# Notations de Dirac-Schützenberger, calcul des décalages et Physique


**Gérard H. E. Duchamp**

**LIPN**, ***Université de Paris XIII, France***

***<u>Collaborators</u>***
***(PVI-PXIII- Group – "CIP" = Combinatorics Informatics and Physics):***

**Pawel Blasiak**, ***Instit. of Nucl. Phys., Krakow, Pologne***

**Andrzej Horzela**, ***Instit. of Nucl. Phys., Krakow, Pologne***

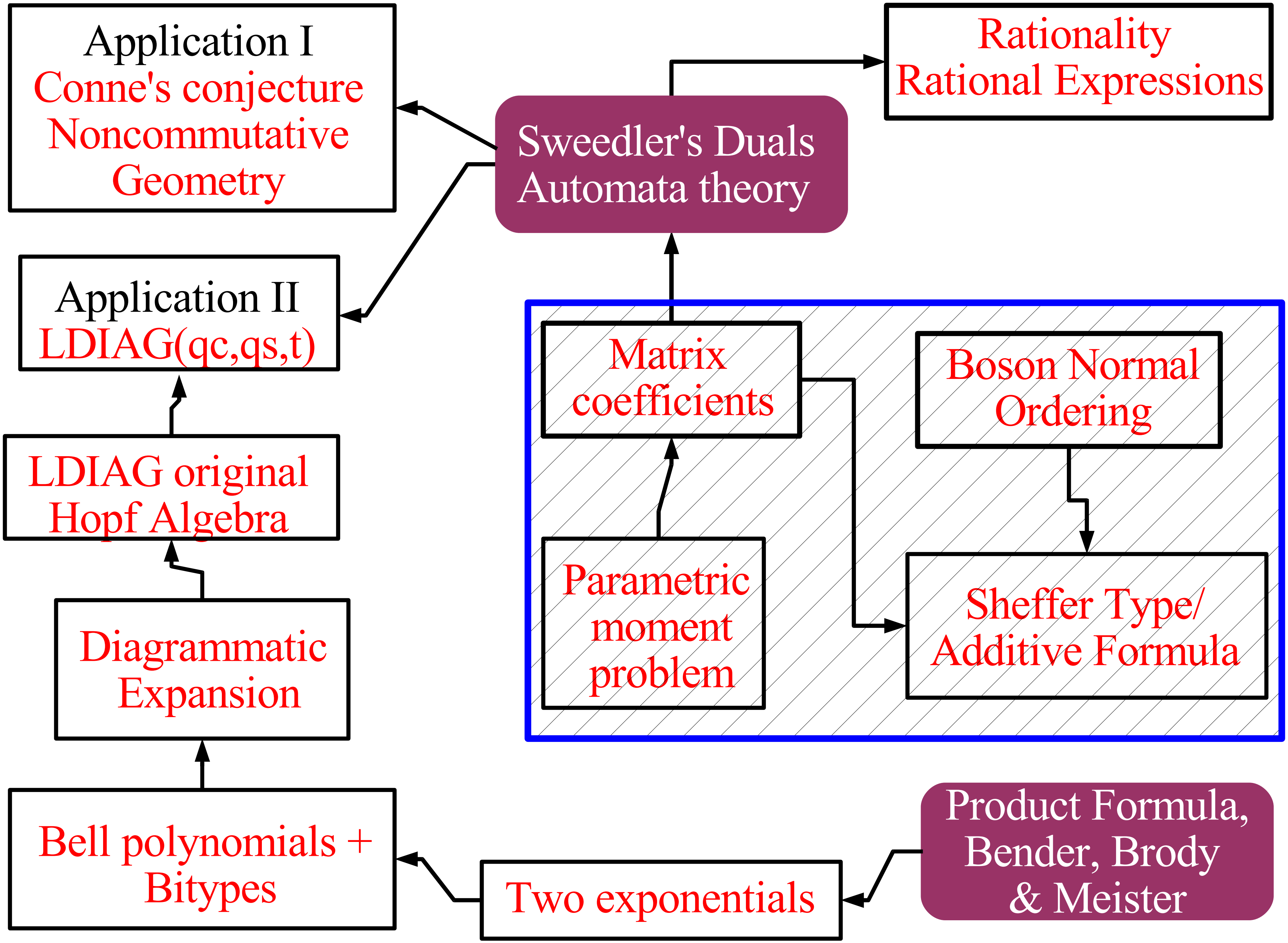
**Karol A. Penson**, **LPTMC**, ***Université de Paris VI, France***

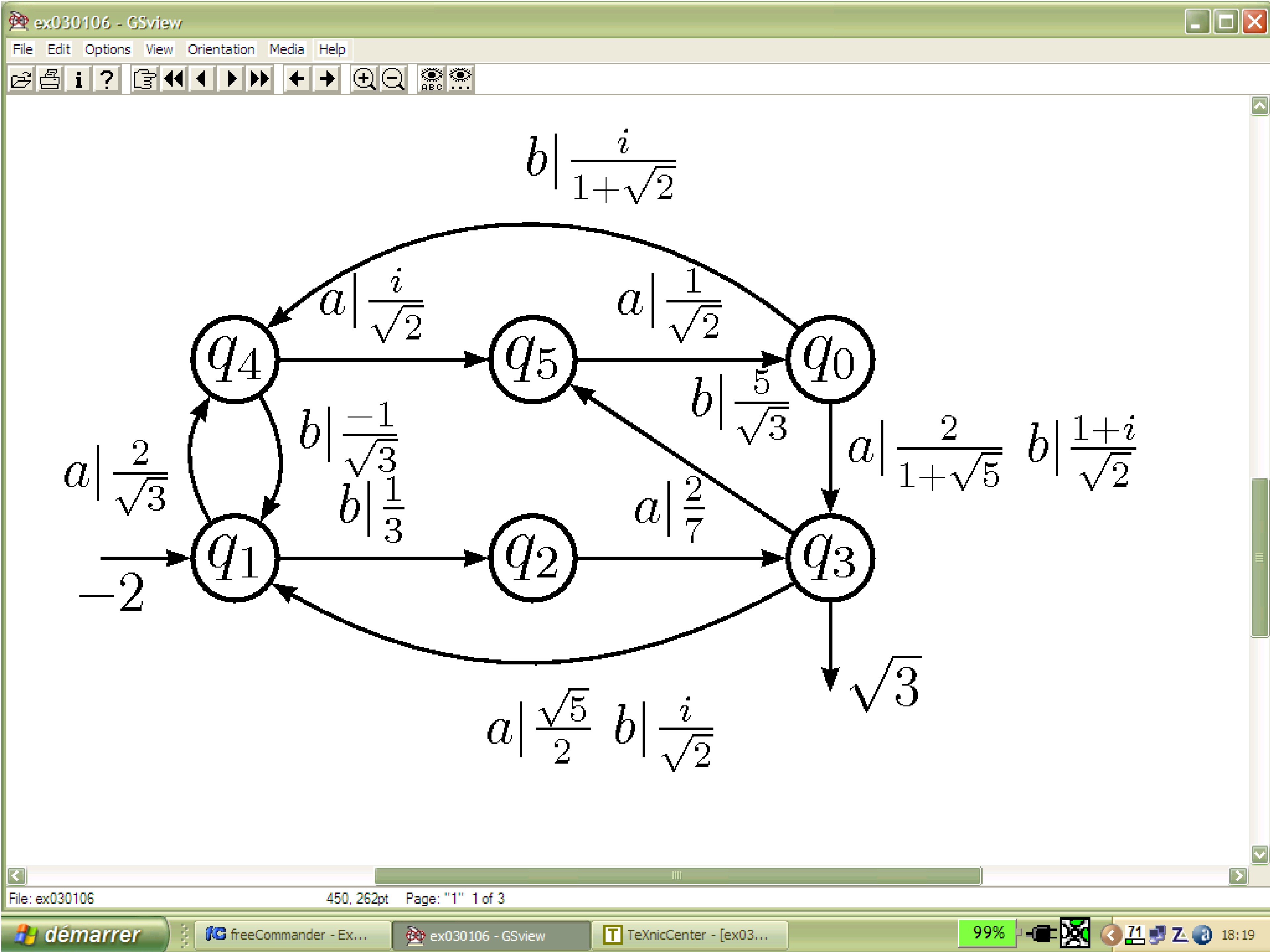
**Allan I. Solomon**, ***The Open University, United Kingdom***

**Christophe Tollu**, ***LIPN, Université de Paris XIII, France***

**Frédéric Toumazet**, ***LIPN, Université de Paris XIII, France***


LITIS/Rouen – Séminaire – 21.06.07


Application I
Conne's conjecture
Noncommutative
Geometry
Sweedler's Duals
Automata theory
Rationality
Rational Expressions
Application II
LDIAG(qc,qs,t)
LDIAG original
Hopf Algebra
Matrix
coefficients
Boson Normal
Ordering
Parametric
moment
problem
Sheffer Type/
Additive Formula
Diagrammatic
Expansion
Bell polynomials +
Bitypes
Two exponentials
Product Formula,
Bender, Brody
& Meister


ex030106 - GSview
File Edit Options View Orientation Media Help
$b|\frac{i}{1+\sqrt{2}}$
$a|\frac{i}{\sqrt{2}}$
$a|\frac{1}{\sqrt{2}}$
$q_4$
$q_5$
$q_0$
$b|\frac{5}{\sqrt{3}}$
$b|\frac{-1}{\sqrt{3}}$
$a|\frac{2}{\sqrt{3}}$
$a|\frac{2}{1+\sqrt{5}}$
$b|\frac{1+i}{\sqrt{2}}$
$b|\frac{1}{3}$
$a|\frac{2}{7}$
$q_1$
$q_2$
$q_3$
$-2$
$\sqrt{3}$
$a|\frac{\sqrt{5}}{2}$
$b|\frac{i}{\sqrt{2}}$
File: ex030106
450, 262pt
Page: "1" 1 of 3
démarrer
freeCommander - Ex...
ex030106 - GSview
TeXnicCenter - [ex03...
99%
18:19

# (A part of) The legacy of Schützenberger or how to compute efficiently in Sweedler's duals using Automata Theory

## Sweedler's dual of a Hopf algebra

*i) Multiplication*

$$\mathcal{A}\otimes\mathcal{A} \xrightarrow{\ \mu\ } \mathcal{A}$$

*ii)* By dualization one gets

$$(\mathcal{A})^* \xrightarrow{\ {}^t\mu\ } (\mathcal{A}\otimes\mathcal{A})^*$$

but not a "stable calculus" as

$$(\mathcal{A})^*\otimes(\mathcal{A})^* \subseteq (\mathcal{A}\otimes\mathcal{A})^*$$

(strict in general). We ask for elements $x\in\mathcal{A}$ such that

$${}^t\mu(x)\in(\mathcal{A})^*\otimes(\mathcal{A})^*$$

These elements are easily characterized as the "representative linear forms" (see also the Group-Theoretical formulation in the talk of Pierre Cartier)

**Proposition :** TFAE (the notations being as above)

i) ${}^t\mu(c) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$

ii) There are functions $f_i, g_i$ $i=1,2..n$ such that

$$c(xy)=\sum_{i=1}^{n} f_i(x)\, g_i(y)$$

forall x,y in $\mathcal{A}$.

iii) There is a morphism of algebras $\mu: \mathcal{A} \to k^{n \times n}$ (square matrices of size n x n), a line $\lambda$ in $k^{1 \times n}$ and a column $\xi$ in $k^{n \times 1}$ such that, for all z in $\mathcal{A}$,

$$c(z)=\lambda\mu(z)\xi$$

In many "Combinatorial" cases, we are concerned with the case $\mathcal{A}$= k<A> (non-commutative polynomials with coefficients in a field k).

Indeed, one has the following theorem (the beginning can be found in [ABE : Hopf algebras]) and the end is one of the starting points of Schützenberger's school of automata and language theory.

**Theorem A:** TFAE (the notations being as above)

i) ${}^t\mu(c) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$

ii) There are functions $f_i$ ,$g_i$ i=1,2..n such that

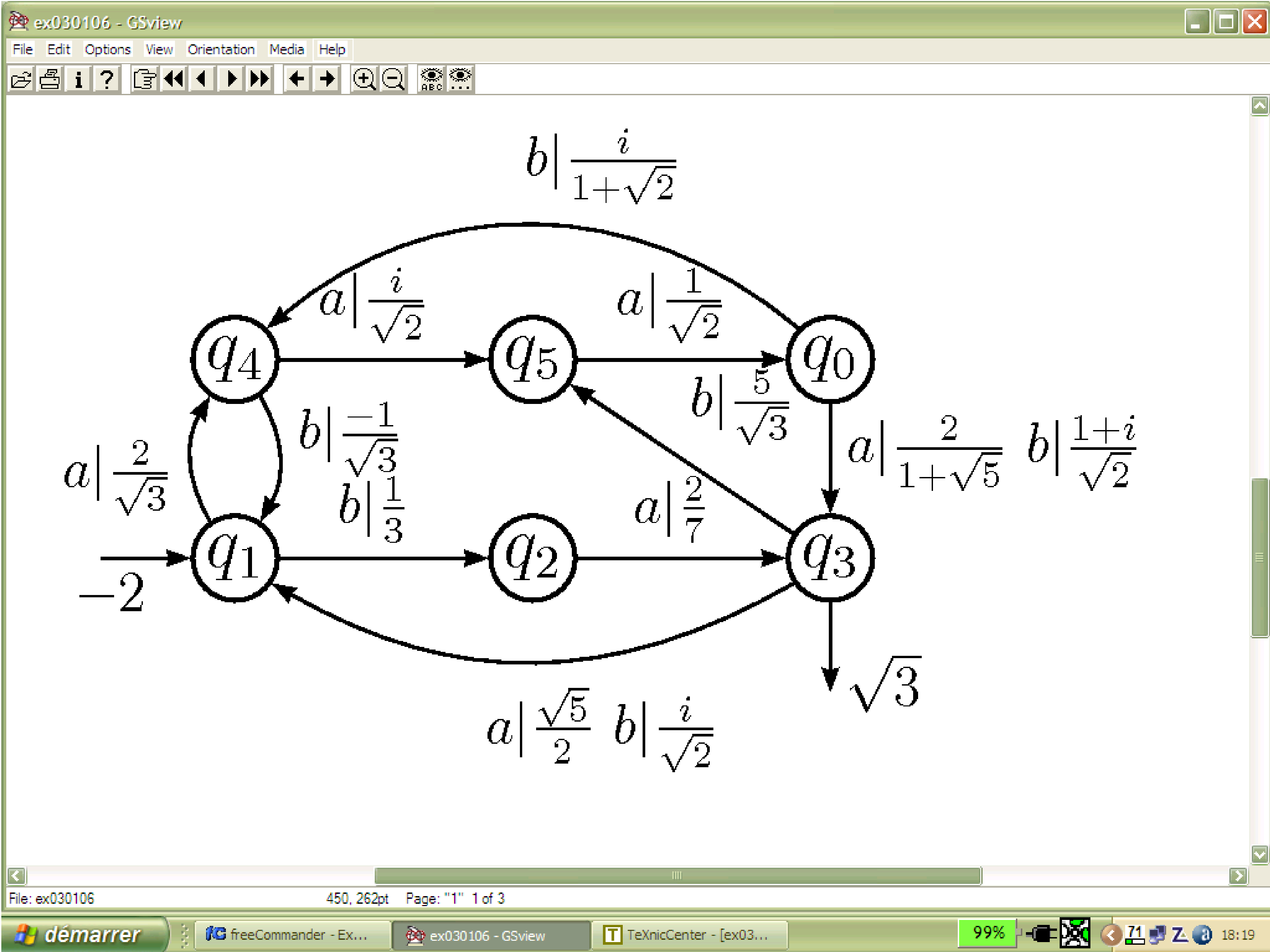
$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in $A^*$ (the free monoid of alphabet A).

iii) There is a morphism of monoids $\mu: A^* \dashrightarrow k^{n \times n}$ (square matrices of size n x n), a row $\lambda$ in $k^{1 \times n}$ and a column $\xi$ in $k^{n \times 1}$ such that, for all word w in $A^*$

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If A is finite) c lies in the rational closure of A within the algebra k<<A>>.


ex030106 - GSview
File Edit Options View Orientation Media Help
$b|\frac{i}{1+\sqrt{2}}$
$a|\frac{i}{\sqrt{2}}$
$a|\frac{1}{\sqrt{2}}$
$q_4$
$q_5$
$q_0$
$b|\frac{5}{\sqrt{3}}$
$b|\frac{-1}{\sqrt{3}}$
$a|\frac{2}{\sqrt{3}}$
$a|\frac{2}{1+\sqrt{5}}$
$b|\frac{1+i}{\sqrt{2}}$
$b|\frac{1}{3}$
$a|\frac{2}{7}$
$q_1$
$q_2$
$q_3$
$-2$
$\sqrt{3}$
$a|\frac{\sqrt{5}}{2}$ $b|\frac{i}{\sqrt{2}}$
File: ex030106
450, 262pt Page: "1" 1 of 3
démarrer
freeCommander - Ex...
ex030106 - GSview
TeXnicCenter - [ex03...
99%
18:19

We can safely apply the first three conditions of **Theorem A** to *Ldiag.* The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more “basic” functions. The modification reads

iv) (A is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in A} p(a)\, a$$

within the algebra k<<A>>.

*iii) Schützenberger's* theorem (known as the theorem of Kleene-Schützenberger) could be rephrased in saying that functions in a Sweedler's dual are behaviours of finite (state and alphabet) automata.

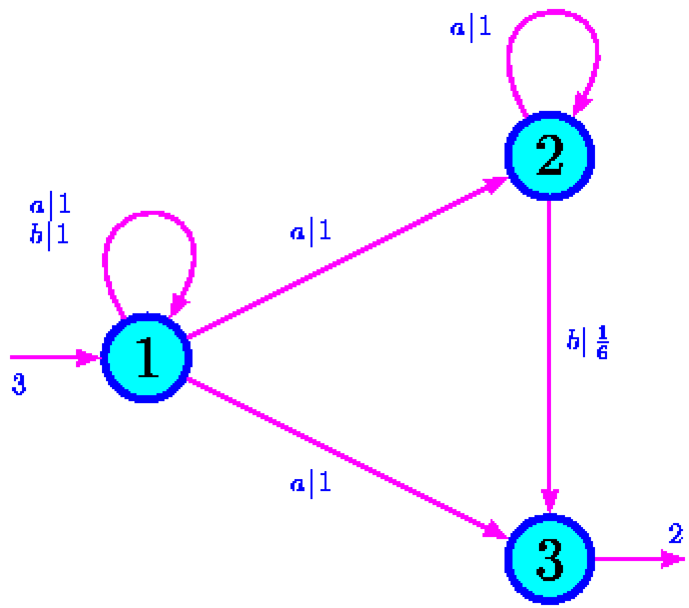
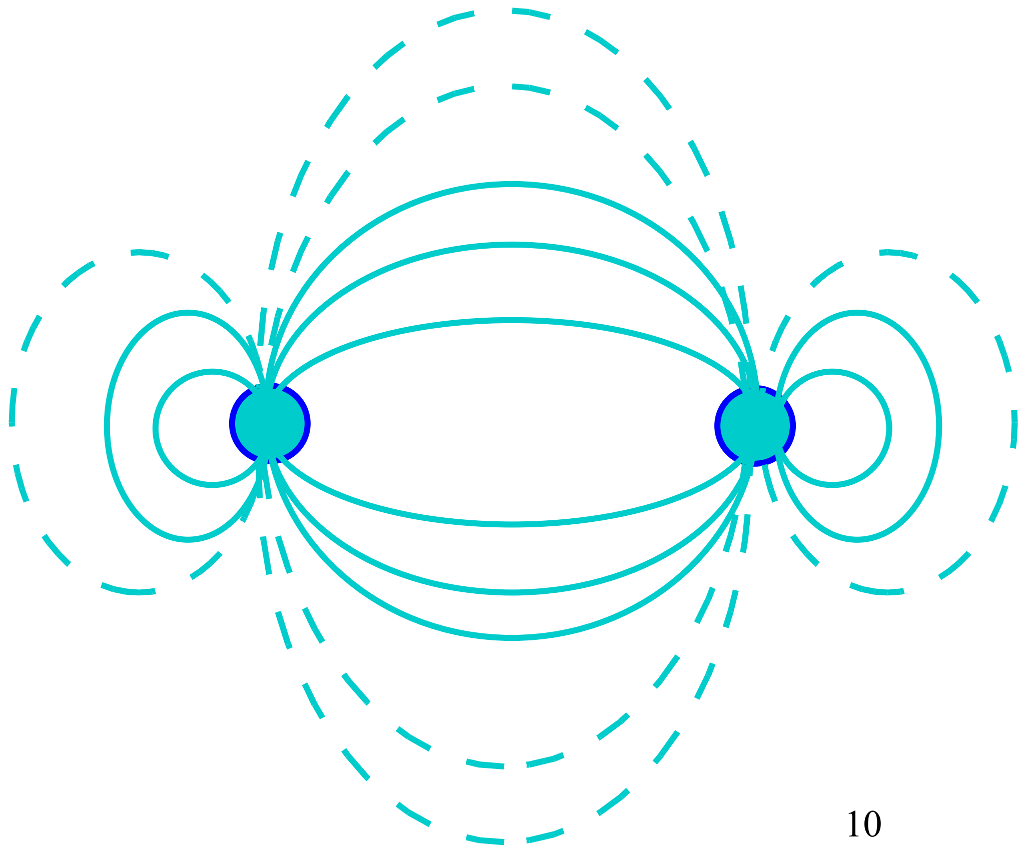



FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6+a^*b).$$

In our case, we are obliged to allow infinitely many edges.



# Computations in $K\langle A\rangle^0$, Sweedler's dual of $K\langle A\rangle$

Summability : We say that a family $(f_i)_{i\in I}$ ($I$ finite or not, $f_i$ in $K\langle\langle A\rangle\rangle$) is summable if, for each $w\in A^*$, the family $(\langle f_i|w\rangle)_{i\in I}$ is finitely supported and we set

$$\left(\sum_{i\in I} f_i\right) : w \to \left(\sum_{i\in I} \langle f_i|w\rangle\right)$$

Identifying each word with the Dirac linear form located at the word, one has then, for each $f\in K\langle\langle A\rangle\rangle$

$$f=\sum_{w\in A^*} f(w)w$$

If $f \in K^{rat}\langle\langle A \rangle\rangle$, it exists a morphism of monoids $\mu: A^* \to K^{n \times n}$ (square matrices of size n x n), a row $\lambda$ in $k^{1 \times n}$ and a column $\xi$ in $k^{n \times 1}$ such that, for all word w in $A^*$, $f(w)=\lambda\mu(w)\xi$. Then

$$f=\sum_{w\in A^*} f(w)w=\sum_{w\in A^*} \lambda\mu(w)\xi\, w=\lambda\Big(\sum_{w\in A^*} \mu(w)w\Big)\xi=$$

$$\lambda\Big(\sum_{w\in A^*} \mu(w)w\Big)\xi=\lambda\Big(\sum_{m\geq 0} \sum_{|w|=m} \mu(w)w\Big)\xi$$

But, as words and scalars commute (it is so by construction of the convolution algebra $K^{n \times n}\langle\langle A \rangle\rangle$), one has

$$\sum_{m\geq 0} \sum_{|w|=m} \mu(w)w=\sum_{m\geq 0} \Big(\sum_{a\in A} \mu(a)a\Big)^m=\Big(\sum_{a\in A} \mu(a)a\Big)^*$$

hence

$$f=\lambda\Big(\sum_{a\in A} \mu(a)a\Big)^*\xi$$

where the “star” stands for the sum of the geometric series.

If $Q$ is a finite set, the space $k^{Q\times Q}$ of square matrices with indices in $Q$ and coefficients in $k$ has a natural semiring structure with the usual operations (sum and product). A (right) star of $M \in k^{Q\times Q}$ (when it exists) is a solution of the equation $MY + 1_{Q\times Q} = Y$ (where $1_{Q\times Q}$ is the identity matrix). Let $M \in k^{Q\times Q}$ be given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} \in k^{Q_1\times Q_1}$, $a_{12} \in k^{Q_1\times Q_2}$, $a_{21} \in k^{Q_2\times Q_1}$ and $a_{22} \in k^{Q_2\times Q_2}$ such that $Q_1 + Q_2 = Q$. Let $N \in k^{Q\times Q}$ given by

$$N = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = (a_{11} + a_{12}a_{22}{}^{*}a_{21})^{*} \qquad (1)$$

$$A_{12} = a_{11}{}^{*}a_{12}A_{22} \qquad (2)$$

$$A_{21} = a_{22}{}^{*}a_{21}A_{11} \qquad (3)$$

$$A_{22} = (a_{22} + a_{21}a_{11}{}^{*}a_{12})^{*} \qquad (4)$$

$$A_{11} = (a_{11} + a_{12}a_{22}{}^{*}a_{21})^{*} \quad (1)$$

$$A_{12} = a_{11}{}^{*}a_{12}A_{22} \quad (2)$$

$$A_{21} = a_{22}{}^{*}a_{21}A_{11} \quad (3)$$

$$A_{22} = (a_{22} + a_{21}a_{11}{}^{*}a_{12})^{*} \quad (4)$$

Sketch of the proof

- Prove the equivalence

equational star <=> iterative star

- Prove that the block-matrix $A_{ij}$ solves the equational star equation

# A (short) word on automata theory.

- The formulas (for the star* of a matrix) above are sufficiently “expressive” to be the crucial fact in the resolution of a conjecture in Noncommutative Geometry.

- For applications, automata theory had to cope with spaces of coefficients much more general than that of a field ... even the “minus” operation of the rings had to disappear to be able to cope with problems like shortest path or the Noncommutative problem or the shortest path with list of minimal arcs .

The emerging structure is that of a semiring. Think of a ring without the “minus” operation, nevertheless “transfer” matrix computations can be performed.

The input alphabet being set by the automaton under consideration, we will here rather focus on the definition of semirings providing transition coefficients. For convenience, we first begin with various laws on $\mathbb{R}_+ := [0, +\infty[$ including

1. $+$ (ordinary sum)
2. $\times$ (ordinary product)
3. min (if over $[0, 1]$, with neutral 1, otherwise must be extended to $[0, +\infty]$ and then, with neutral $+\infty$) or max
4. $+_a$ defined by $x +_a y := log_a(a^x + a^y)$ $(a > 0)$
5. $+_{[n]}$ (Hölder laws) defined by $x +_{[n]} y := \sqrt[n]{x^n + y^n}$
6. $+^s$ (shifted sum, $x +^c y := x + y - 1$, over whole $\mathbb{R}$, with neutral 1)
7. $\times^c$ (complemented product, $x + y - xy$, can be extended also to whole $\mathbb{R}$, stabilizes the range of probabilities or fuzzy $[0, 1]$ and is distributive over the shifted sum)

As (useful) examples, one has ([0,+∞], min, +), ([0,+∞[, max, +) or its (commutative or not) variants.

# What remains for K<A> ? (free algebra)

- K semiring :

  - Universal properties (comprising – little known - tensor products)

  - Complete semiring K<<A>>, summability is defined by pointwise convergence (see computation above).

  - Rational closures and Kleene-Schützenberger Thm

  - Rational expressions, Brzozowski theorem

  - Automata theory, theory of codes

  - Lazard's monoidal elimination

In fact, one call pull the operations on the functions back to the level of automata.

**Proposition 2** *Let $R : \mathcal{A}_r = (\lambda^r, \mu^r, \gamma^r)$ (resp. $S : \mathcal{A}_s = (\lambda^s, \mu^s, \gamma^s)$) of rank $n$ (resp. $m$). The linear representations of the sum, the concatenation and the star are respectively*
$R + S$ :

$$\mathcal{A}_r \boxplus \mathcal{A}_s = \left( \begin{pmatrix} \lambda^r & \lambda^s \end{pmatrix}, \left( \begin{array}{c|c} \mu^r(a) & 0_{n\times m} \\ \hline 0_{m\times n} & \mu^s(a) \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^r \\ \gamma^s \end{pmatrix} \right) , \qquad (1)$$

$R.S$ :

$$\mathcal{A}_r \boxdot \mathcal{A}_s = \left( \begin{pmatrix} \lambda^r & 0_{1\times m} \end{pmatrix}, \left( \begin{array}{c|c} \mu^r(a) & \gamma^r\lambda^s\mu^s(a) \\ \hline 0_{m\times n} & \mu^s(a) \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^r\lambda^s\gamma^s \\ \gamma^s \end{pmatrix} \right) , \qquad (2)$$

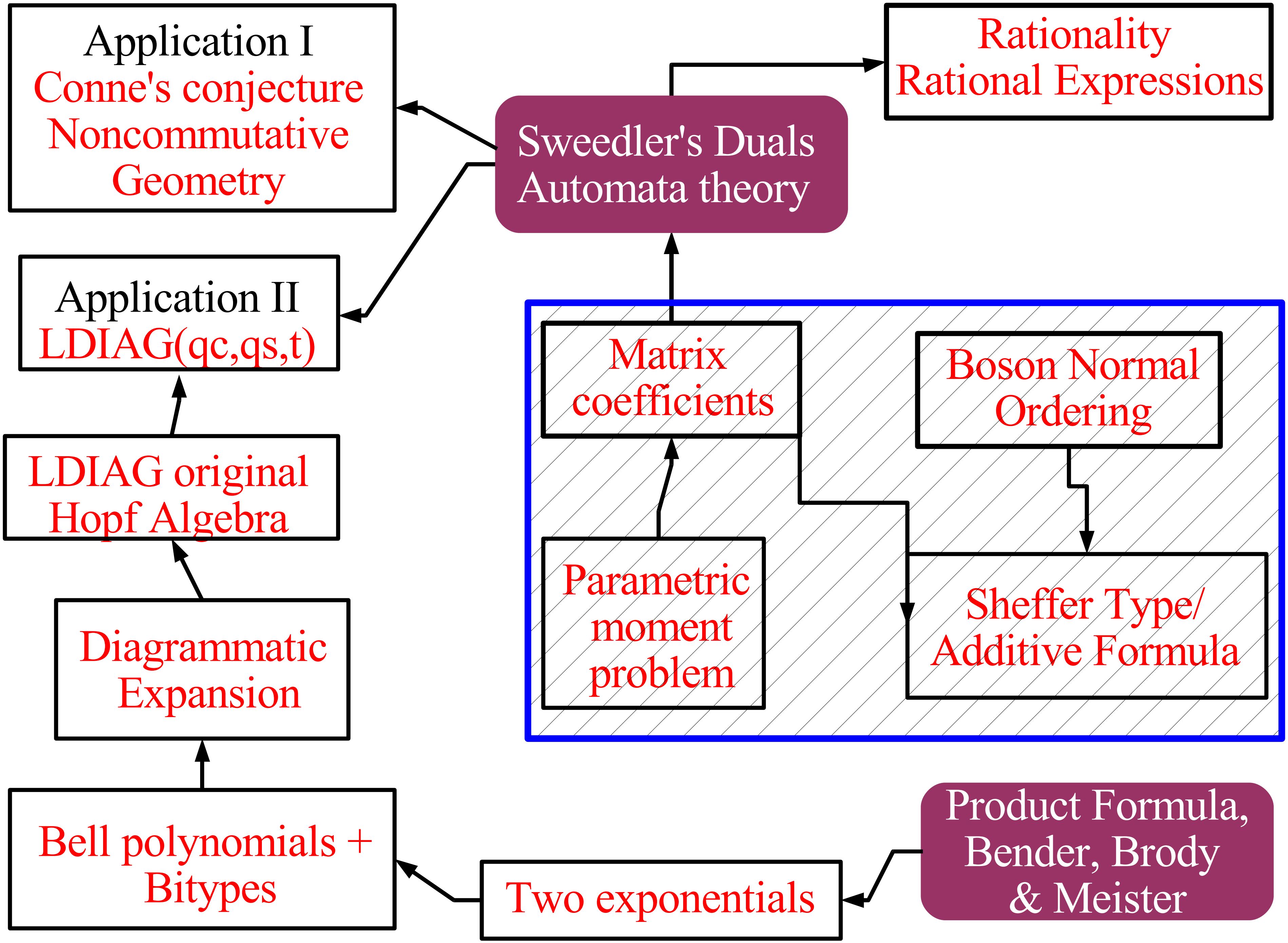
*If* $\lambda^s\gamma^s = 0$, $S^*$ :

$$\mathcal{A}_s^{\circledast} = \left( \begin{pmatrix} 0_{1\times m} & 1 \end{pmatrix}, \left( \begin{array}{c|c} \mu^s(a) + \gamma^s\lambda^s\mu^s(a) & 0_{m\times 1} \\ \hline \lambda^s\mu^s(a) & 0 \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^s \\ 1 \end{pmatrix} \right) . \qquad (3)$$

## Remarks

1) The question of pulling back other operations (like coproducts) on representative functions at the level of some equivalence class of automata remains open.

2) In case $\mathcal{A}=\ell^2(G,\mathbb{C})$ where $G$ is a compact group (endowed with any Haar measure), one can prove that the space of representative continuous functions is dense in $\mathcal{A}$ (Peter-Weyl's theorem).



Application I
Conne's conjecture
Noncommutative
Geometry
Sweedler's Duals
Automata theory
Rationality
Rational Expressions
Application II
LDIAG(qc,qs,t)
Matrix
coefficients
Boson Normal
Ordering
LDIAG original
Hopf Algebra
Parametric
moment
problem
Sheffer Type/
Additive Formula
Diagrammatic
Expansion
Bell polynomials +
Bitypes
Two exponentials
Product Formula,
Bender, Brody
& Meister

# Application I

## Conne's conjecture about rationality phenomena in Noncommutative Geometry

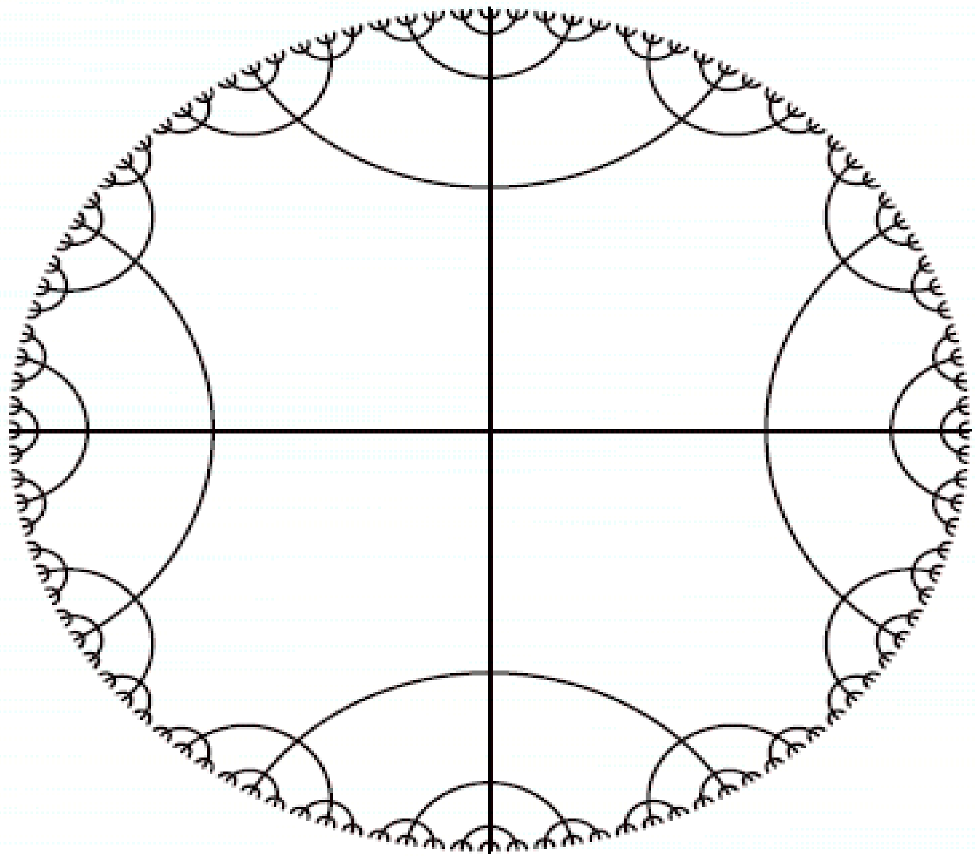


FIGURE 5. Tree

As an application we shall give a new proof of the beautiful result of M. Pimsner and D. Voiculescu that the reduced $C^*$-algebra of the free group on two generators does not contain any nontrivial idempotent [448]. This settled a long-standing conjecture of R.V. Kadison. We shall use a specific Fredholm module $(\mathcal{H}, F)$ over the reduced $C^*$-algebra of the free group which already appears in [448] and in the simplified proof of J. Cuntz [145], and whose geometric meaning in terms of trees was clarified by P. Julg and A. Valette in [315], [316], [317].

Remark 3. The proof of Lemma 1 shows that for any $a \in \mathbb{C}\Gamma$ the quantum differential $da$ is a finite-rank operator. Then let $(\mathbb{C}\Gamma)^{\sim}$ be the smallest subalgebra $\mathcal{B}$ of $C_r^*(\Gamma)$ containing $\mathbb{C}\Gamma$ and having the property for any $n \in \mathbb{N}$

$$x \in M_n(\mathcal{B}) \cap M_n(C_r^*(\Gamma))^{-1} \Rightarrow x \in M_n(\mathcal{B})^{-1}.$$

5. FREDHOLM MODULES AND RANK-ONE DISCRETE GROUPS 351

One easily checks that for any $a \in (\mathbb{C}\Gamma)^{\sim}$, the operator $da$ is of finite rank. It is natural to conjecture that the converse holds. This can be proved when the free group

Ensuite (p 342), il définit une clôture de $\mathbb{C}\Gamma \subset C_r^*(\Gamma)$ par adjonction de coefficients d'inverses de matrices. Nous en rappellons la construction ci-dessous.

Soit $(C_r^*(\Gamma))_n$ la plus petite sous-algèbre $\mathcal{B} \subset C_r^*(\Gamma)$ telle que $\Gamma \subset \mathcal{B}$ et que

$$x \in M_n(\mathcal{B}) \cap [M_n(C_r^*(\Gamma))]^{-1} \Longrightarrow x \in [M_n(\mathcal{B})]^{-1}$$

On a évidemment

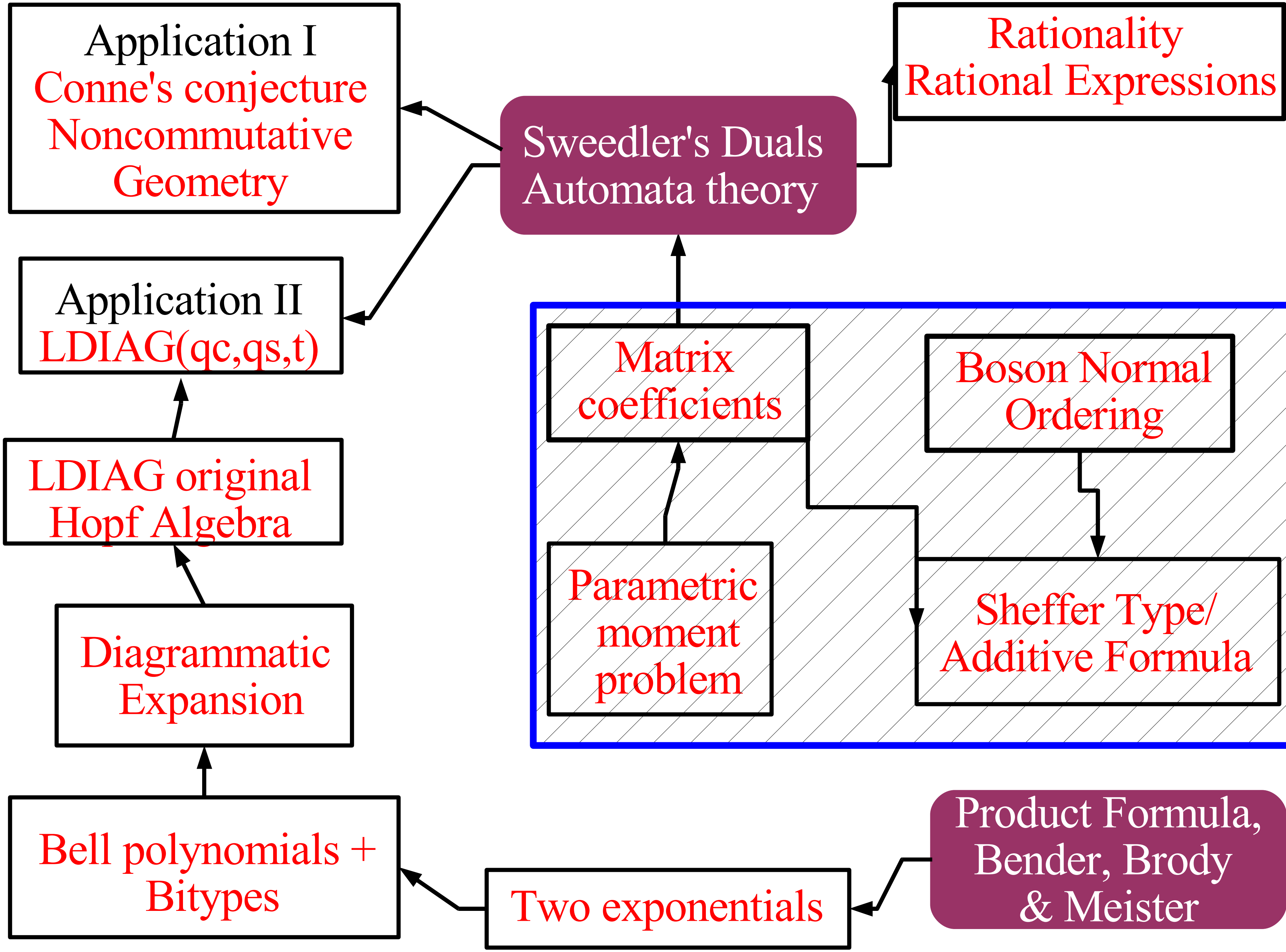
$$(C_r^*(\Gamma))_1 \subset (C_r^*(\Gamma))_2 \subset \cdots (C_r^*(\Gamma))_n \subset \cdots$$

la clôture annoncée est $(C_r^*(\Gamma))^{\sim} = \cup_{n\geq 1}(C_r^*(\Gamma))_n$.

Enfin (au paragraphe IV.5) est défini un module de FREDHOLM (pour la définition exacte voir le paragraphe 2.2) qui conduit à la caractérisation d'une sous-algèbre $(C_r^*(\Gamma))_{fin} \subset C_r^*(\Gamma)$ par une condition de finitude de rang.

A. CONNES remarque que $(C_r^*(\Gamma))^{\sim} \subset (C_r^*(\Gamma))_{fin}$ et conjecture l'inclusion inverse (p342 remarque 3).

Dans ce qui suit nous démontrons que $(C_r^*(\Gamma))^{\sim} = (C_r^*(\Gamma))_1$, ce qui entraine la conjecture et l'égalité de tous les $(C_r^*(\Gamma))_n$.


Application I
Conne's conjecture
Noncommutative
Geometry
Sweedler's Duals
Automata theory
Rationality
Rational Expressions
Application II
LDIAG(qc,qs,t)
Matrix
coefficients
Boson Normal
Ordering
LDIAG original
Hopf Algebra
Parametric
moment
problem
Sheffer Type/
Additive Formula
Diagrammatic
Expansion
Bell polynomials +
Bitypes
Two exponentials
Product Formula,
Bender, Brody
& Meister

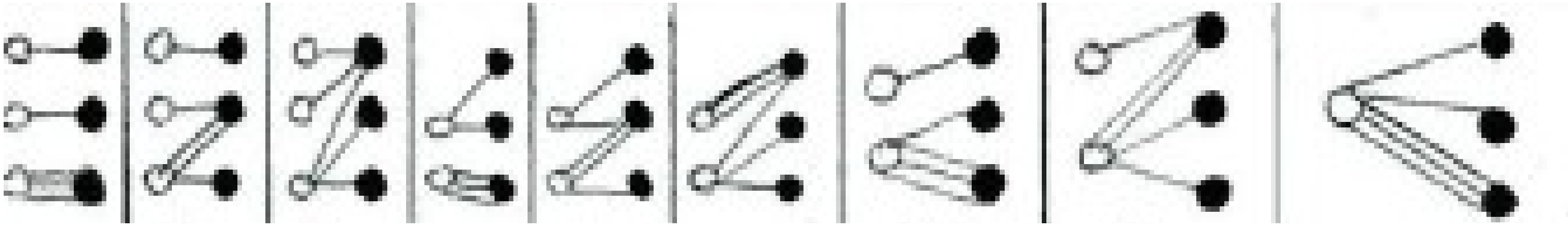
# Application II
# LDIAG Hopf algebra

In a relatively recent paper Bender, Brody and Meister (*) introduce a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs (see third Part of this talk).

These graphs label monomials and are obtained in the case of special interest when the two EGF have a constant term equal to unity.

---

***Bender, C.M, Brody, D.C. and Meister,***
***Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999***



***Some 5-line diagrams***

If we write these functions as exponentials, we are led to witness a surprising interplay between the following aspects: **algebra** (of normal forms or of the exponential formula, Hopf structure), **geometry** (of one-parameter groups of transformations and their conjugates) and **analysis** (parametric Stieltjes moment problem and convolution of kernels). Today, we will first focus on the algebra. If time permits, we will touch on the other aspects.

# Construction of the Hopf algebra LDIAG

# How these diagrams arise and which data structures are around them

Let F, G be two EGFs.

$$F = \sum_{n \geq 0} a_n \frac{y^n}{n!};\ G = \sum_{m \geq 0} b_m \frac{y^m}{m!};\ \mathcal{H}(F,G) := \sum_{n \geq 0} a_n b_n \frac{y^n}{n!}$$

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0}$$

Called « product formula » in the QFTP of
Bender, Brody and Meister.

In case *F(0)=G(0)=1*, one can set

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

and then,

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

with $\alpha, \beta \in \mathbb{N}^{(\mathbb{N}^*)}$ multiindices

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

We will adopt the notation

$$\alpha = 1^{a_1} 2^{a_2} \cdots r^{a_r}$$

for the *type* of a (set) partition which means that there are $a_1$ singletons $a_2$ pairs $a_3$ 3-blocks $a_4$ 4-blocks and so on.

The number of set partitions of type $\alpha$ as above is well known (see Comtet for example)

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Then, with

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

one has

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

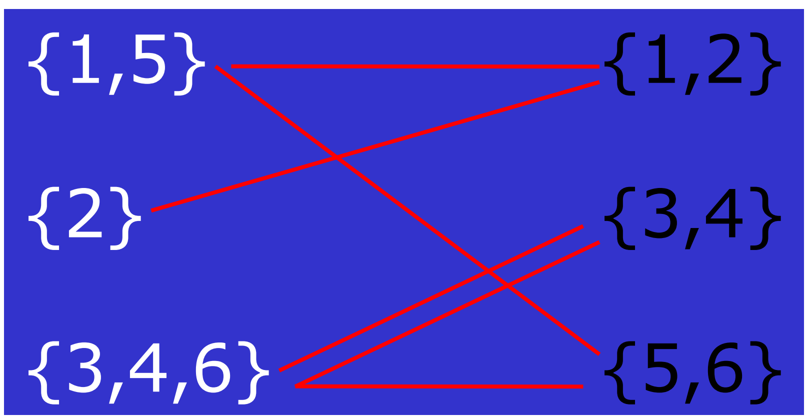
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^{\alpha}\mathbb{V}^{\beta}$$

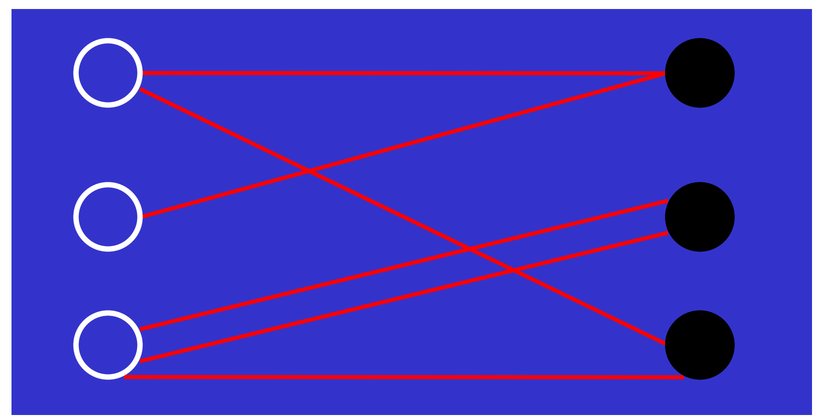
Now, one can count in another way the term *numpart(α)numpart(β)*. Remarking that this is the number of pairs of set partitions (P1,P2) with type(P1)=α, type(P2)=β. But every pair of partitions (P1,P2) has an intersection matrix ...

| | {1,5} | {2} | {3,4,6} |
|---|---|---|---|
| {1,2} | 1 | 1 | 0 |
| {3,4} | 0 | 0 | 2 |
| {5,6} | 1 | 0 | 1 |

Classes of packed matrices see NCSF VI (GD, Hivert, and Thibon)

Feynman-type diagram (Bender & al.)



Now the product formula for EGFs reads

$$\mathcal{H}(F,G) = \sum_{d\ FB-diagram} \frac{y^{|d|}}{|d|!} \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

$$\mathcal{H}(F,G) = \sum_{d \in \mathbf{diag}} \frac{y^{|d|}}{|d|!} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)}$$

The main interest of these new forms is that we can impose rules on the counted graphs and we can call these (and their relatives) graphs : Feynman-Bender Diagrams of this theory (here, the simplified model of Quantum Field Theory of Partitions).

One has now 3 types of diagrams :

- the diagrams with labelled edges (from 1 to |d|). Their set is denoted (see above) FB-diagrams.

- the unlabelled diagrams (where permutations of black and white spots are allowed). Their set is denoted (see above) **diag.**

- the diagrams, as drawn, with black (resp. white) spots ordered i.e. labelled. Their set is denoted **ldiag.**

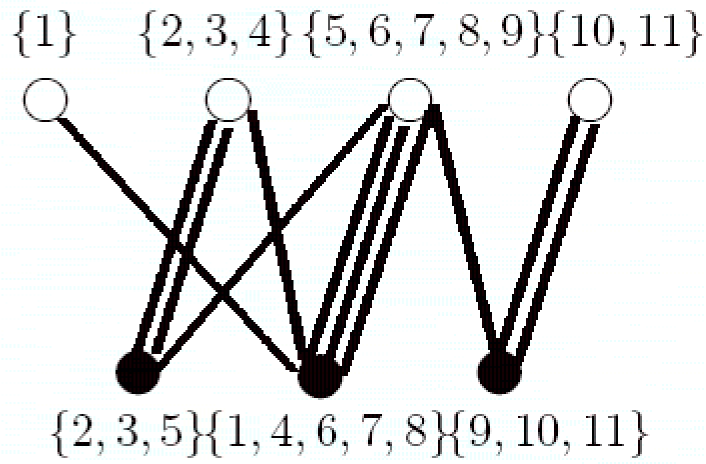
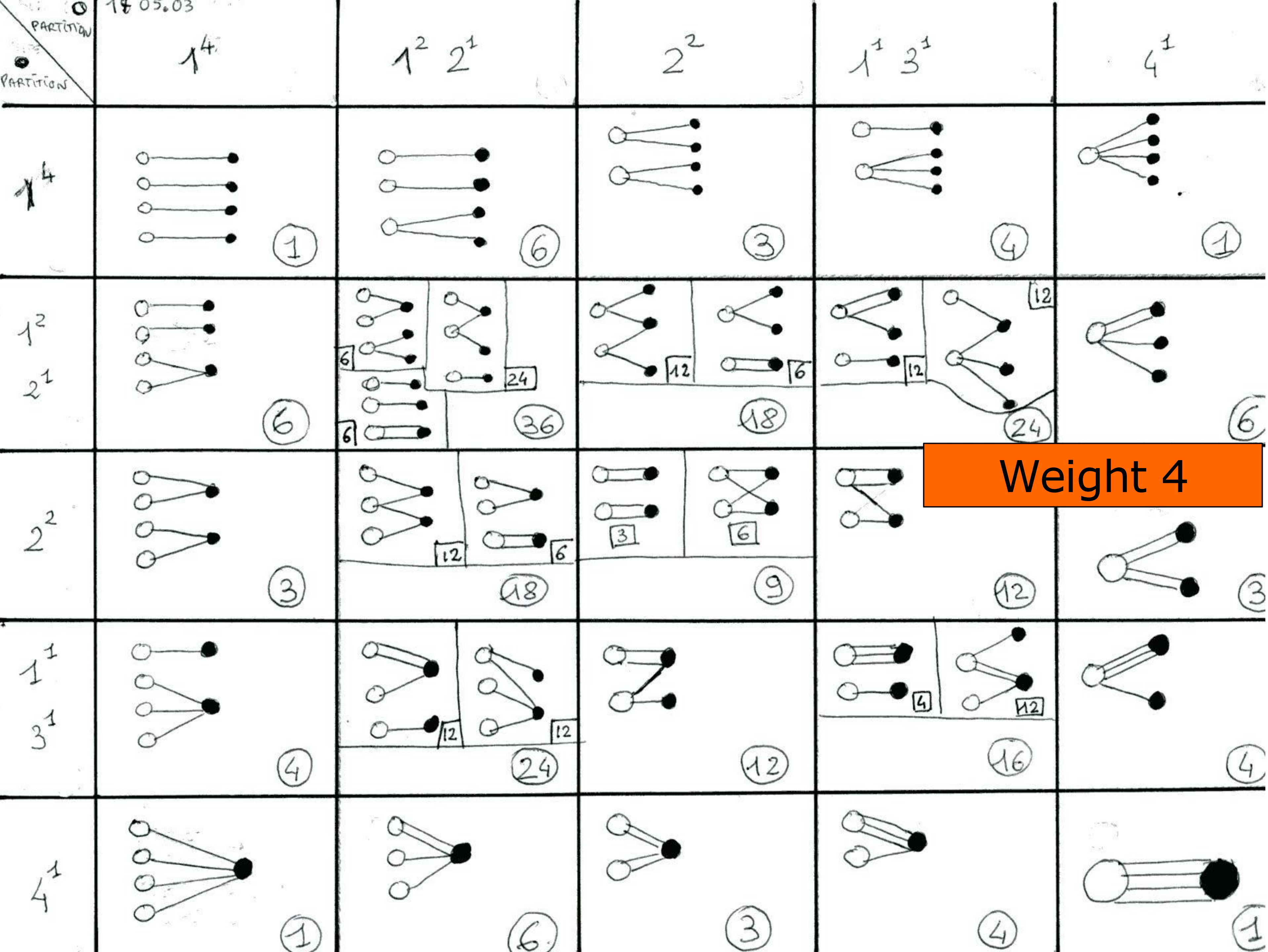
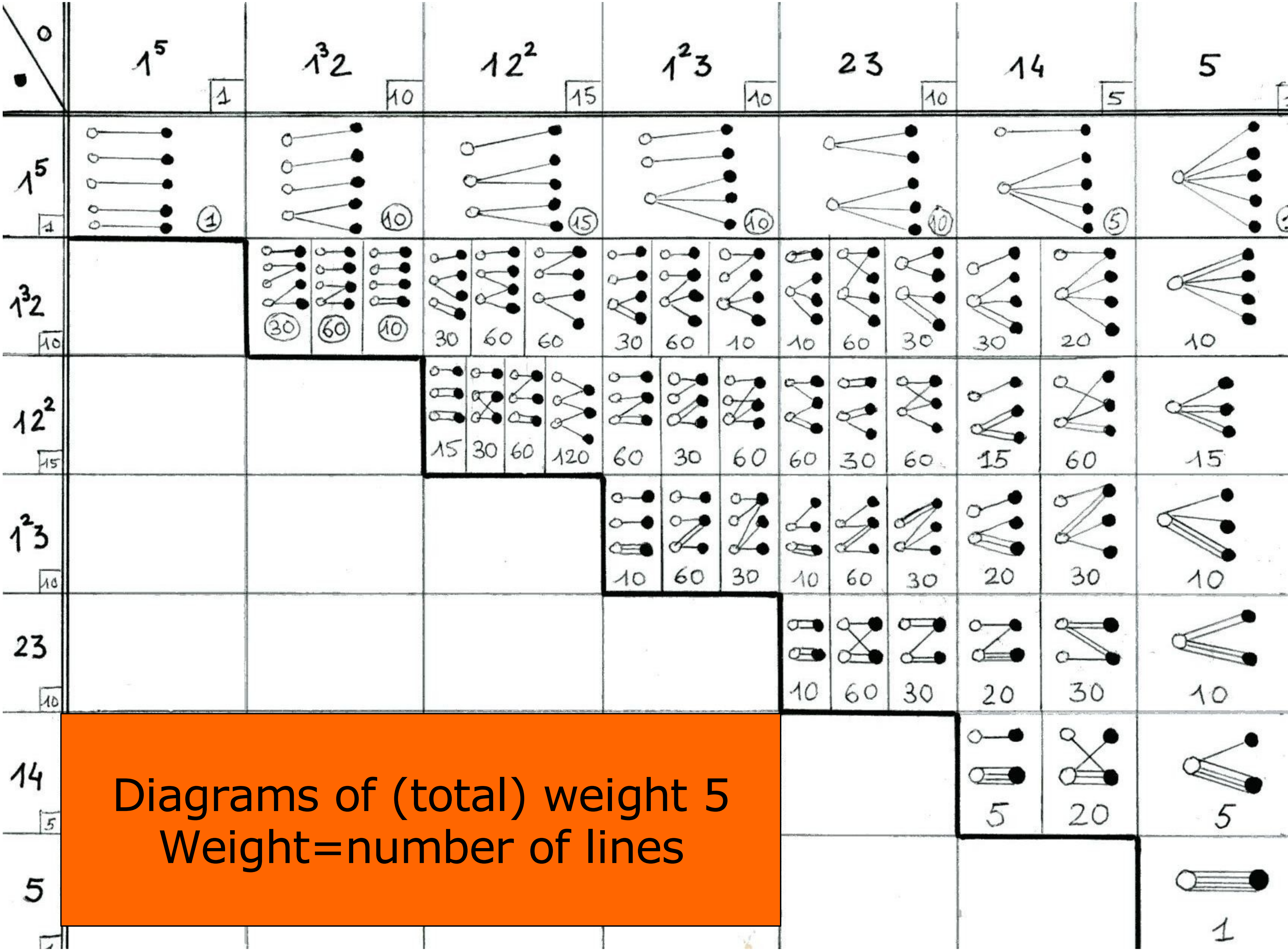



**Fig 1**. — *Diagram from $P_1$, $P_2$ (set partitions of $[1 \cdots 11]$).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for $P_1$ and white spots for $P_2$).*
*The incidence matrix corresponding to the diagram (as drawn) or these partitions is* $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. *But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*


18.05.03
PARTITION
PARTITION
$1^4$
$1^2 2^1$
$2^2$
$1^1 3^1$
$4^1$
1
6
3
4
1
6
6
24
6
36
12
6
18
12
12
24
6
12
6
18
3
6
9
12
3
4
12
12
24
12
4
12
16
4
1
6
3
4
1

Weight 4



Diagrams of (total) weight 5
Weight=number of lines

# Hopf algebra structure

$$(H,\mu,\Delta,1_H,\varepsilon,\alpha)$$

Satisfying the following axioms

- $(H,\mu,1_H)$ is an associative k-algebra with unit (here k will be a – commutative - field)
- $(H,\Delta,\varepsilon)$ is a coassociative k-coalgebra with counit
- $\Delta : H \to H\otimes H$ is a morphism of algebras
- $\alpha : H \to H$ is an anti-automorphism (the antipode) which is the inverse of Id for convolution.

Convolution is defined on End(H) by

$$\varphi \bullet \psi = \mu\, (\varphi \otimes \psi)\, \Delta$$

with this law End(H) is endowed with a structure of associative algebra with unit $1_H\varepsilon$.

First step: Defining the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d \qquad LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

(functions with finite supports on the set of diagrams). At this stage, we have a natural arrow *LDiag* → *Diag*.

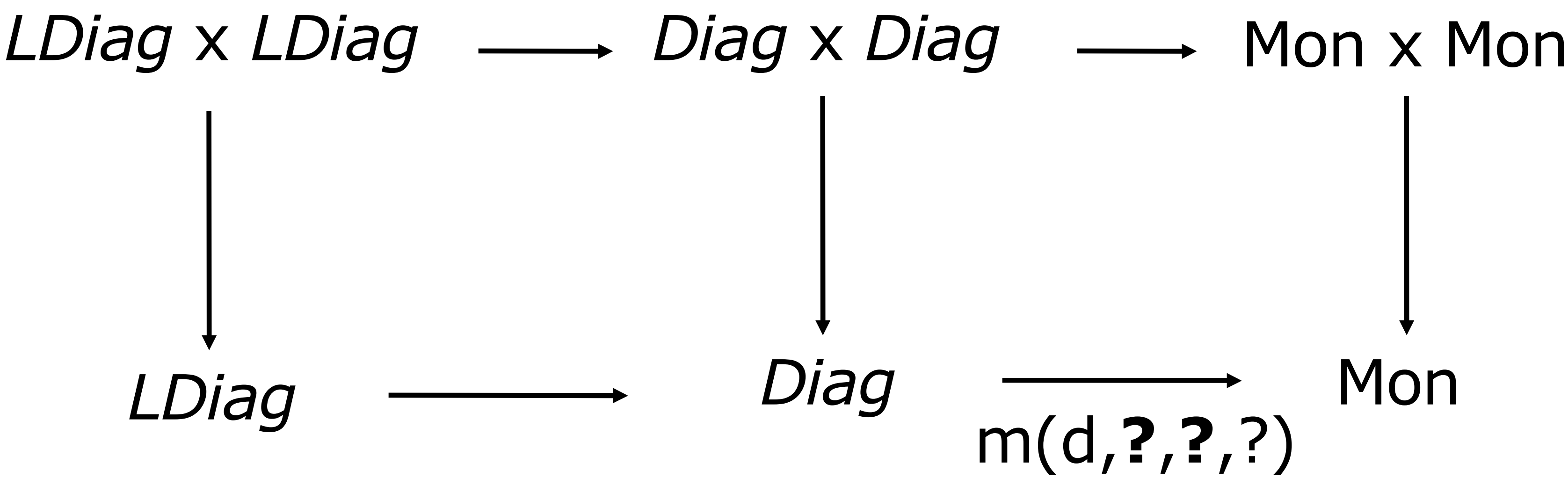
Second step: The product on *Ldiag* is just the concatenation of diagrams

$$d_1 \star d_2 = d_1 \; d_2$$

And, setting $m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\mathbf{V}^{\beta(d)} z^{|d|}$
one gets

$$m(d_1 * d_2,\mathbf{L},\mathbf{V},z) = m(d_1,\mathbf{L},\mathbf{V},z)\, m(d_2,\mathbf{L},\mathbf{V},z)$$

This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which, in turn, is compatible with the product of monomials.

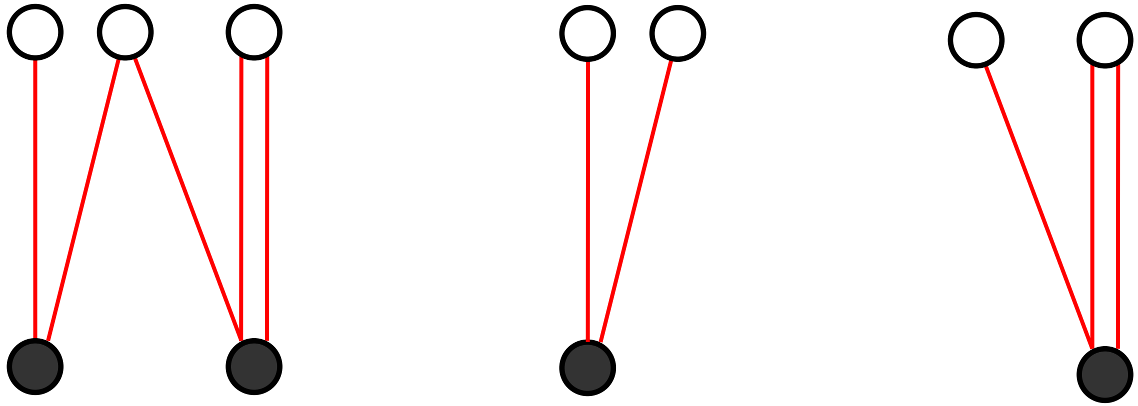


The coproduct needs to be compatible with m(d,**?**,**?**,?). One has two symmetric possibilities (black spots and white spots). The « black spots co-product » reads

$$\Delta_{BS}(d)=\sum d_I \otimes d_J$$

the sum being taken over all the decompositions, (I,J) of the Black Spots of d into two subsets.

For example, with the following diagrams d, $d_1$ and $d_2$



one has $\Delta_{BS}(d)=d\otimes\varnothing + \varnothing\otimes d + d_1\otimes d_2 + d_2\otimes d_1$

If we concentrate on the multiplicative structure of *Ldiag*, we remark that the objects are in one-to-one correspondence with the so-called packed matrices of NCSFVI (Hopf algebra MQSym), but the product of MQSym is given (w.r.t. a certain basis **MS**) according to the following example

$$\mathbf{MS}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{MS}_{[3\ 1]} =$$

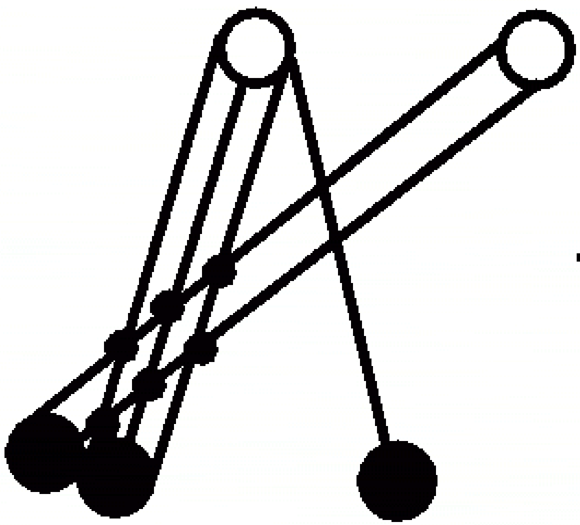
$$\mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 0 & 0 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

It is possible to (re)connect these Hopf algebras to MQSym and others of interest for physicists, by deforming the product with two parameters.
The double deformation goes as follows

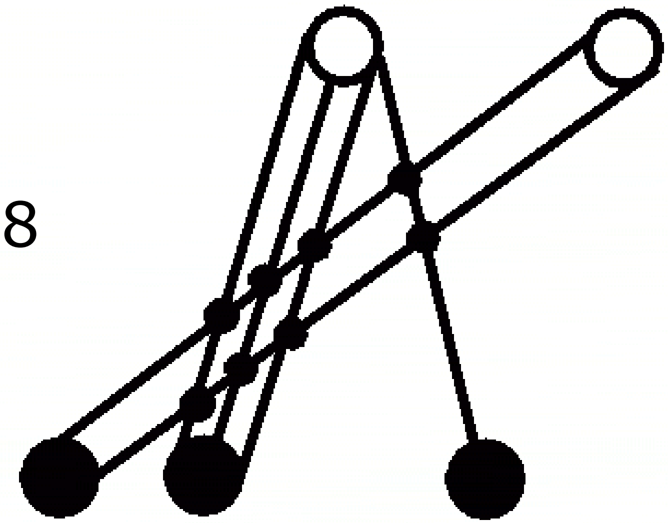
- Concatenate the diagrams
- Develop according to the rules :
  - Every crossing "pays" a $q_c$
  - Every node-stacking "pays" a $q_s$

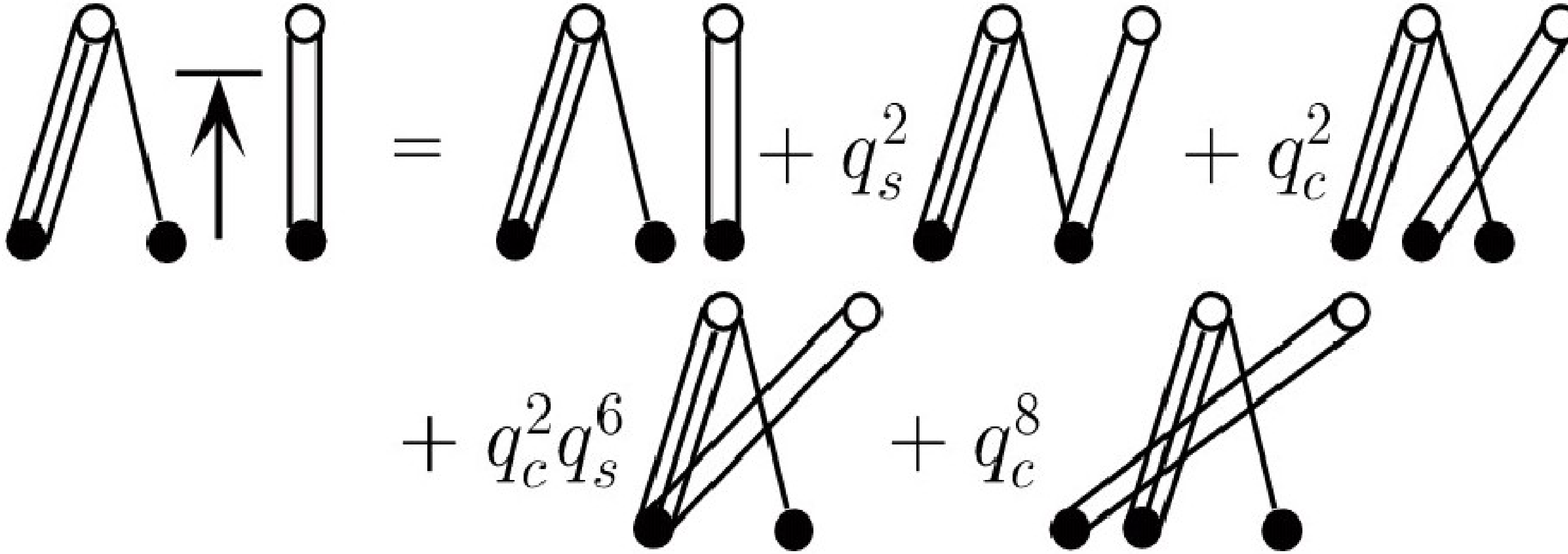
In the expansion, the weights are given by the intersection numbers.

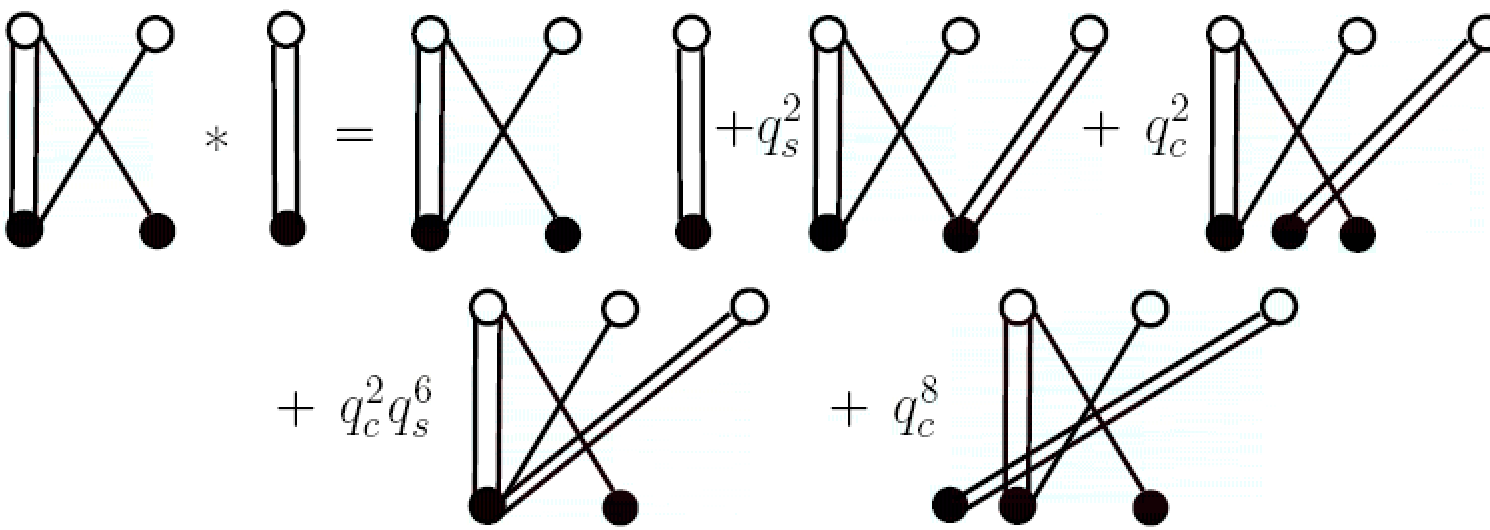
$q_c^{2} q_s^{6}$



$+q_c^{8}$




=
$+ q_s^2$
$+ q_c^2$
$+ q_c^2 q_s^6$
$+ q_c^8$


*
=
$+ q_s^2$
$+ q_c^2$
$+ q_c^2 q_s^6$
$+ q_c^8$

We could check that this law is associative (now three independent proofs). For example, direct computation reads

$$(au \uparrow bv) \uparrow cw = \left(a(u \uparrow bv) + q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v) + q^{|au||b|} b(au \uparrow v)\right) \uparrow cw$$

$$\left[a((u \uparrow bv) \uparrow cw) + q^{(|u|+|bv|)|c|} t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} ((u \uparrow bv) \uparrow w) + q^{(|au|+|bv|)|c|} c(a(u \uparrow bv) \uparrow w)\right]$$

$$\left[q^{|u||b|} t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|u||b|+(|u|+|v|)|c|} t^{|a||b|} t^{(|a|+|b|)|c|} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w\right.$$

$$\left. q^{|u||b|+(|au|+|bv|)|c|} t^{|a||b|} c\left(\left(\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)\right) \uparrow w\right)\right]$$

$$\left[q^{|au||b|} b((au \uparrow v) \uparrow cw) + q^{|au||b|+(|au|+|v|)|c|} t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w) + q^{|au||b|+(|au|+|bv|)|c|} c(b(au \uparrow v) \uparrow w\right.$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \Big(b(v \uparrow cw) + q^{|v||c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|}c(bv \uparrow w)\Big) =$$

$$\Big[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|}b(au \uparrow v \uparrow cw)\Big] +$$

$$\Big[q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) +$$

$$q^{|v||c|+|au|(|b|+|c|)}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\Big] +$$

$$\Big[q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w)\Big] \quad (3)$$

dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

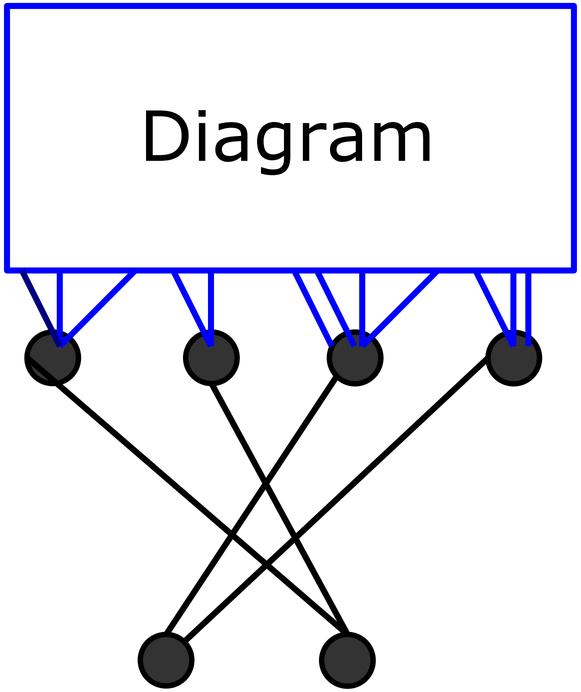
$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

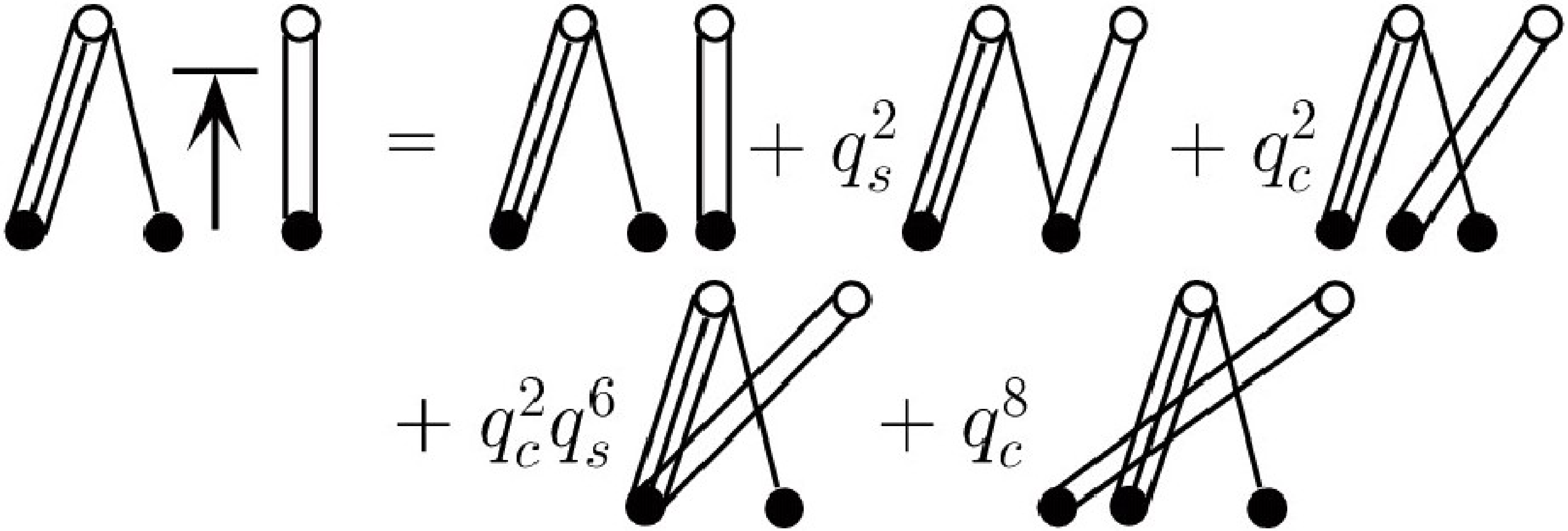
dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$

$$q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w) \quad (5)$$

This amounts to use  a monoidal action with two parameters. Associativity provides an identity in an algebra which acts on a diagram as the algebra of the sum of symmetric semigroups. Here, it is the symmetric semigroup which acts on the black spots





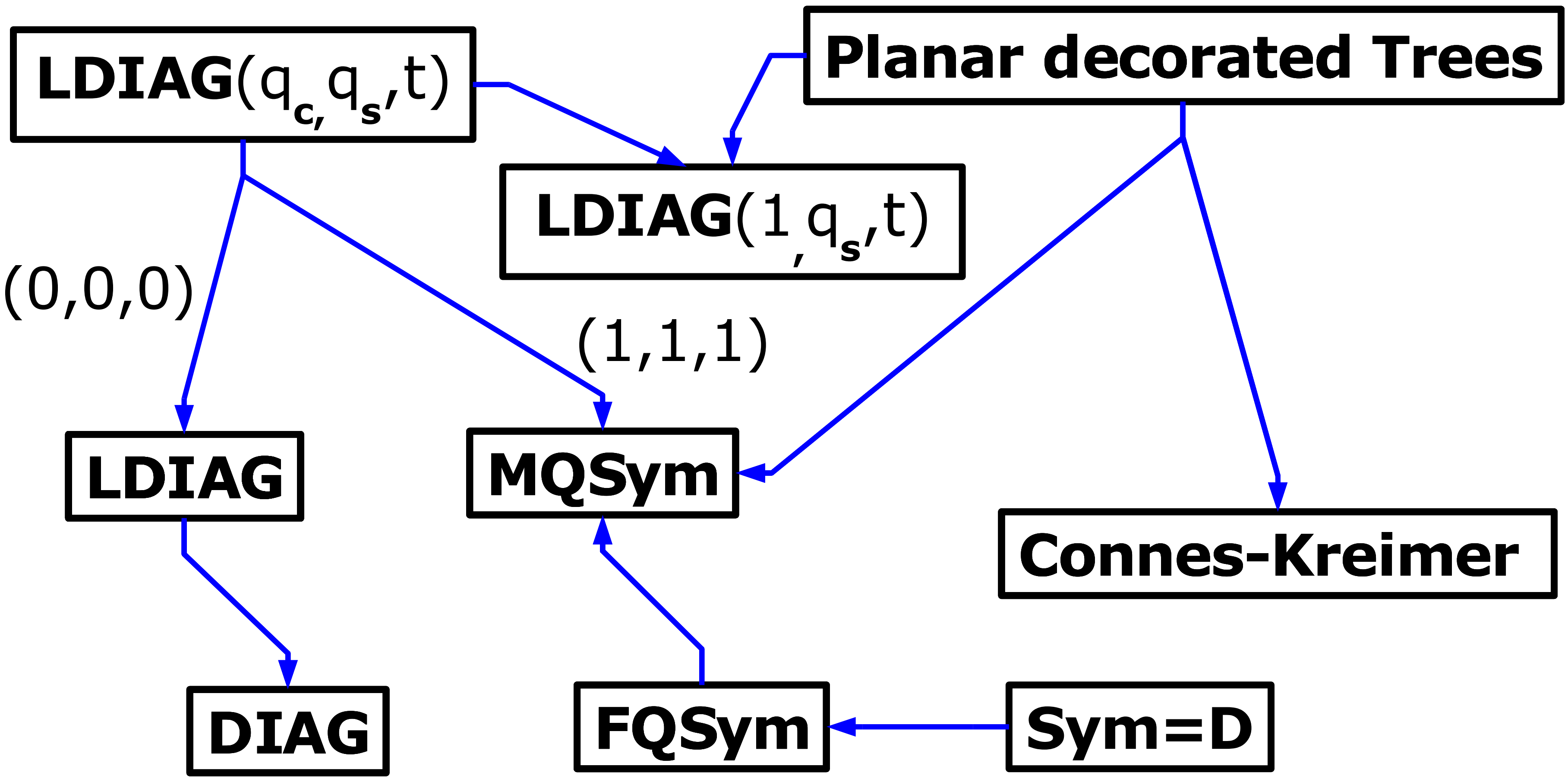

The labelled diagrams are in one to one correspondence with the packed matrices of MQSym and we can see easily that the product of the latter is obtained for

$$q_c = 1 = q_s$$

**Hopf interpolation** : One can see that the more intertwined the diagrams are the fewer connected components they have. This is the main argument to prove that LDIAG($q_{c,}$ $q_s$) is free on indecomposable diagrams. Therefore one can define a coproduct on these generators by

$$\Delta_t = (1\text{-}t)\Delta_{BS} + t\,\Delta_{MQSym}$$

$t \in \{0,1\}$ this is LDIAG($q_{c,}$ $q_s$,t).




Notes :

i) The arrow *Planar Dec. Trees* → *LDIAG(1,$q_s$,t)* is due to L. Foissy

ii) **LDIAG**, through a noncommutative alphabetic realization shows to be a bidendriform algebra (FPSAC07 paper by ParisXIII & Monge).

# Concluding remarks and future

*i)* The diagrams of diag are well suited to EGFs. What are the good data structures for other ones ?

*ii)* One can change the constants $V_k=1$ to a condition with level (i.e. $V_k=1$ for $k\leq N$ and $V_k= 0$ for $k>N$). We obtain then sub-Hopf algebras of the one constructed above. These can apply to the manipulation of partition functions of physical models including Free Boson Gas, Kerr model and Superfluidity.

# Concluding remarks and future (cont'd)

*iii) The deformation above is likely to be decomposed in two deformation processes ; twisting (already investigated in NCSFIII) and shifting (ongoing work with JGL and al.). Also, it could have a connection with other well known associators.*

*iv) The identity on the symmetric semigroup can be lifted to a more general monoid which takes into account the operations of concatenation and stacking which are so familiar to Computer Scientists (ongoing work in LIPN).*

# References

***Interactions with physicists :***

***1) G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,***

***One-parameter groups and combinatorial physics,***

***Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), November 2003. arXiv : quant-ph/0401126.***

***--> Formal groups and local groups of sequence transformations.***

***2) K A Penson, P Blasiak, G. H. E. Duchamp, A Horzela and A. I. Solomon,***

***Hierarchical Dobinski-type relations via substitution and the moment problem,***

***J. Phys. A: Math. Gen. 37 (2004)***

***--> Sheffer-type sequences, coherent states and Dobinski-type formulas***

***3) G H E Duchamp, P Blasiak, A Horzela, K A Penson and A. I. Solomon***

***Feynman graphs and related Hopf algebras***

***Journal of Physics: Conference Series, SSPCM'05, arXiv : cs.SC/0510041***

***--> Construction of LDIAG***

## *Interactions with mathematicians :*

*1) I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon,*

*Noncommutative symmetric functions.*

*Adv. in Math. 112 (1995), 218–348.*

*--> A realisation of Solomon's algebra of Descents as a Noncommutative analogue of Symmetric Functions*

*2) Duchamp G., Klyachko A., Krob D., Thibon J.-Y.,*

*Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras.*

*Discrete Mathematics and Theoretical Computer Science Vol. 2 (1998).*

*--> Construction of FQSYM and a model of q-deformation (for crossings)*

*3) Duchamp G., Reutenauer C.,*

*Un critère de rationalité provenant de la géométrie noncommutative.*

*Inventiones Math.*, 128, *613-622. (1997)*

*--> Application of rational calculus in Sweedler's duals.*

## *Algebraic Combinatorics :*

*1) Duchamp G., Hivert F., Thibon J.Y. :*

*Non commutative symmetric functions VI: Free quasi-symmetric functions and related algebras.*

*International Journal of Algebra and Computation Vol 12, No 5 (2002).*

*--> Construction of the Hopf algebra of Matrix Quasi-Symmetric Functions*

*2) Gérard H. E. Duchamp, Jean-Gabriel Luque, K. A. Penson, Christophe Tollu,*

*Free Quasi-Symmetric functions, product actions and Quantum Field Theory of Partitions.*

*Séminaire Lotharingien de Combinatoire 54A (2007).*

*--> Talk of Christophe Tollu : a new product on the algebra of Symmetric Functions and structure theorem*

*3) G. H. E. Duchamp a , J-G. Luque b , J-C. Novelli b , C. Tollu a , F. Toumazet,*

*Hopf algebras of diagrams*

**Formal Power Series and Algebraic Combinatorics, Tianjin, China 2007**

*--> A realisation of LDIAD(qc,qs,t) with proof of « bidendriformity ».*

## Automata Theory :


*1)) J. Berstel, C. Reutenauer,*

*Rational series and their languages.*

*EATCS Monographs on Theoretical Computer Science. Springer (1988).*

*--> Link between rational series and languages*

*2) M.P. Schützenberger,*

*On the definition of a family of automata.*

*Information and Control,* **4**, *(1961).*

*--> Fundation of Automata theory*

*3) S. Eilenberg,*

*Automata, languages and machines*

*Volumes A & B, Acad. Press*

*--> A systematic exposition of the theory of automata and transducers.*

*4) Duchamp G., Flouret M., Laugerotte \'E., Luque J-G.,*

*Direct and dual laws for automata with multiplicities, T.C.S.267, 105-120 (2001).*

Thank You