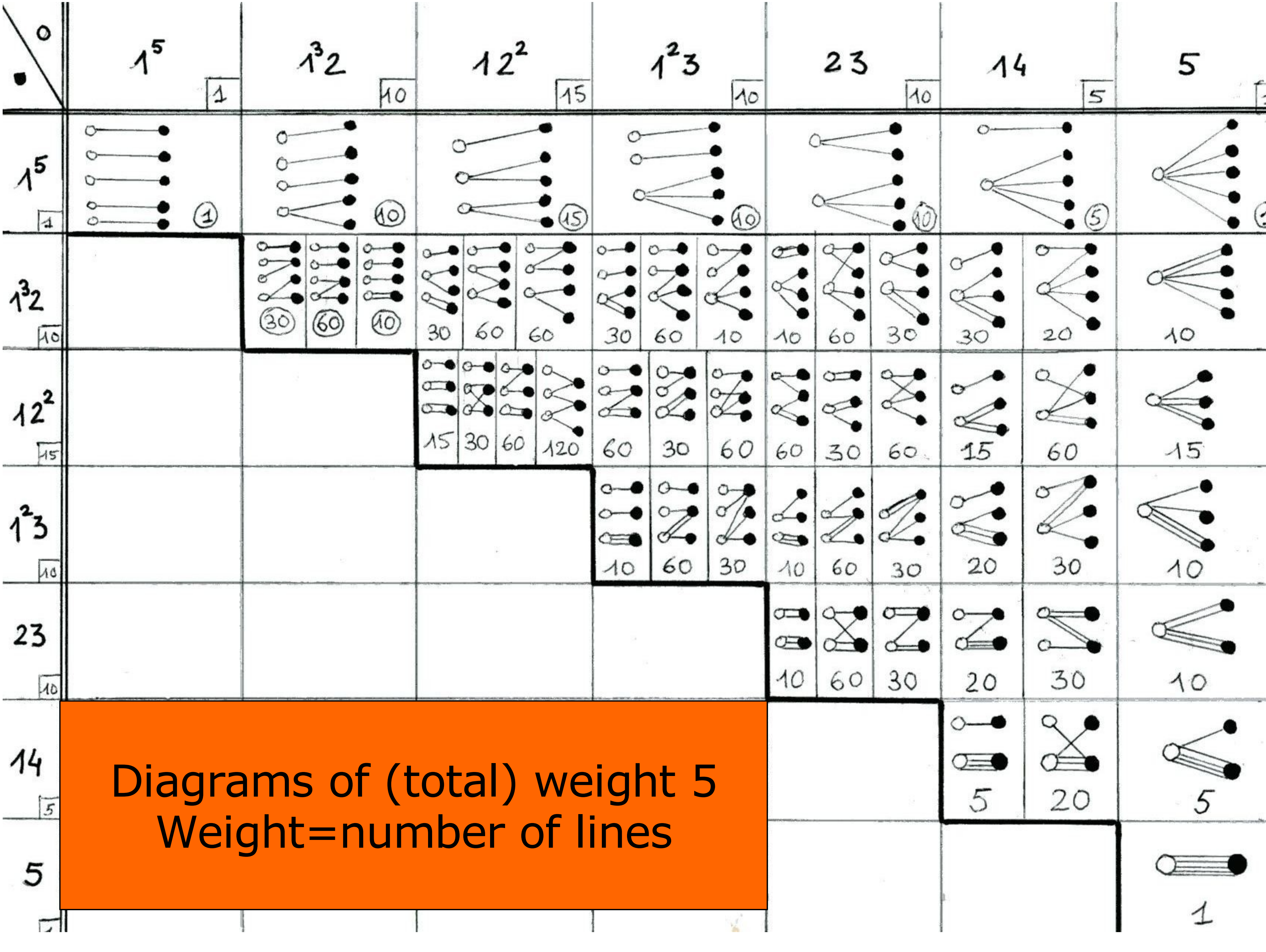


Fig 1. — *Diagram from* P_1, P_2 *(set partitions of* $[1 \cdots 11]$*).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for* P_1 *and white spots for* P_2*).*
The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. *But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*

Weight 4



18.05.03
PARTITION
PARTITION
1^4
$1^2\ 2^1$
2^2
$1^1\ 3^1$
4^1
1 6 3 4 1
6 36 18 24 6
3 18 9 12 3
4 24 12 16 4
1 6 3 4 1
6 24 6 36
12 6 18
12 12 24
12 6 3 6 9
12 12 24
4 12 16



Diagrams of (total) weight 5
Weight=number of lines

Hopf algebra structure

$$(H,\mu,\Delta,1_H,\varepsilon,\alpha)$$

Satisfying the following axioms

- $(H,\mu,1_H)$ is an associative k-algebra with unit (here k will be a – commutative - field)
- (H,Δ,ε) is a coassociative k-coalgebra with counit
- $\Delta : H \to H\otimes H$ is a morphism of algebras
- $\alpha : H \to H$ is an anti-automorphism (the antipode) which is the inverse of Id for convolution.

Convolution is defined on End(H) by

$$\varphi \bullet \psi = \mu\,(\varphi \otimes \psi)\,\Delta$$

with this law End(H) is endowed with a structure of associative algebra with unit $1_H\varepsilon$.

First step: Defining the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d \qquad LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

(functions with finite supports on the set of diagrams). At this stage, we have a natural arrow *LDiag* → *Diag*.

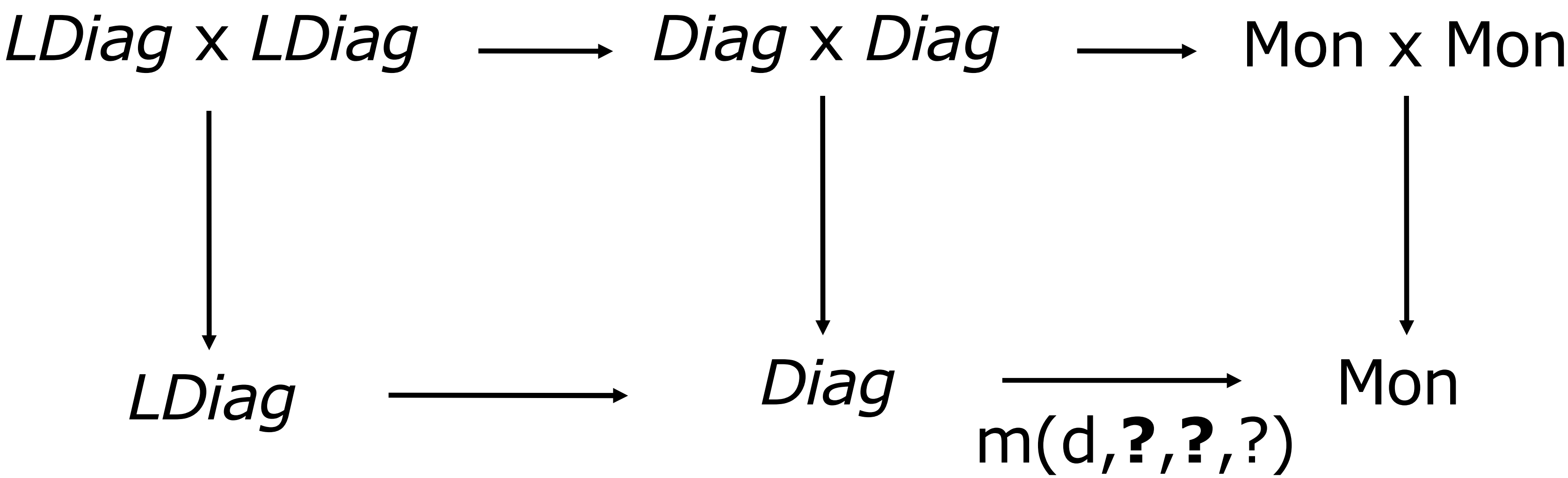
Second step: The product on *Ldiag* is just the concatenation of diagrams

$$d_1 \star d_2 = d_1\ d_2$$

And, setting $m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\mathbf{V}^{\beta(d)}\, z^{|d|}$
one gets

$$m(d_1 * d_2,\mathbf{L},\mathbf{V},z)= m(d_1,\mathbf{L},\mathbf{V},z)m(d_2,\mathbf{L},\mathbf{V},z)$$

This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which, in turn, is compatible with the product of monomials.

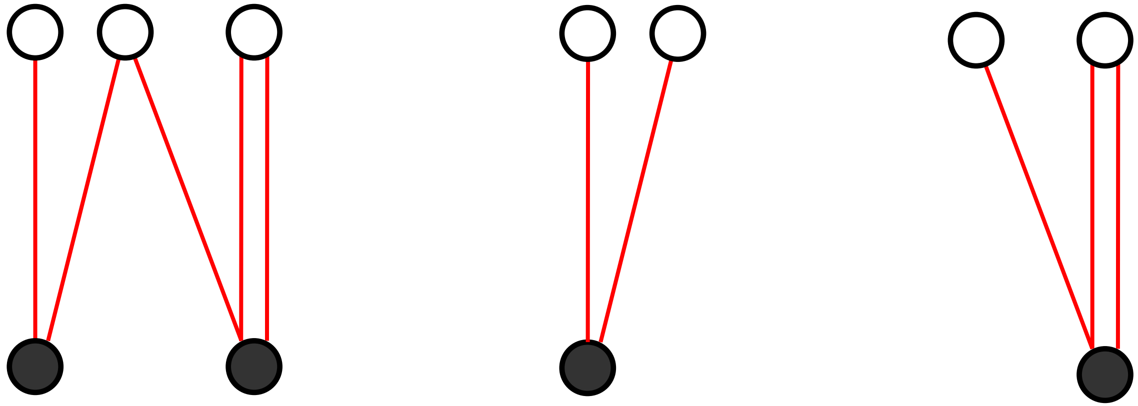


The coproduct needs to be compatible with m(d,**?**,**?**,?). One has two symmetric possibilities (black spots and white spots). The « black spots co-product » reads

$$\Delta_{BS}(d)=\sum d_I \otimes d_J$$

the sum being taken over all the decompositions, (I,J) of the Black Spots of d into two subsets.

For example, with the following diagrams d, d_1 and d_2



one has $\Delta_{BS}(d)=d\otimes\emptyset + \emptyset\otimes d + d_1\otimes d_2 + d_2\otimes d_1$

If we concentrate on the multiplicative structure of *Ldiag*, we remark that the objects are in one-to-one correspondence with the so-called packed matrices of NCSFVI (Hopf algebra MQSym), but the product of MQSym is given (w.r.t. a certain basis **MS**) according to the following example

$$\mathbf{MS}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{MS}_{[3\ 1]} =$$

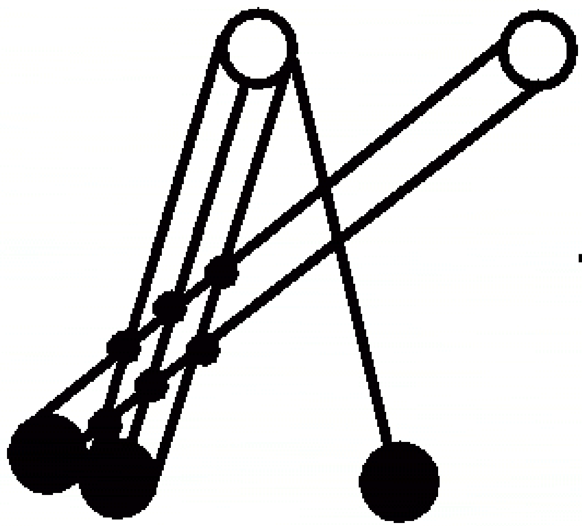
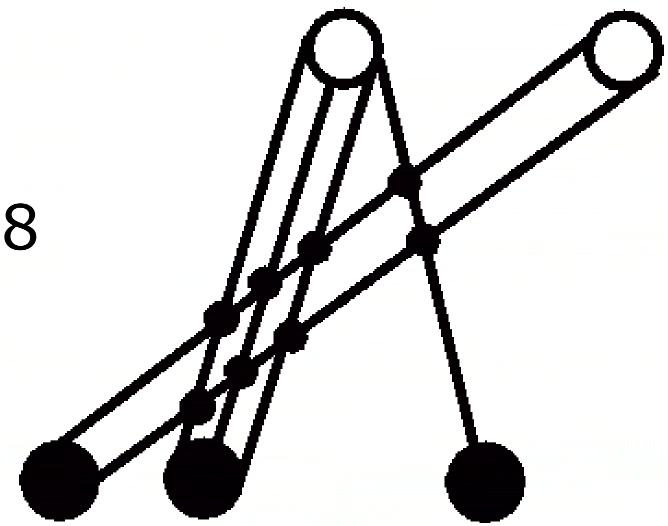
$$\mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

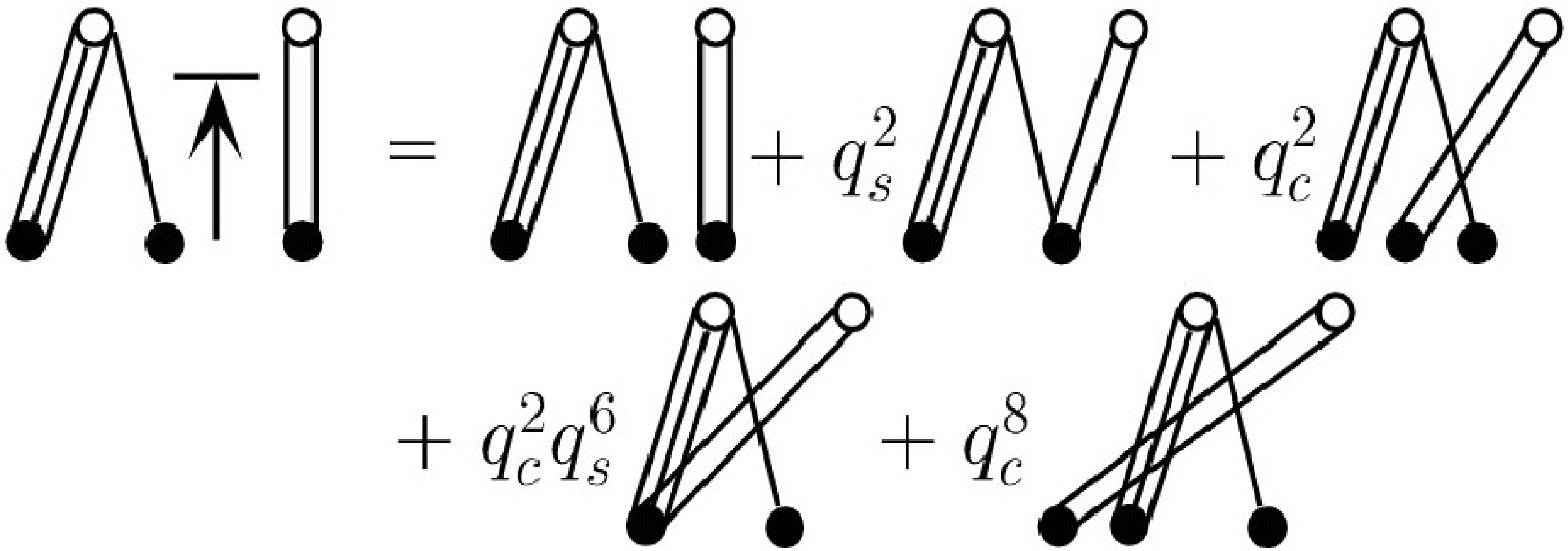
It is possible to (re)connect these Hopf algebras to MQSym and others of interest for physicists, by deforming the product with two parameters.
The double deformation goes as follows

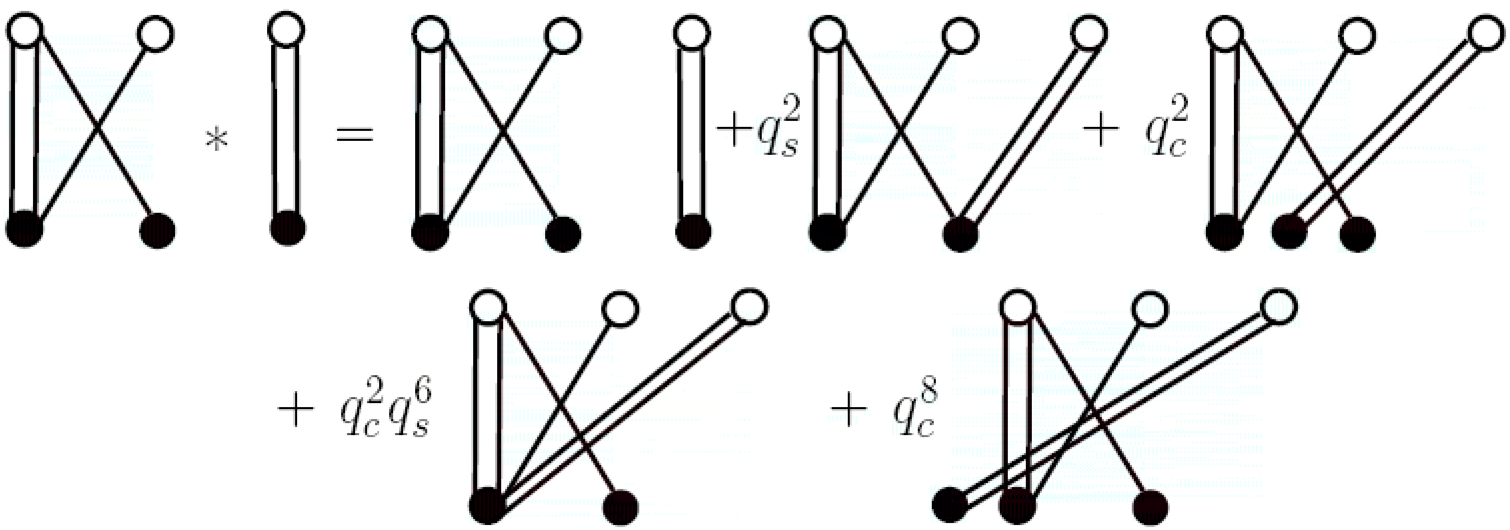
- Concatenate the diagrams

- Develop according to the rules :
  - Every crossing “pays” a q_c
  - Every node-stacking “pays” a q_s

In the expansion, the weights are given by the intersection numbers.

$q_c^{\,2} q_s^{\,6}$  $+q_c^{\,8}$ 


=
+ q_s^2
+ q_c^2
+ $q_c^2 q_s^6$
+ q_c^8


*
=
+q_s^2
+ q_c^2
+ $q_c^2 q_s^6$
+ q_c^8

We could check that this law is associative (now three independent proofs). For example, direct computation reads

$$(au \uparrow bv) \uparrow cw = \Big(a(u \uparrow bv) + q^{|u||b|}t^{|a||b|}\begin{bmatrix}b\\a\end{bmatrix}(u \uparrow v) + q^{|au||b|}b(au \uparrow v)\Big) \uparrow cw$$

$$\Big[a((u \uparrow bv) \uparrow cw) + q^{(|u|+|bv|)|c|}t^{|a||c|}\begin{bmatrix}c\\a\end{bmatrix}((u \uparrow bv) \uparrow w) + q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w)\Big]$$

$$\Big[q^{|u||b|}t^{|a||b|}\begin{bmatrix}b\\a\end{bmatrix}(u \uparrow v \uparrow cw) + q^{|u||b|+(|u|+|v|)|c|}t^{|a||b|}t^{(|a|+|b|)|c|}\begin{bmatrix}c\\b\\a\end{bmatrix}(u \uparrow v \uparrow w$$

$$q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix}b\\a\end{bmatrix}(u \uparrow v)) \uparrow w)\Big]$$

$$\Big[q^{|au||b|}b((au \uparrow v) \uparrow cw) + q^{|au||b|+(|au|+|v|)|c|}t^{|b||c|}\begin{bmatrix}c\\b\end{bmatrix}(au \uparrow v \uparrow w) + q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \left(b(v \uparrow cw) + q^{|v||c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|}c(bv \uparrow w)\right) =$$

$$\left[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|}b(au \uparrow v \uparrow cw)\right] +$$

$$\left[q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) + \right.$$

$$\left. q^{|v||c|+|au|(|b|+|c|)}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\right] +$$

$$\left[q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w)\right] \quad (3)$$

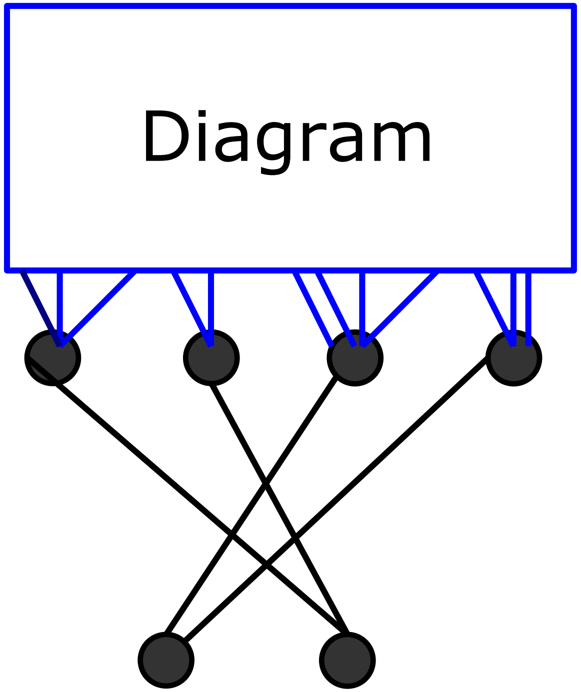
dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

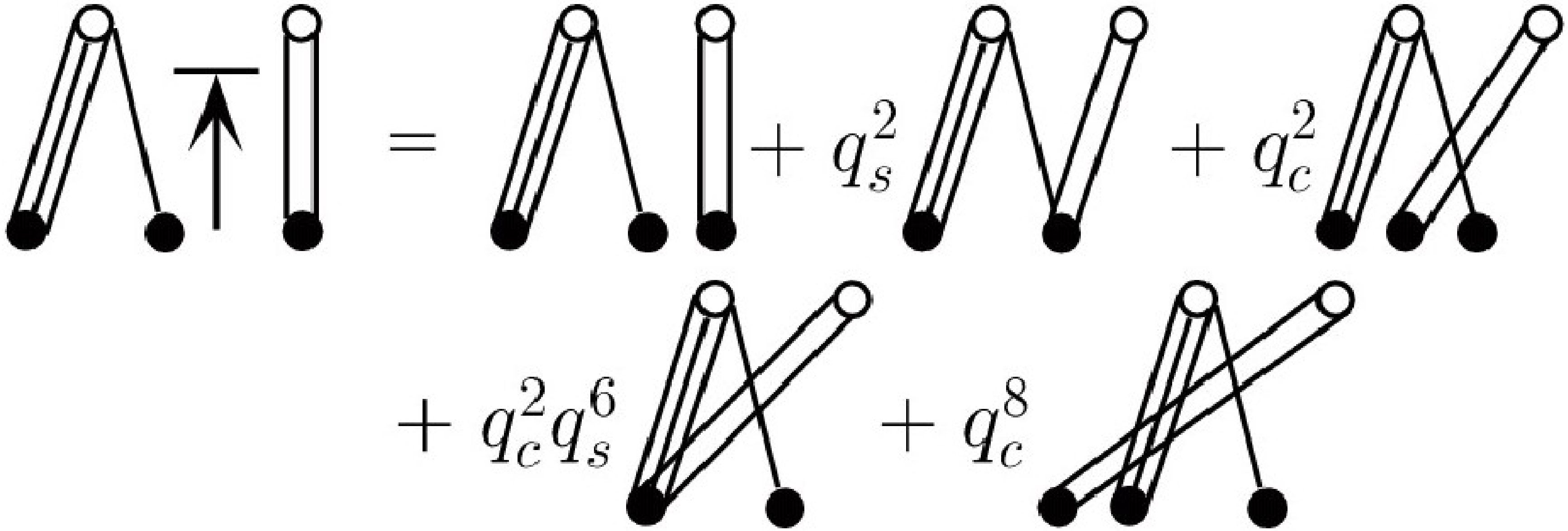
$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$
$$q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w) \quad (5)$$

This amounts to use a monoidal action with two parameters. Associativity provides an identity in an algebra which acts on a diagram as the algebra of the sum of symmetric semigroups. Here, it is the symmetric semigroup which acts on the black spots





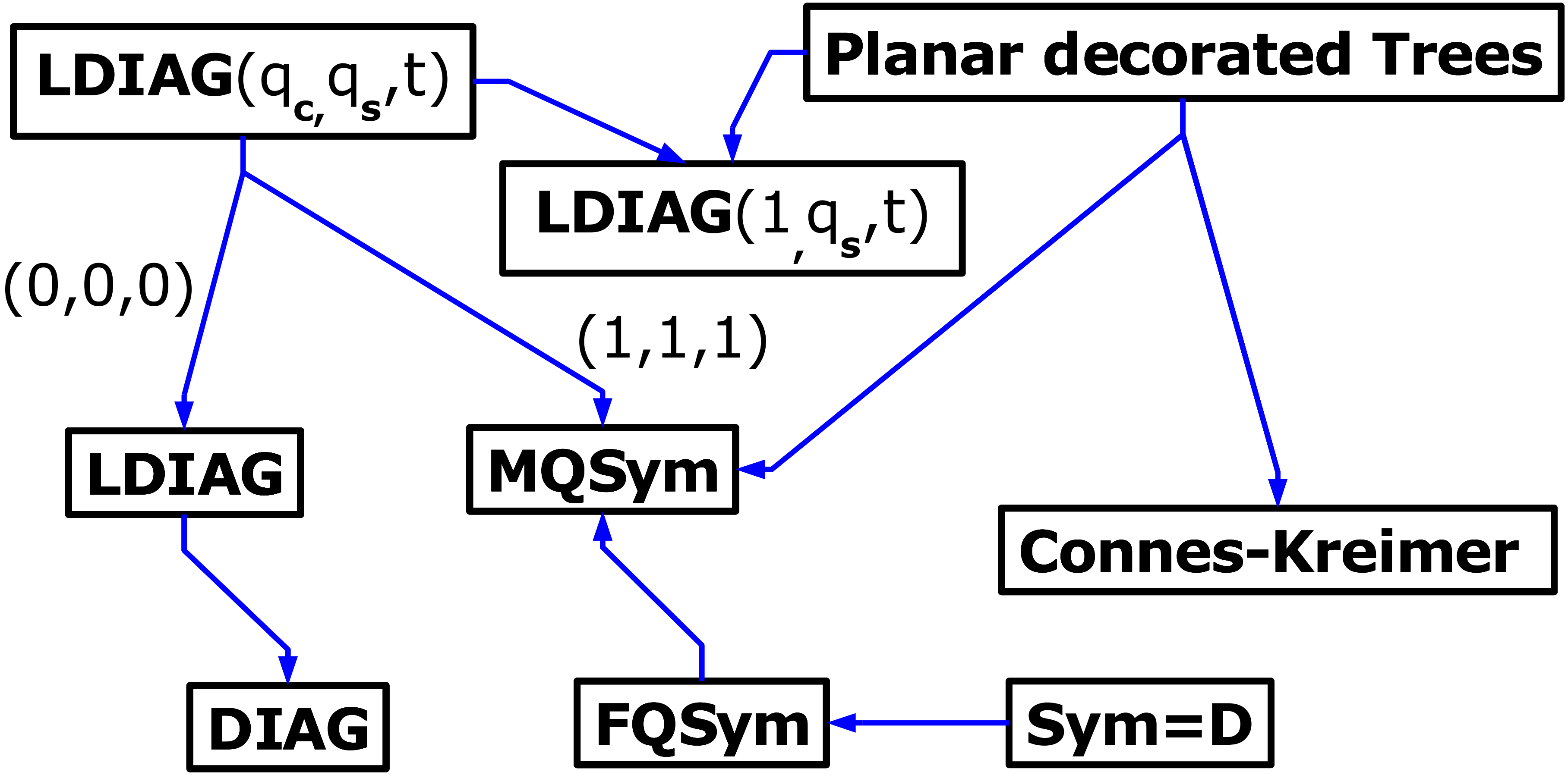
The labelled diagrams are in one to one correspondence with the packed matrices of MQSym and we can see easily that the product of the latter is obtained for

$$q_c = 1 = q_s$$

Hopf interpolation : One can see that the more intertwined the diagrams are the fewer connected components they have. This is the main argument to prove that LDIAG($q_{c,}$ q_s) is free on indecomposable diagrams. Therefore one can define a coproduct on these generators by

$$\Delta_t=(1-t)\Delta_{BS}+t\ \Delta_{MQSym}$$

this is LDIAG($q_{c,}$ q_s,t).



Notes :

i) The arrow *Planar Dec. Trees* → *LDIAG(1,q_s,t)* is due to L. Foissy

ii) **LDIAG**(q_c, q_s, t), through a noncommutative alphabetic realization shows to be a bidendriform algebra (FPSAC07 paper by ParisXIII & Monge).

(A part of) The legacy of Schützenberger or how to compute efficiently in Sweedler's duals using Automata Theory

Sweedler's dual of a Hopf algebra

i) Multiplication

$$\mathcal{A}\otimes\mathcal{A} \xrightarrow{\mu} \mathcal{A}$$

ii) By dualization one gets

$$(\mathcal{A})^* \xrightarrow{{}^t\mu} (\mathcal{A}\otimes\mathcal{A})^*$$

but not a "stable calculus" as

$$(\mathcal{A})^* \otimes (\mathcal{A})^* \subseteq (\mathcal{A}\otimes\mathcal{A})^*$$

(strict in general). We ask for elements $x \in \mathcal{A}$ such that

$${}^t\mu(x) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$$

These elements are easily characterized as the "representative linear forms" (see also the Group-Theoretical formulation in the last talk of Pierre Cartier)

Proposition : TFAE (the notations being as above)

i) ${}^t\mu(c) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$

ii) There are functions f_i, g_i i=1,2..n such that

$$c(xy)=\sum_{i=1}^{n} f_i(x)\, g_i(y)$$

forall x,y in $\mathcal{A}$.

iii) There is a morphism of algebras $\mu: \mathcal{A} \dashrightarrow k^{n \times n}$ (square matrices of size n x n), a line λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all z in $\mathcal{A}$,

$$c(z)=\lambda\mu(z)\xi$$

In many “Combinatorial” cases, we are concerned with the case $\mathcal{A}$= k<A> (non-commutative polynomials with coefficients in a field k).

Indeed, one has the following theorem (the beginning can be found in [ABE : Hopf algebras]) and the end is one of the starting points of Schützenberger's school of automata and language theory.

Theorem A: TFAE (the notations being as above)

i) ${}^{t}\mu(c) \in (\mathcal{A})^{*} \otimes (\mathcal{A})^{*}$

ii) There are functions f_i ,g_i i=1,2..n such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in A^* (the free monoid of alphabet A).

iii) There is a morphism of monoids μ: A^* --> $k^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If A is finite) c lies in the rational closure of A within the algebra k<<A>>.

We can safely apply the first three conditions of **Theorem A** to *Ldiag.* The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more “basic” functions. The modification reads

iv) (A is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in A} p(a)\, a$$

within the algebra k<<A>>.

iii) Schützenberger's theorem (known as the theorem of Kleene-Schützenberger) could be rephrased in saying that functions in a Sweedler's dual are behaviours of finite (state and alphabet) automata.

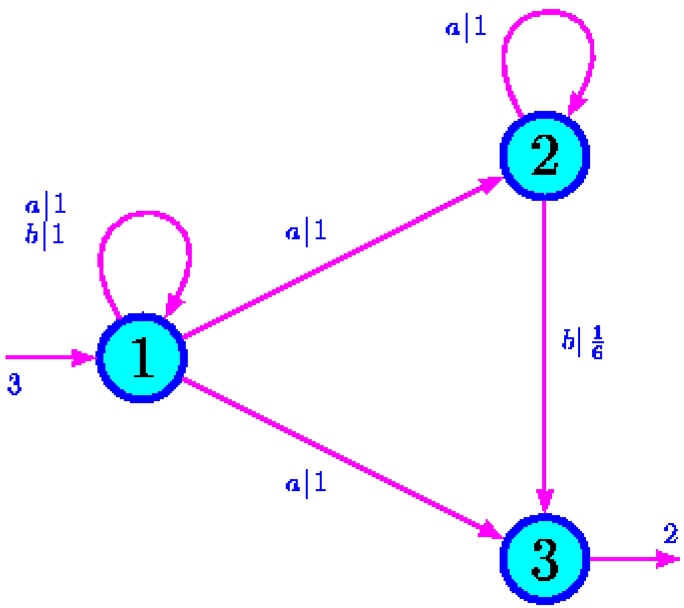
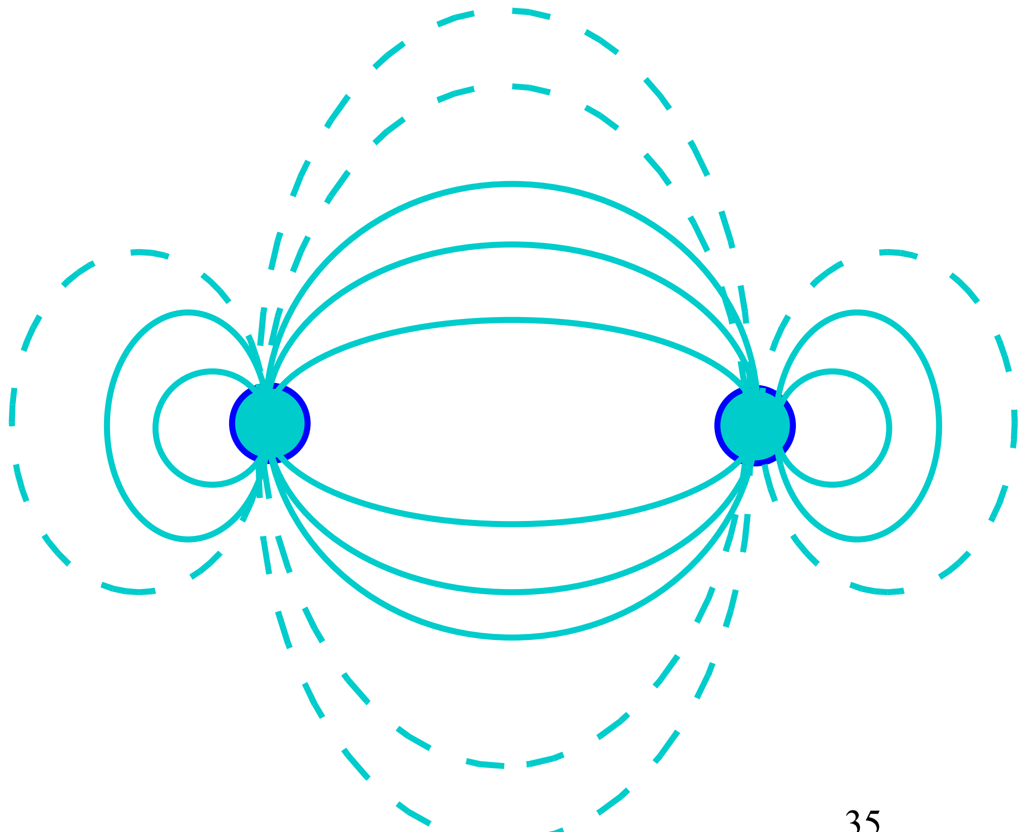


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6+a^*b).$$

In our case, we are obliged to allow infinitely many edges.



Computations in $K^{rat}<<A>>$, Sweedler's dual of $K<A>$

Summability : We say that a family $(f_i)_{i\in I}$ (I finite or not, f_i in $K^{rat}<<A>>$) is summable if, for each $w\in A^*$, the family $(<f_i|w>)_{i\in I}$ is finitely supported and we set

$$(\sum_{i\in I} f_i) : w \to (\sum_{i\in I} <f_i|w>)$$

Identifying each word with the Dirac linear form located at the word, one has then, for each $f\in K<<A>>$

$$f= \sum_{w\in A^*} f(w)w$$

If $f \in K^{rat}\langle\langle A \rangle\rangle$, it exists a morphism of monoids $\mu: A^* \to K^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*, $f(w)=\lambda\mu(w)\xi$. Then

$$f = \sum_{w \in A^*} f(w)w = \sum_{w \in A^*} \lambda\mu(w)\xi\, w = \lambda\Big(\sum_{w \in A^*} \mu(w)w\Big)\xi =$$

$$\lambda\Big(\sum_{w \in A^*} \mu(w)w\Big)\xi = \lambda\Big(\sum_{n \geq 0} \sum_{|w|=n} \mu(w)w\Big)\xi$$

But, as words and scalars commute (it is so by construction of the convolution algebra $K^{n \times n}\langle\langle A \rangle\rangle$), one has

$$\sum_{n \geq 0} \sum_{|w|=n} \mu(w)w = \sum_{n \geq 0} \Big(\sum_{a \in A} \mu(a)a\Big)^n = \Big(\sum_{a \in A} \mu(a)a\Big)^*$$

hence

$$f = \lambda\Big(\sum_{a \in A} \mu(a)a\Big)^*\xi$$

where the "star" stands for the sum of the geometric series.

If Q is a finite set, the space $k^{Q\times Q}$ of square matrices with indices in Q and coefficients in k has a natural semiring structure with the usual operations (sum and product). A (right) star of $M \in k^{Q\times Q}$ (when it exists) is a solution of the equation $MY + 1_{Q\times Q} = Y$ (where $1_{Q\times Q}$ is the identity matrix). Let $M \in k^{Q\times Q}$ be given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} \in k^{Q_1\times Q_1}$, $a_{12} \in k^{Q_1\times Q_2}$, $a_{21} \in k^{Q_2\times Q_1}$ and $a_{22} \in k^{Q_2\times Q_2}$ such that $Q_1 + Q_2 = Q$. Let $N \in k^{Q\times Q}$ given by

$$N = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = (a_{11} + a_{12}a_{22}{}^{*}a_{21})^{*} \qquad (1)$$

$$A_{12} = a_{11}{}^{*}a_{12}A_{22} \qquad (2)$$

$$A_{21} = a_{22}{}^{*}a_{21}A_{11} \qquad (3)$$

$$A_{22} = (a_{22} + a_{21}a_{11}{}^{*}a_{12})^{*} \qquad (4)$$

A (short) word on automata theory.

- The formulas (for the star* of a matrix) above are sufficiently "expressive" to be the crucial fact in the resolution of a conjecture in Noncommutative Geometry.

- For applications, automata theory had to cope with spaces of coefficients much more general than that of a field ... even the "minus" operation of the rings had to disappear to be able to cope with problems like shortest path or the Noncommutative problem or the shortest path with list of minimal arcs .

The emerging structure is that of a semiring. Think of a ring without the "minus" operation, nevertheless "transfer" matrix computations can be performed.

The input alphabet being set by the automaton under consideration, we will here rather focus on the definition of semirings providing transition coefficients. For convenience, we first begin with various laws on $\mathbb{R}_+ := [0, +\infty[$ including

1. $+$ (ordinary sum)
2. $\times$ (ordinary product)
3. min (if over $[0, 1]$, with neutral 1, otherwise must be extended to $[0, +\infty]$ and then, with neutral $+\infty$) or max
4. $+_a$ defined by $x +_a y := log_a(a^x + a^y)$ $(a > 0)$
5. $+_{[n]}$ (Hölder laws) defined by $x +_{[n]} y := \sqrt[n]{x^n + y^n}$
6. $+^s$ (shifted sum, $x +^c y := x + y - 1$, over whole $\mathbb{R}$, with neutral 1)
7. $\times^c$ (complemented product, $x + y - xy$, can be extended also to whole $\mathbb{R}$, stabilizes the range of probabilities or fuzzy $[0, 1]$ and is distributive over the shifted sum)

As (useful) examples, one has ([0,+∞], min, +), ([0,+∞[, max, +) or its (commutative or not) variants.

What remains for K<A> ? (free algebra)

- K semiring :

  - Universal properties (comprising – little known - tensor products)

  - Complete semiring K<<A>>, summability is defined by pointwise convergence (see computation above).

  - Rational closures and Kleene-Schützenberger Thm

  - Rational expressions, Brzozowski theorem

  - Automata theory, theory of codes

  - Lazard's monoidal elimination

Concluding remarks and future

i) The diagrams of diag are well suited to EGFs. What are the good data structures for other ones ?

ii) One can change the constants $V_k=1$ to a condition with level (i.e. $V_k=1$ for $k \leq N$ and $V_k= 0$ for $k>N$). We obtain then sub-Hopf algebras of the one constructed above. These can apply to the manipulation of partition functions of physical models including Free Boson Gas, Kerr model and Superfluidity.

Concluding remarks and future (cont'd)

iii) The deformation above is likely to be decomposed in two deformation processes ; twisting (already investigated in NCSFIII) and shifting (ongoing work with JGL and al.). Also, it could have a connection with other well known associators.

iv) The identity on the symmetric semigroup can be lifted to a more general monoid which takes into account the operations of concatenation and stacking which are so familiar to Computer Scientists (ongoing work in LIPN).

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Thank You

A single exponential

The normal ordering problem goes as follows.

- Weyl (two-dimensional) algebra defined as

$$< a^+, a \; ; [a \, , a^+]=1 > \;|\; aa^+ \;{-}{-}{-}{>}\; a^+a +1$$

- Known to have no (faithful) representation by bounded operators in a Banach space.

There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

$$a \;{-}{-}{-}{>}\; d/dx \; ; \; a+ \;{-}{-}{-}{>}\; x$$

where a has degree -1 and a^+ has degree 1.

A typical element in the Weyl algebra is of the form (normal form).

$$\Omega = \sum_{k,l\geq 0} c(k,l)(a^{+})^{k} a^{l}$$

When Ω is a single monomial, a word i.e. a product of generators a^{+}, a, there is solution to the normal ordering problem (and thus, by linearity to the general problem) using rook numbers.

Today, we will be interested with the use of matrix coefficients in two instances :

normal ordering --> infinite matrices (--> moments)

finite representations --> Sweedler's dual and automata

A word (boson string) and more generally an homogeneous operator (for the grading where a has degree -1 and a^+ has degree 1) of degree e reads

$$\Omega = \sum_{\substack{k,l \geq 0 \\ k-l=e}} c(k,l)(a^+)^k a^l$$

Due to the symmetry of the Weyl algebra, we can suppose, with no loss of generality that $e \geq 0$. For homogeneous operators one can define generalized Stirling numbers (GSN) by

$$\Omega^n = (a^+)^{ne} \sum_{k \geq 0} S_\Omega(n,k)(a^+)^k a^k \quad \text{(Eq1)}$$

The case of a pure string is of special interest for physics and can be solved combinatorally. The recipe, for a string W is the following:

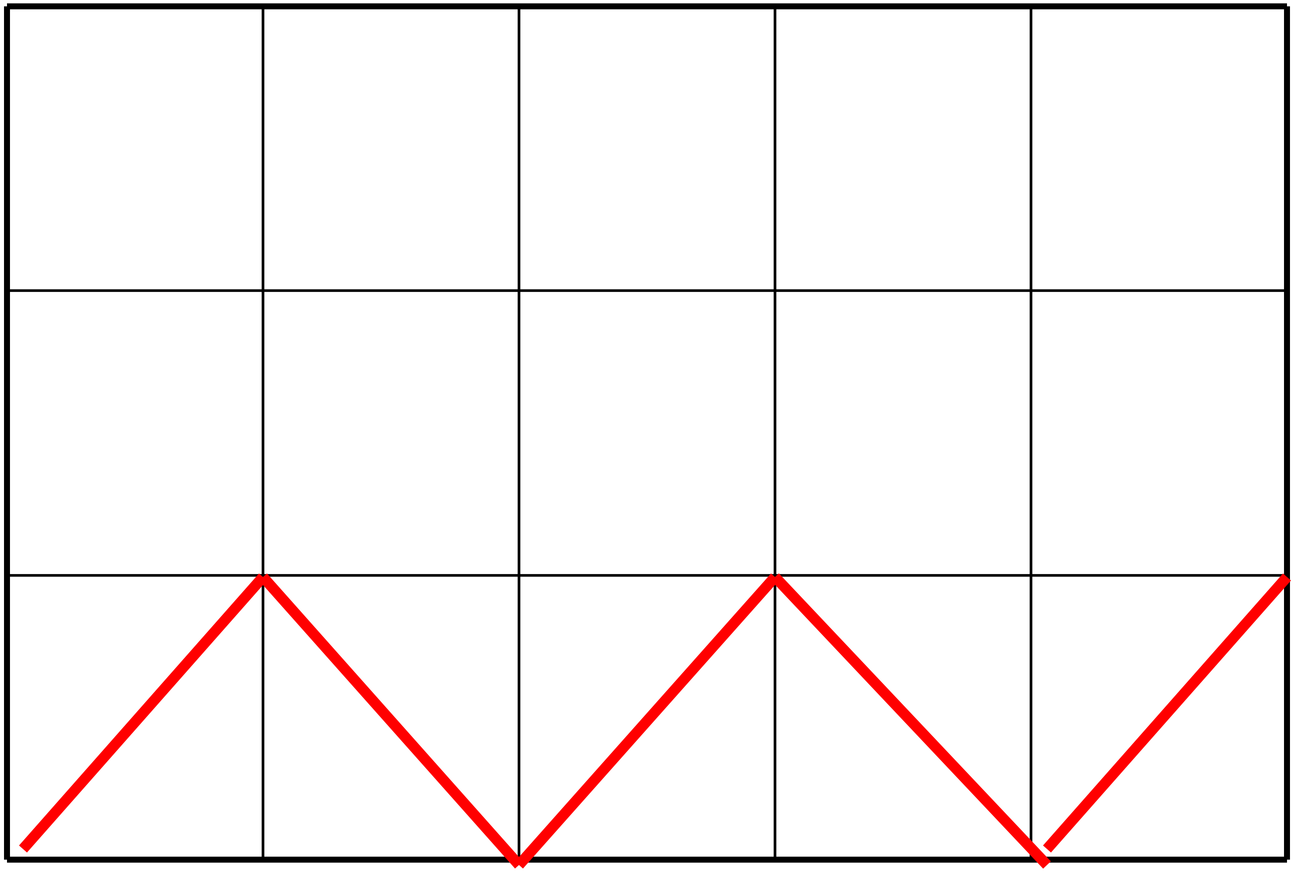
- associate a path with north east steps for every a^+ and a south east step for every a.
- construct the Ferrers diagram B over this path

The normal form of W is

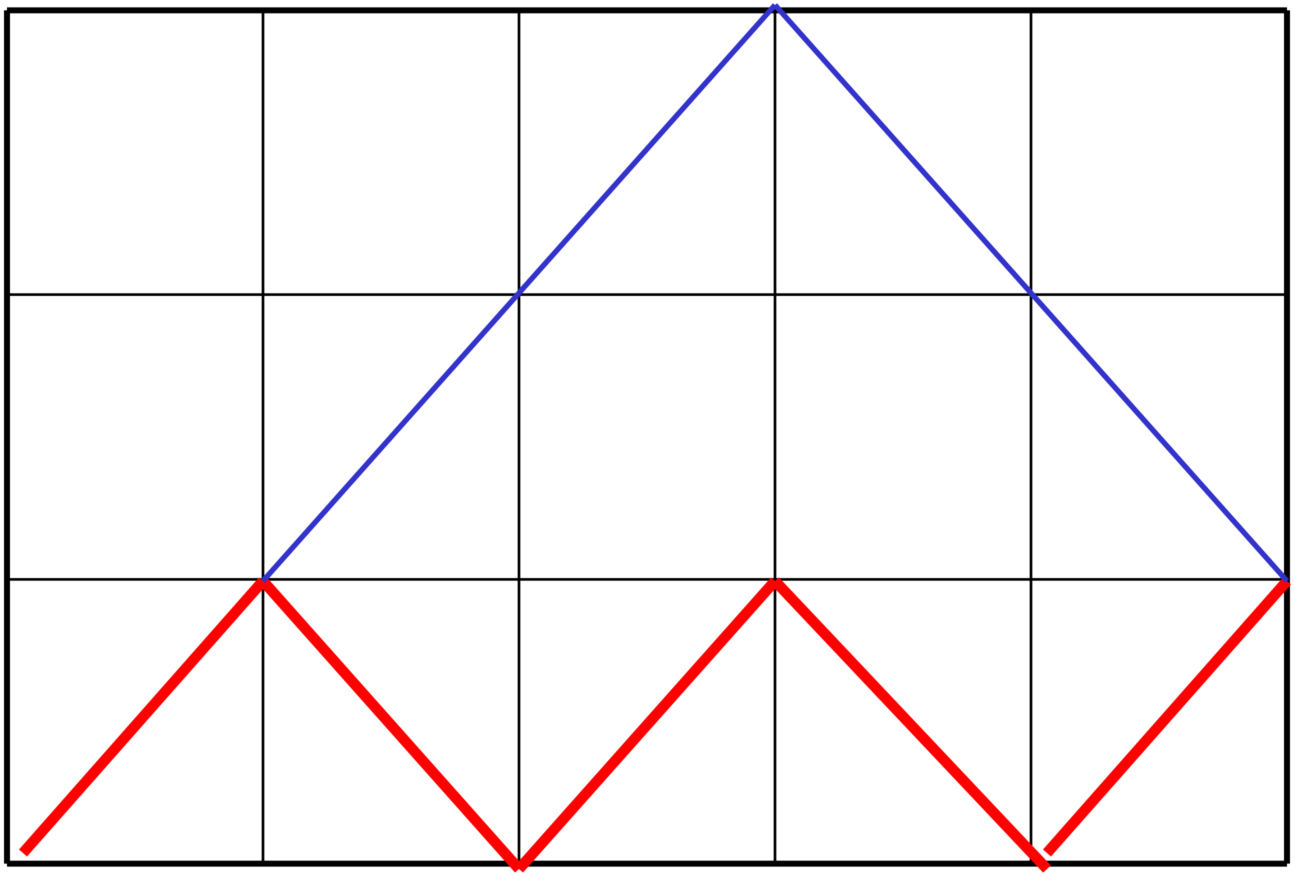
$$W = \sum_{k \geq 0} R(B,k)(a^{+})^{r-k} a^{s-k}$$

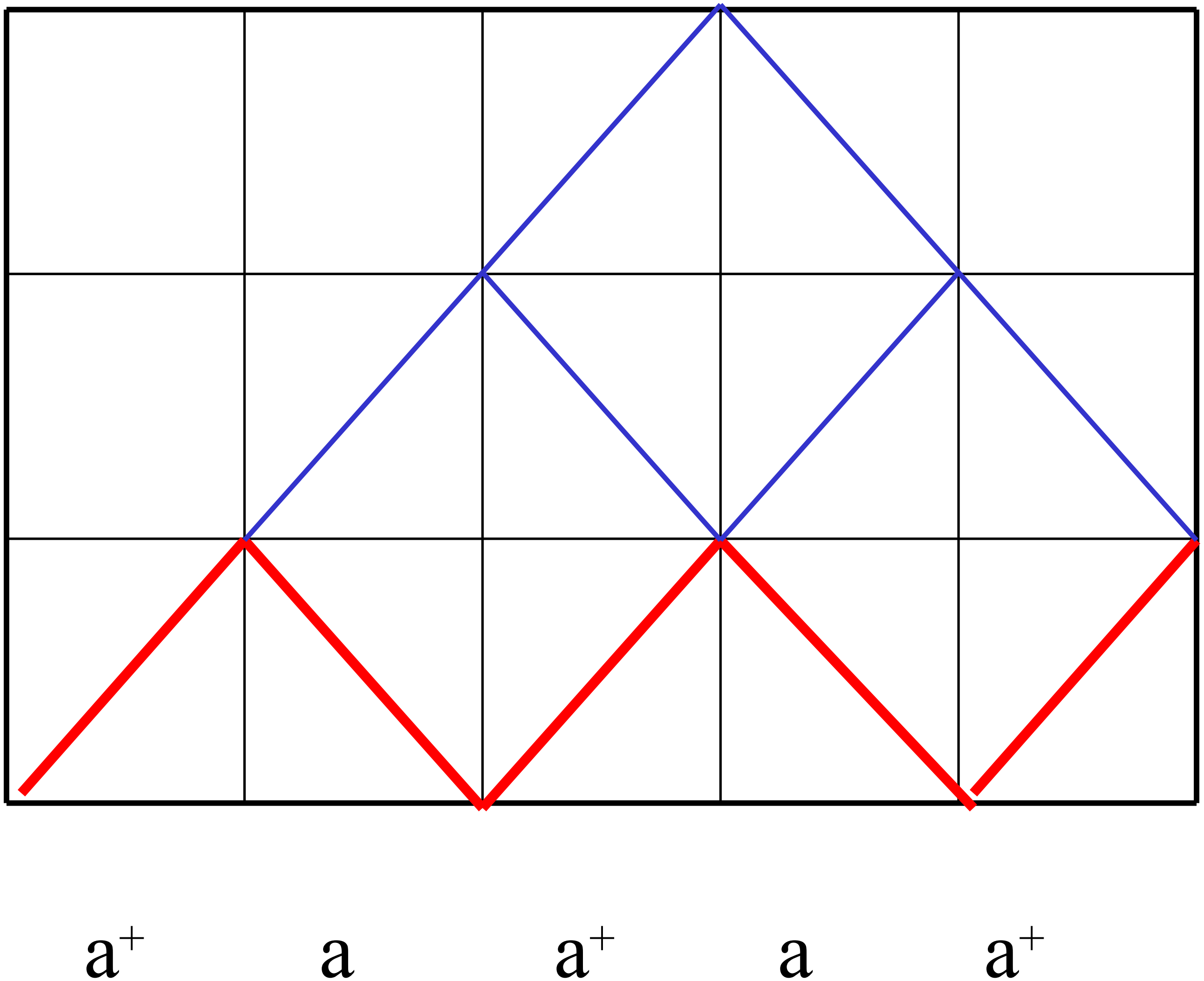
where R(B,k) is the k-th rook number of the board B.

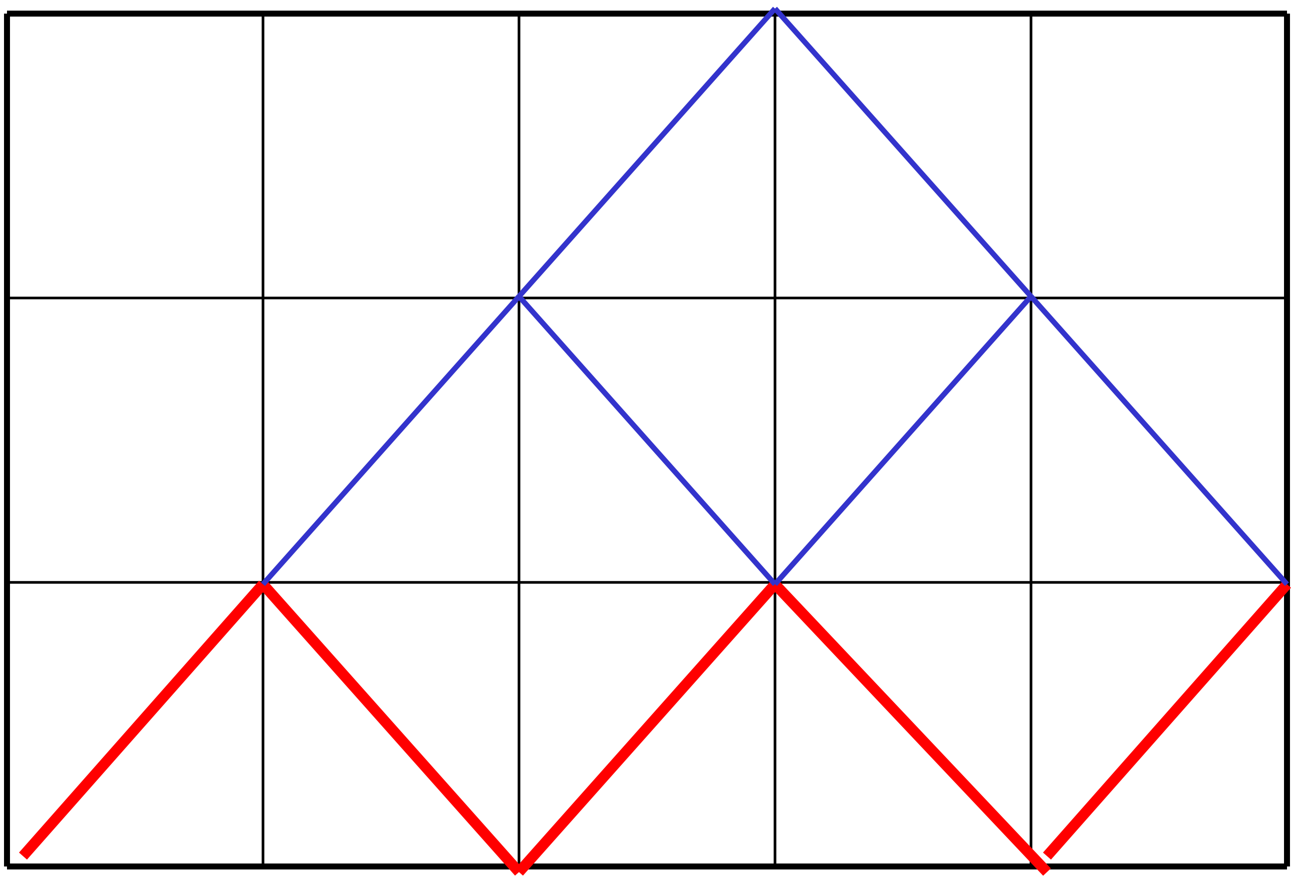
Example with $\Omega = a^{+}\, a\; a^{+}\, a\, a^{+}$



a^{+} a a^{+} a a^{+}


a^+ a a^+ a a^+


a^+ a a^+ a a^+



a^+ a a^+ a a^+

$$a^+aa^+aa^+ = 1\ a^+a^+a^+aa + 3\ a^+a^+a + 1\ a^+$$

In particular, the boson string $w=[(a^+)^r a^s]$ was considered by Penson, Solomon, Blasiak and al. In this case the GSN will be denoted by $S_{r,s}(n,k)$ and, due to the particular form of W, one has

$$\left[(a^+)^r a^s\right]^n = (a^+)^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k)(a^+)^k a^k$$

For $w = a^+ a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3)$$

For $w = a^+ a a^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+ a a a^+ a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

Setting

$$B_{r,s}(n,y) = \sum_{k=0}^{\infty} S_{r,s}(n,k) y^k$$

for the generating polynomials of the lines of the generalized Stirling matrix, one has the formulas

$$B_{r,s}(n,y) = \sum_{k=s}^{ns} S_{r,s}(n,k) y^k$$

$$= e^{-y} \sum_{k=s}^{\infty} \frac{1}{k!} \prod_{j=1}^{n} [(k+(j-1)(r-s))(k+(j-1)(r-s)-1)$$

$$\cdots (k+(j-1)(r-s)-s+1)] y^k.$$

... and, when s=1, the EGF of these polynomials is an exponential which gives an additive formula in the variable y (see the paper One-parameter Groups or below)

$$e^{y(e^x-1)} = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{1,1}(n,k)y^k\right)\frac{x^n}{n!}$$

and

$$\exp\left[y\left(\frac{1}{\sqrt[r-1]{1-(r-1)x^{r-1}}}-1\right)\right] = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n} S_{r,1}(n,k)y^k\right)\frac{x^n}{n!} \qquad r = 2, 3, \ldots.$$

For which, we have Dobi'nski-type relations

$$\frac{1}{e^y}\sum_{k=1}^{\infty}\frac{k^n}{k!}y^k = \sum_{k=1}^{n} S_{1,1}(n,k)y^k \qquad n = 0, 1, \ldots$$

$$\frac{(r-1)^n}{e^y}\sum_{k=1}^{\infty}\frac{\Gamma\left(n+\frac{k}{r-1}\right)}{k!\,\Gamma\left(\frac{k}{r-1}\right)}y^k = \sum_{k=1}^{n} S_{r,1}(n,k)y^k \qquad n = 1, 2, \ldots.$$

The matrices of coefficients for expressions with only a single « a » turn to be matrices of substitutions with prefunction factor. This is, in fact, due to a conjugacy phenomenon.

Conjugacy trick: The one-parameter groups associated with the operators of type $\Omega = q(x)d/dx + v(x)$ are conjugate to vector fields on the line.

Let $u_2 = \exp(\int(v/q))$ and $u_1 = q/u_2$ then

$u_1 u_2 = q$; $u_1 u'_2 = v$ and the operator $q(a^+)a + v(a^+)$

reads, via the Bargmann-Fock correspondence

$(u_2 u_1)d/dx + u_1 u'_2 = u_1(u'_2 + u_2 d/dx) = u_1 d/dx\, u_2 =$

$$1/u_2\,(u_1\, u_2\, d/dx)\, u_2$$

Which is conjugate to a vector field and integrates as a substitution with prefunction factor.

Example: The expression $\Omega = a^{+2}a\, a^{+} + a^{+}a\, a^{+2}$ above corresponds to the operator (the line below ω is in form q(x)d/dx+v(x))

$$\omega = x^2 \frac{d}{dx} x + x \frac{d}{dx} x^2 =$$

$$2x^3 \frac{d}{dx} + 3x^2 = x^{-3/2}(2x^3 \frac{d}{dx})x^{3/2} = x^{-3/2}(\phi)x^{3/2}$$

Now, ϕ is a vector field and its one-parameter group acts by a one parameter group of substitutions. We can compute the action by another conjugacy trick which amounts to straightening ϕ to a constant field.

Thus set

$$\exp(\lambda\,\phi)[f(x)] = f(u^{-1}(u(x)+\lambda))$$ for some u …

By differentiation w.r.t. λ at ($\lambda=0$) one gets

$$u' = 1/(2x^3)\ ;\ u = -1/(4x^2)\ ;\ u^{-1}(y) = (-4y)^{-1/2}$$

Maple 9 - [Untitled (1) - [Server 1]]

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One parameter group by f(v(u(x)+λ)); v is reciprocal of u

```
> T1(lambda,x):=(-4*(-1/(4*x^2)+lambda))^(-1/2);
```

$$T1(\lambda, x) := \frac{1}{\sqrt{\frac{1}{x^2} - 4\lambda}}$$

We suppose x>0

```
> T1:=(lambda,x) -> x/((1-4*lambda*x^2)^(1/2));
```

$$T1 := (\lambda, x) \rightarrow \frac{x}{\sqrt{1 - 4\lambda x^2}}$$

Checking the tangent vector

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))^2-T1(lambda1+lambda2,x)^2);
```

$$0$$

Time: 0.2s Bytes: 1.94M Available: 1.33G

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In view of the conjugacy established previously we have that exp(λ ω)[f(x)] acts as

$$U_\lambda(f) = x^{-\frac{3}{2}} f(s_\lambda(x)).(s_\lambda(x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

which explains the prefactor. Again we can check by computation that the composition ($U_\mu U_\lambda$) amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of Eq1 (slide 24) we must have

$$\exp(\lambda\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k \left(\frac{d}{dx}\right)^k$$

Hence, introducing the eigenfunctions of the derivative (a method which is equivalent to the computation with coherent states) one can recover the mixed generating series of $S_{\omega}(n,k)$ from the knowledge of the

one-parameter group of transformations.

$$\exp(\lambda\omega)\left[e^{yx}\right] = \left(\sum_{n\geq 0}\frac{\lambda^{n}}{n!}x^{ne}\sum_{k=0}^{ne}S_{\omega}(n,k)x^{k}y^{k}\right)e^{yx}$$

Thus, one can state

Proposition (*): With the definitions introduced, the following conditions are equivalent (where f → $U_\lambda[f]$ is the one-parameter group exp(λω)).

$$1.\ \sum_{n,k\geq 0} S_\omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)}$$

$$2.\ U_\lambda[f](x) = g(\lambda x^e)f(x(1+\phi(\lambda x^e)))$$

Remark : Condition 1 is known as saying that $S_\omega(n,k)$ is of « Sheffer » type.

Example : With $\Omega = a^{+2} a\, a^{+} + a^{+} a\, a^{+2}$ (Slide 11), we had e=2 and

$$U_\lambda[f](x) = \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt[2]{\frac{x^2}{1-4\lambda x^2}}\right)$$

Then, applying the preceding correspondence one gets

$$\sum_{n,k\geq 0} S_\omega(n,k) \frac{x^n}{n!} y^k = \sqrt[4]{\frac{1}{(1-4x)^3}}\, \mathrm{e}^{y\left(\sqrt{\frac{1}{1-4x}}-1\right)} =$$

$$\sqrt[4]{\frac{1}{(1-4x)^3}}\, \mathrm{e}^{y\left(\sum_{n\geq 1} c_n x^n\right)}$$

Where $c_n = \binom{2n}{n}$ are the central binomial coefficients.

```
> E1:=(1/((1-4*x)^3))^(1/4)*exp(y*(1/(1-4*x)^(1/2)-1));
```

$$E1 := \left(\frac{1}{(1-4\,x)^3}\right)^{(1/4)} \mathbf{e}^{\left(y\left(\frac{1}{\sqrt{1-4\,x}}-1\right)\right)}$$

```
> T1:=taylor(E1,x=0,6);
```

$$T1 := 1 + (2\,y+3)\,x + \left(12\,y + 2\,y^2 + \frac{21}{2}\right)x^2 + \left(59\,y + 18\,y^2 + \frac{4}{3}y^3 + \frac{77}{2}\right)x^3 + \left(270\,y + 115\,y^2 + 16\,y^3 + \frac{2}{3}y^4 + \frac{1155}{8}\right)x^4 + \left(\frac{4389}{8} + \frac{4767}{4}y + 637\,y^2 + 126\,y^3 + 10\,y^4 + \frac{4}{15}y^5\right)x^5 + \mathrm{O}(x^6)$$

```
> seq([sort(coeff(T1,x,n)*n!)],n=1..5);
```

$$[2\,y+3],\ [4\,y^2+24\,y+21],\ [8\,y^3+108\,y^2+354\,y+231],\ [16\,y^4+384\,y^3+2760\,y^2+6480\,y+3465],\ [32\,y^5+1200\,y^4+15120\,y^3+76440\,y^2+143010\,y+65835]$$

```
> M1:=matrix(5,5,(n,k)->coeff(coeff(T1,x,n)*n!,y,k));
```

$$M1 := \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 24 & 4 & 0 & 0 & 0 \\ 354 & 108 & 8 & 0 & 0 \\ 6480 & 2760 & 384 & 16 & 0 \\ 143010 & 76440 & 15120 & 1200 & 32 \end{bmatrix}$$

Autre exemple : transformation idempotente.

I(n,k)=nombre d'endofonctions de [1..n] idempotentes avec k points fixes.

$$\sum_{n,k} I(n,k)\frac{x^n}{n!}y^k = \sum_{n\geq 0}\sum_{k\leq n}\binom{n}{k}k^{(n-k)}\frac{x^n}{n!}y^k =$$

$$\sum_{k\leq n}\sum_{n\geq k}\binom{n}{k}k^{(n-k)}\frac{x^{(n-k)}x^k}{n!}y^k = e^{yxe^x}$$

ceci est un cas particulier de la « formule exponentielle »

Maple 9 - [calc1.mws - [Server 1]]

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```
> g1:=exp(y*x*exp(x));d1:=taylor(g1,x=0,7);
```

$$g1 := \mathbf{e}^{(y\,x\,\mathbf{e}^{x})}$$

$$d1 := 1 + y\,x + \left(y + \frac{1}{2}y^2\right)x^2 + \left(\frac{1}{2}y + y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{1}{6}y + y^2 + \frac{1}{2}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{24}y + \frac{2}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{6}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{1}{120}y + \frac{1}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{3}y^4 + \frac{1}{24}y^5 + \frac{1}{720}y^6\right)x^6 + \mathrm{O}(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 \end{bmatrix}$$

Substitutions and the « connected graph theorem» « exponential formula»

A great, powerful and celebrated result:
(For certain classes of graphs)
If $C(x)$ is the EGF of CONNECTED graphs, then *exp(C(x))* is the EGF of ALL (non void) graphs.
(Touchard, Uhlenbeck, Mayer,…)

This implies that the matrix

M(n,k)=number of graphs with n vertices and having k connected components

is the matrix of a substitution (like $S_\Omega(n,k)$ previously but without prefactor).

Maple 9 - [Untitled (1) - [Server 1]]

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```
> f1:=exp(y*(x+x^2/2)*exp(x));d1:=taylor(f1,x=0,7);
```

$$f1 := e^{(y\,(x+1/2\,x^2)\,e^x)}$$

$$d1 := 1 + y\,x + \left(\frac{3}{2}y + \frac{1}{2}y^2\right)x^2 + \left(y + \frac{3}{2}y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{5}{12}y + \frac{17}{8}y^2 + \frac{3}{4}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{8}y + \frac{23}{12}y^2 + \frac{13}{8}y^3 + \frac{1}{4}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{7}{240}y + \frac{5}{4}y^2 + \frac{109}{48}y^3 + \frac{35}{48}y^4 + \frac{1}{16}y^5 + \frac{1}{720}y^6\right)x^6 + O(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 9 & 1 & 0 & 0 & 0 \\ 0 & 10 & 51 & 18 & 1 & 0 & 0 \\ 0 & 15 & 230 & 195 & 30 & 1 & 0 \\ 0 & 21 & 900 & 1635 & 525 & 45 & 1 \end{bmatrix}$$

One can prove, using a Zariski-like argument, that, if *M* is such a matrix (with identity diagonal) then, all its powers (positive, negative and fractional) are substitution matrices and form a one-parameter group of substitutions, thus coming from a vector field on the line which could (in theory) be computed.

For example, to begin with the Stirling substitution $z \to e^z - 1$. We know that there is a unique one-parameter group of substitutions $s_\lambda(z)$ such that, for λ integer, one has the value ($s_2(z)$ ←→ partition of partitions)

$$s_2(z) = e^{(e^z-1)} - 1;\ s_3(z) = e^{(e^{(e^z-1)}-1)} - 1;\ s_{-1}(z) = log(1+z)$$

But we have no nice description of this group nor of the vector field generating it.

Product formula

The Hadamard product of two sequences

$$(a_n)_{n\geq 0} \quad (b_n)_{n\geq 0}$$

is given by the pointwise product

$$(a_n b_n)_{n\geq 0}$$

We can at once transfer this law on EGFs by

$$F=\sum_{n\geq 0} a_n \frac{y^n}{n!};\ G=\sum_{m\geq 0} b_m \frac{y^m}{m!};\ \mathcal{H}(F,G):=\sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

but, here, as $\displaystyle \frac{(y\frac{d}{dx})^n}{n!}\frac{x^m}{m!}\Big|_{x=0}=\delta_{mn}\frac{y^n}{n!}$

we get $\displaystyle \mathcal{H}(F,G)=F(y\frac{d}{dx})G(x)|_{x=0}$

- Writing *F* and *G* as free exponentials we shall see that these diagrams are in fact labelling monomials. We are then in position of imposing two types of rule:

  - On the diagrams (Selection rules) : on the outgoing, ingoing degrees, total or partial weights.
  - On the set of diagrams (Composition and Decomposition rules) : product and coproduct of diagram(s)

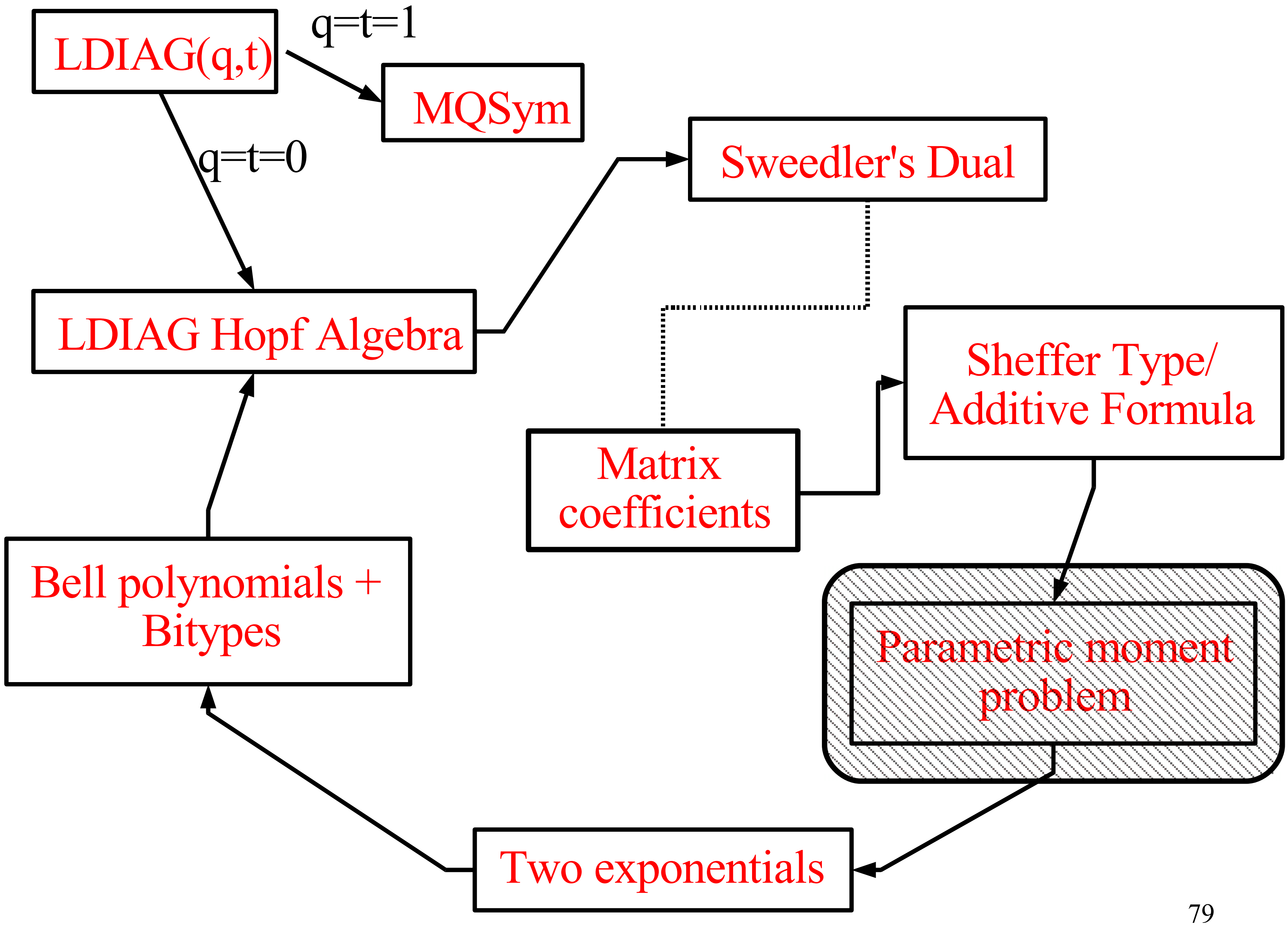
- This leads to structures of Hopf algebras for spaces freely generated by the two sorts of diagrams (labelled and unlabelled).

Labelled diagrams generate the space of Matrix QuasiSymmetric Functions, we thus obtain a new Hopf algebra structure on this space.

Classical Stieltjes moment problem

Consider a sequence of real numbers *B(n)*. The classical Stieltjes moment problem consists in finding a positive measure *W(x)dx* on the half-line]0,+∞[such that

$$B(n) = \int_0^{+\infty} x^n W(x) dx$$


LDIAG(q,t)
q=t=1
MQSym
q=t=0
Sweedler's Dual
LDIAG Hopf Algebra
Sheffer Type/
Additive Formula
Matrix
coefficients
Bell polynomials +
Bitypes
Parametric moment
problem
Two exponentials

Parametric Stieltjes moment problem

Consider a sequence of real functions *B(n,y)*. The parametric Stieltjes moment problem consists in finding a family of positive measures *W(x,y)dx* on the half-line]0,+∞[such that

$$B(n,y) = \int_0^{+\infty} x^n W(x,y) dx$$

Using the first Dobinski relation of slide (10), one can solve the parametric Stieltjes moment problem for the classical Stirling numbers as

$$S_{1,1}(n,y) = \int_0^{+\infty} x^n W_1(x,y)dx$$

with

$$W_1(x,y) = e^{-y} \sum_{k=1}^{\infty} \frac{y^k \delta(x-k)}{k!}$$

which is a Poisson distribution on the half-line]0,+∞[.

Using an inverse Mellin transform, one can solve the second parameric moment problem, which gives, this time, a continuous measure

$$S_{2,1}(n,y) = \int_0^{+\infty} x^n W_2(x,y) dx$$

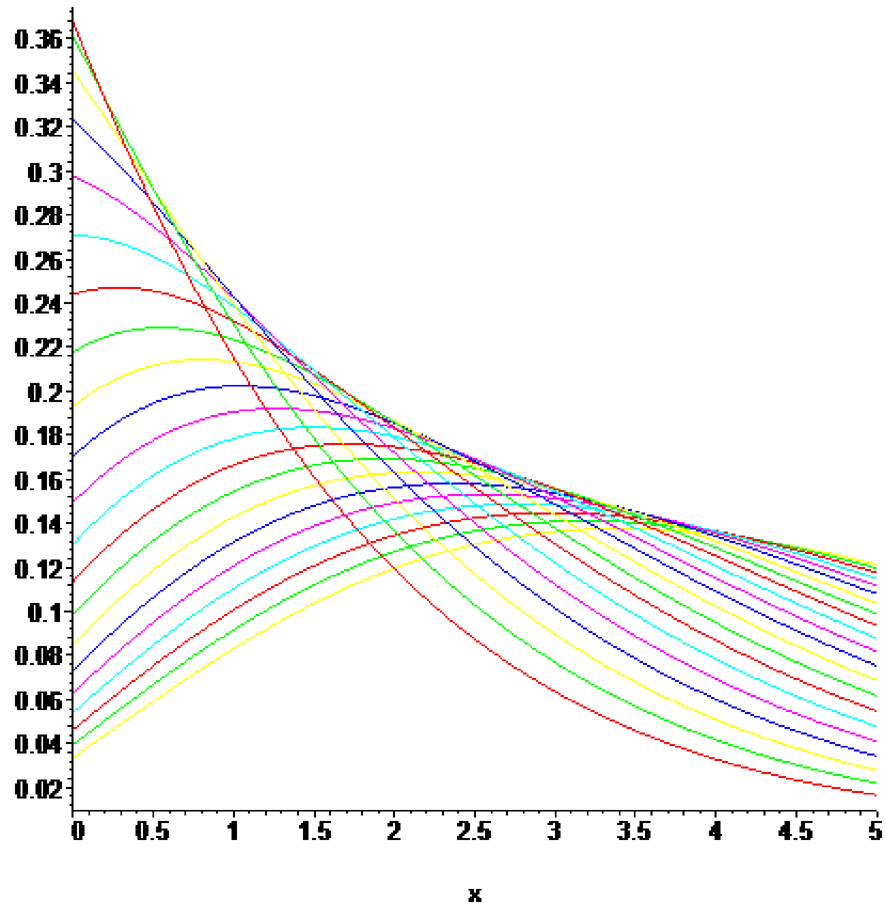
with

$$W_2(x,y) = y\, e^{-(x+y)} \frac{I_1(2\sqrt{xy})}{\sqrt{xy}}$$

```
> f1:=(x,y)->y*exp(-(x+y))*BesselI(1,2*sqrt(x*y))/sqrt(x*y);
```

$$f1 := (x, y) \rightarrow \frac{y\, e^{(-x-y)}\, \mathrm{BesselI}(1, 2\sqrt{x\, y})}{\sqrt{x\, y}}$$

```
> plot([seq(f1(x,1+0.2*k),k=0..20)],x=0..5,tickmarks=[8,10]);
```




```
> seq(evalf(f1(0.001,1+0.2*k),3),k=0..20);
```

0.367, 0.361, 0.345, 0.322, 0.297, 0.271, 0.245, 0.218, 0.193, 0.171, 0.150, 0.131, 0.115, 0.0985, 0.0851, 0.0731, 0.0633, 0.0543, 0.0468, 0.0397, 0.0337

Ongoing work

Realizations of the product for some types of infinite matrices

Convolution of kernels: We first suppose given two infinite matrices F(n,k), G(n,k) (n,k integers) admitting solutions for the parametric moment problem (PMP) which means that there are two (parametric) measures W_F , W_G such that

$$B_F(n,y) = \int_0^{+\infty} x^n W_F(x,y)dx$$

$$B_G(n,y) = \int_0^{+\infty} x^n W_G(x,y)dx$$

Then one can check easily that, if the two kernels W_F *and* W_G are convolable, then FG admits a solution for the PMP and

$$W_{FG}(x,y) = \int_0^{+\infty} W_F(x,z)W_G(z,y)dz$$

Questions: Q1) What are the types of matrices for which there is a PMP solution ?

Q2) Which are the ones for which the kernel is discrete ? Continuous ?

Q3) Are there general laws for convolution of these types of kernels.

Link with grafting: Certain classes of graphs (i.e. closed by relabelling and extraction of connected components) provide lower triangular matrices via

M(n,k)=number of graphs with labels {1, 2,..n} and k connected components

the product of the matrices associated with two classes corresponds to the grafting obtained by considering the connected components of a graph of the first kind as vertices of a graph of the second kind.

Question: What are the legal types of grafting when we change denominators ? Link with renormalisation ?

Substitutions: An infinite matrice F(n,k) with finite rows can be seen as defining a transformation between EGF. The transformation is of the form f --> $u(x)f(v(x))$ with $u(x)=1+\ldots$ and $v(x)=\lambda x+\ldots$ if the sequence of polynomials $B_F(n,y)$ is of Extended Sheffer Type (EST).

There is a « calculus » using vector fields on the half-line and their conjugates. (see SLC Viennot - Lucelle - and Myczcowce talks)

Questions: Q1) Combinatorial fields ? What is the «Stirling field » for instance ?

Q2) Make precise the dictionnaries (formal or analytic)

vector fields ↔ combinatorial matrices

Q3) What are the matrices coming from classes of graphs