Dynamic combinatorics, complex systems and applications to physics

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Mathematics

Chaos Theory

Continuous & Discrete Modelisation

Image Processing

Computer Science

Complex Systems Complexity

Urban Dynamics

Business Banking

Computation Techniques

Decision Making

Physics

Artificial Intelligence

Mechatronics

Electronics

Adaptronics

Abstract ⟷ Applied

Combinatorics
(discrete mathematics)

Information
(computer sci.)

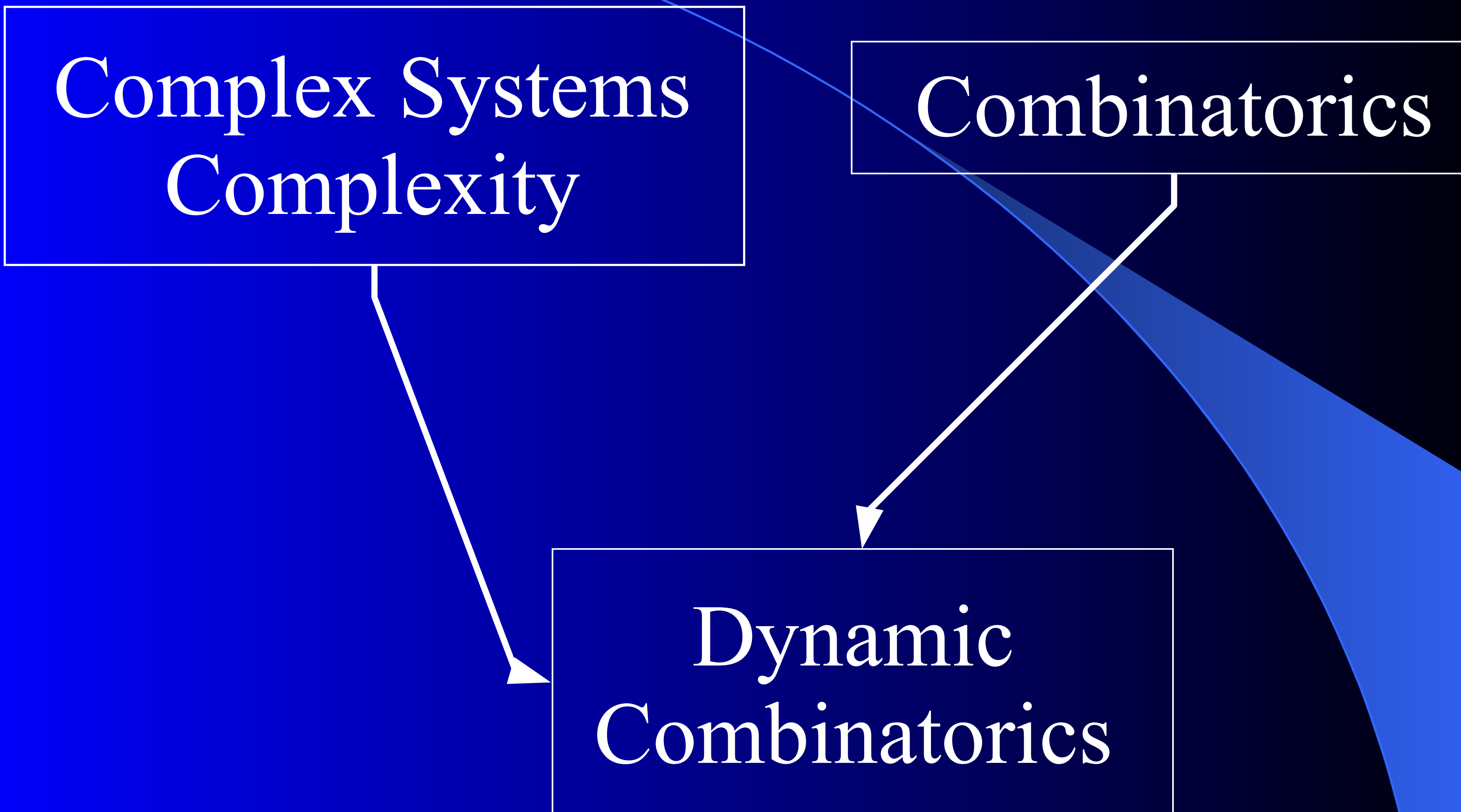
Physics
(classical/quant.)

Combinatorics
(discrete mathematics)

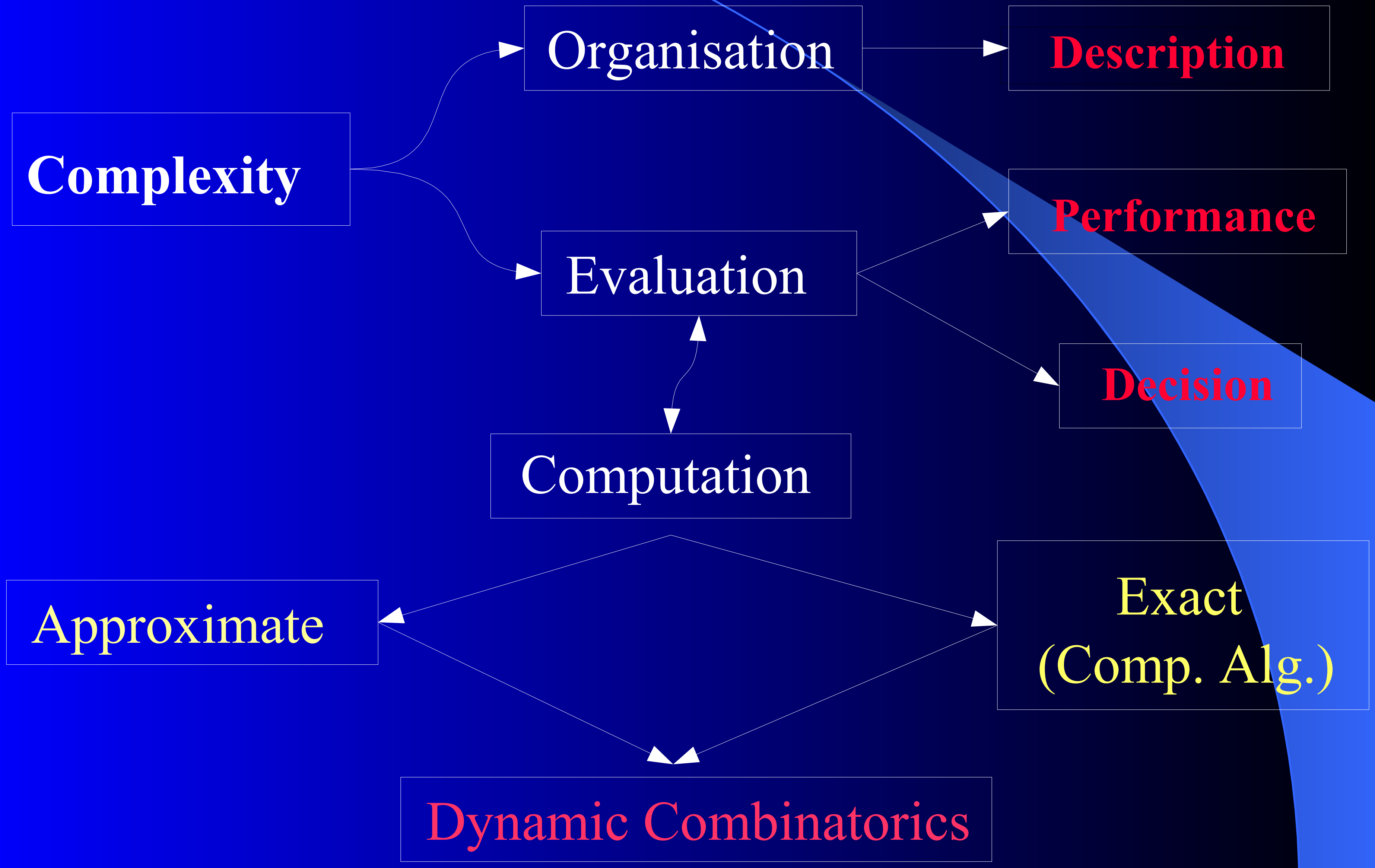
Complex
Systems

Information
(computer sci.)

Physics
(classical/quant.)


Complex Systems Complexity
Combinatorics
Dynamic Combinatorics

Problematics of Dynamic Combinatorics



What is the Legacy ?

Mainly:

- Data Structures
- Programs
- Theorems
- Computation rules
- Experiments
- Simulations

C o m b i n a t o r i c s

… on words

- Langages
- Theory of codes
- Automata
- Transition structures
- Grammars
- Transducers Rational and algebraic expressions
- …

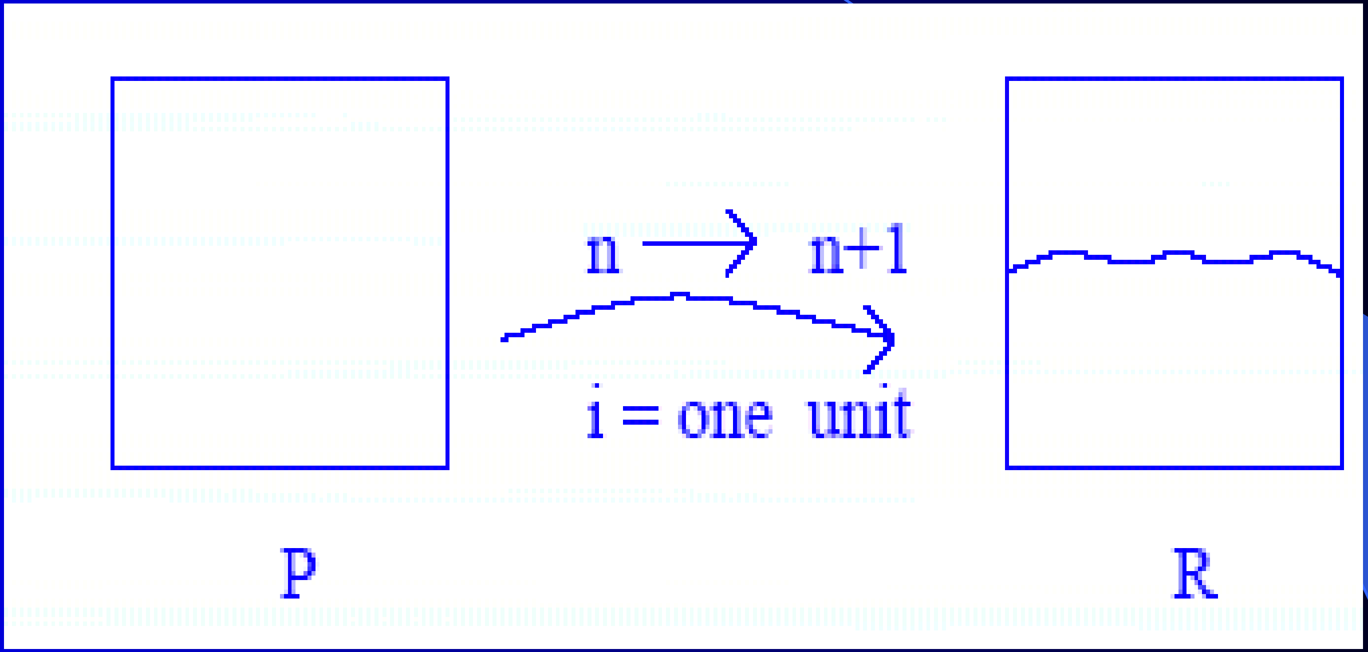
enumerative, analytic

- Polyominos
- Paths (Dyck,…)
- Configurations
- q-grammars
- Generating Functions
- Continued Fractions (mono, multivariate,.)
- Orthogonal Polynomials
- …

algebraic

- Non commutative Continued fractions
- Representations of groups and deformations
- Quantum Groups Functors
- Characters
- Special Functions
- …

A first example . . .


n
n+1
i = one unit
P
R

- $a_1 = 0$
- $a_{i+1} \leq a_i + 1$

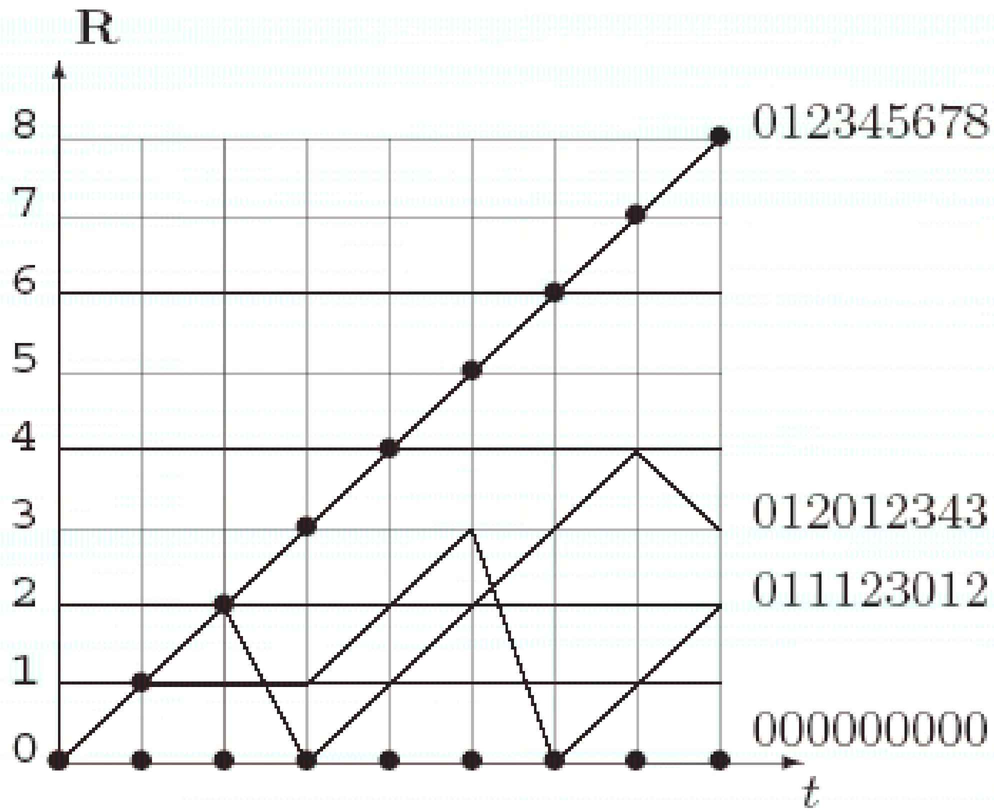
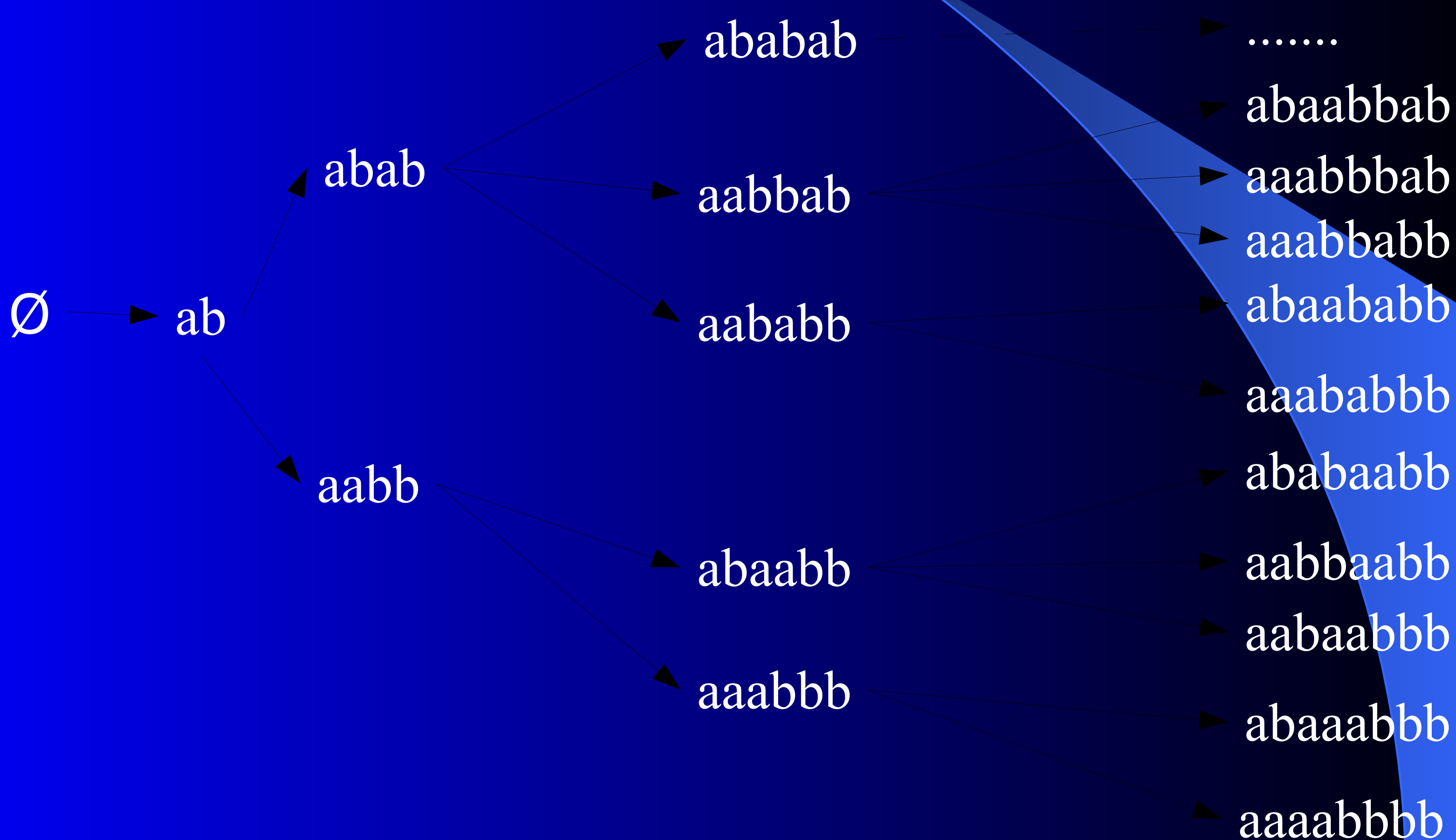
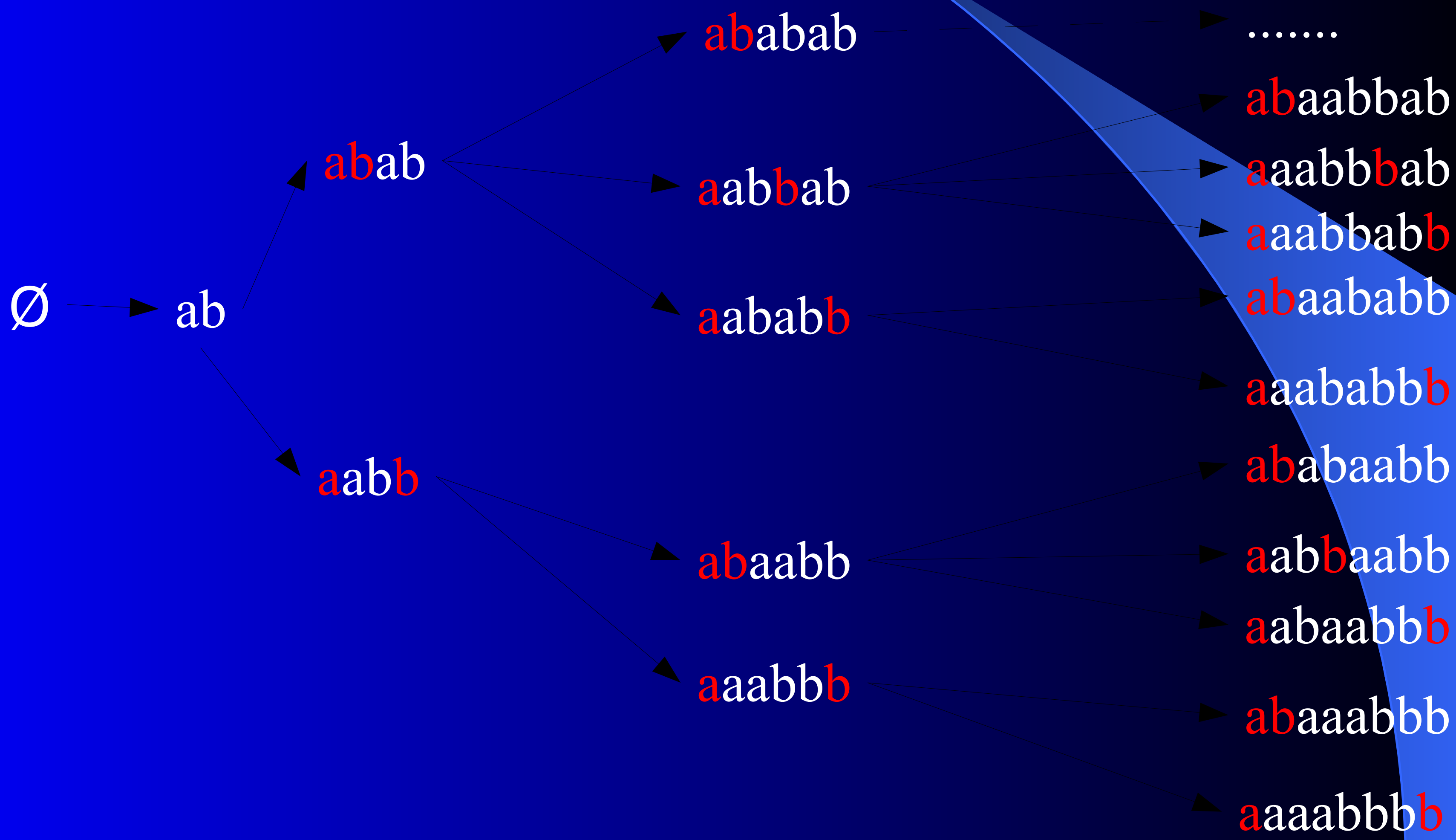


Figure 4.2: Maximal, minimal (dotted) and two intermediate trajectories. Their codes are on the right.

And compare this process with other data structures and codes, for example, with a=(b=)



Explanation: with a=(b=), every « good » word factorizes uniquely aubv with u,v « good » words



In each case, the graph is « graded » (in the first case by the length of the code in the second by half the length of the words) and the « sons » are labelled by a statistics (last entry for the code and number of factors for the words)

----> we can make the structures evolve in the same way.

The « coding theorem » is the following.

Theorem 1. *Let* $\Phi = \{\alpha\} \cup \Phi^+$ *be a data stucture with a bi-variate statistics*

$$\Phi \longrightarrow \mathbb{N} \times X \; ; \; p \mapsto s(p) = (n, k)$$

(X is set of labels) such that

$$s(\Phi^+) \subset \mathbb{N}^+ \times X$$

Set $\Phi_n = \{p \in \Phi | pr_1(s(p)) = n\}$ *and* $l(p) = pr_2(s(p))$ *"label of p". We suppose that there exists a function "return to the father"*

$$d : \Phi_{n+1} \longrightarrow \Phi_n$$

such that $p \mapsto (d(p), l(p))$ *is into. Then, for* $p \in \Phi_n$*, the code*

$$(l(p), l(d(p)), l(d^2(p)), \cdots l(d^{n-1}(p)))$$

is into.

A second (and related) example . . .

Thomas Schelling city segregation model in Urban Dynamics

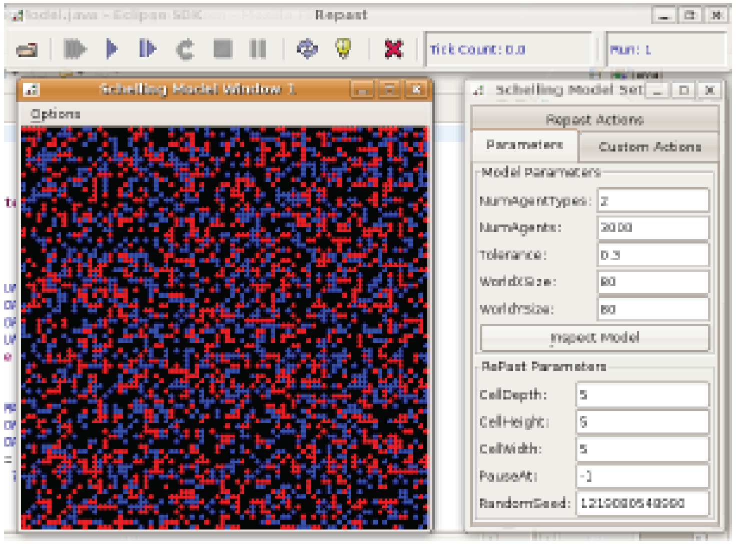
static data and are not adapted for the management or the analysis of dynamic systems, for their adaptive properties for exemple and more generally for their complexity . GIS need to be augmented to be able to be the support of the understanding of the geographical systems evolution. One of the innovative solution to deal with this goal is to mix GIS with multiagent systems (MAS) platform or with swarm intelligence algorithms.

(to be completed)

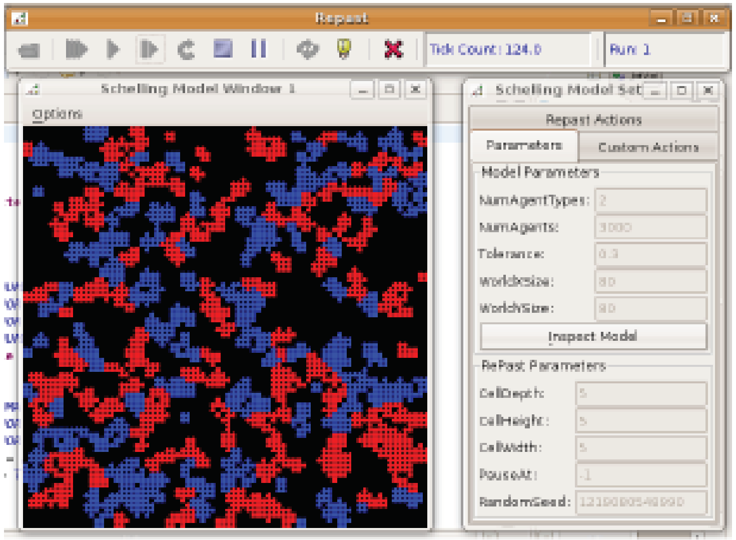
FIRST CASE STUDY: SCHELLING MODEL EXTENSION IN ORDER TO MODEL SELF-ORGANIZATION BY MEANING OF MULTI-CRITERIA SYSTEM SEPARATION

Thomas Schelling's city segregation model illustrates how spatial organizations can emerge from local rules, concerning the spatial distribution of people which belong to different classes. In this model, people can move, depending on their own satisfaction to have neighbours of their own class. Based on this model, a city can be highly segregated even if people have only a mild preference for living among people similar to them.

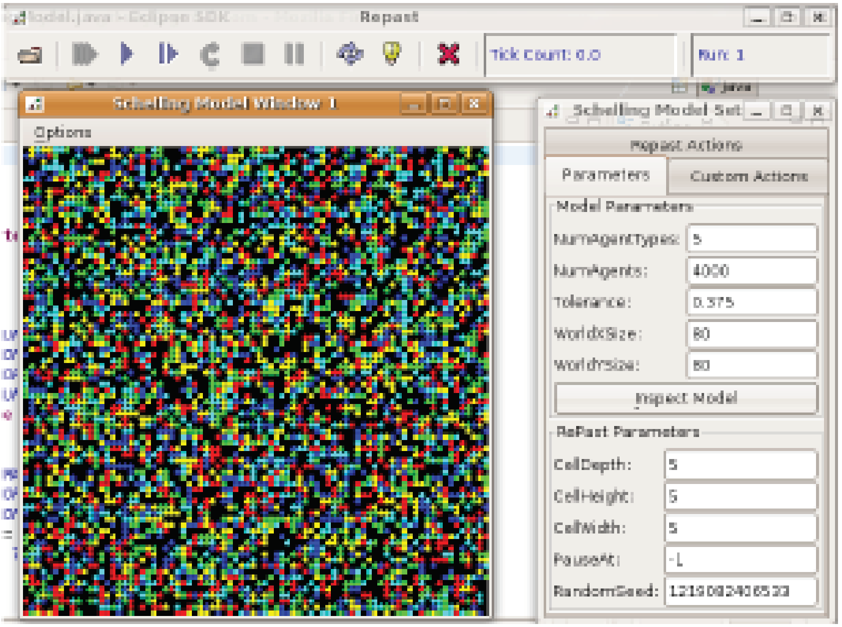
In this model, each person is an agent placed on a 2D grid (in his original presentation, a chessboard was used by Thomas Schelling). Each case can be considered like a house where the agent lives. Each agent cares about the class of his immediate neighbours who are the occupants of the abutting squares of the chessboard. Each



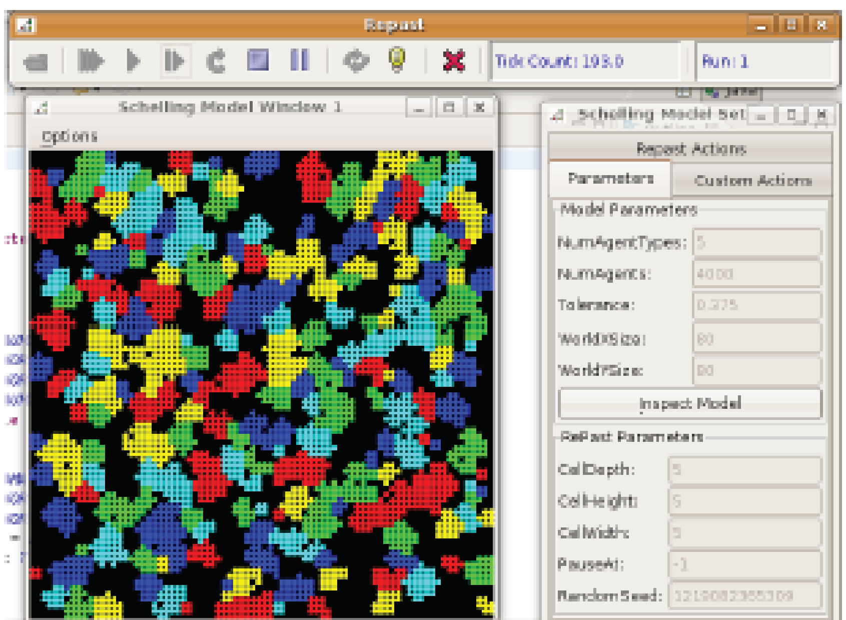
(a) Initial situation



(b) Stable situation after 124 iterations



(a) Initial situation



(b) Final Stable situation for a tolerence rate = 0.375

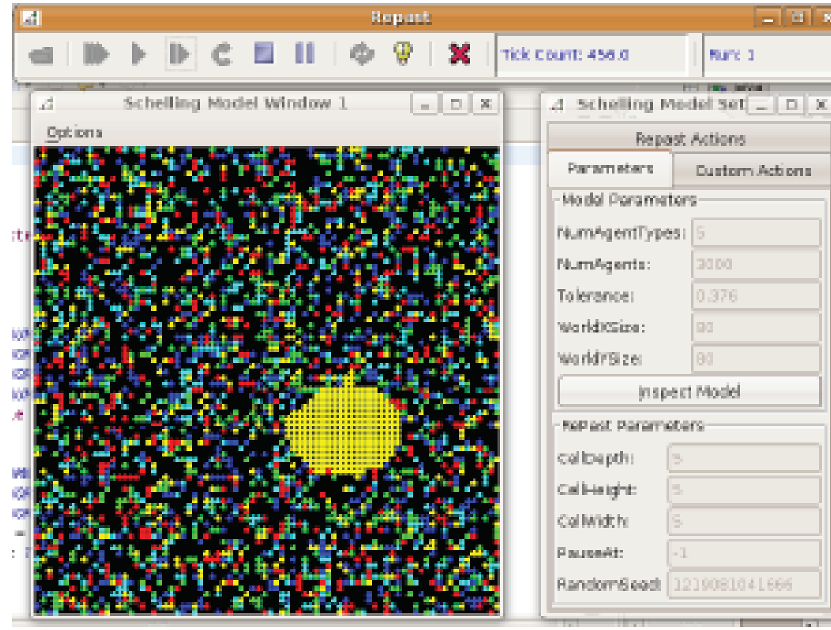
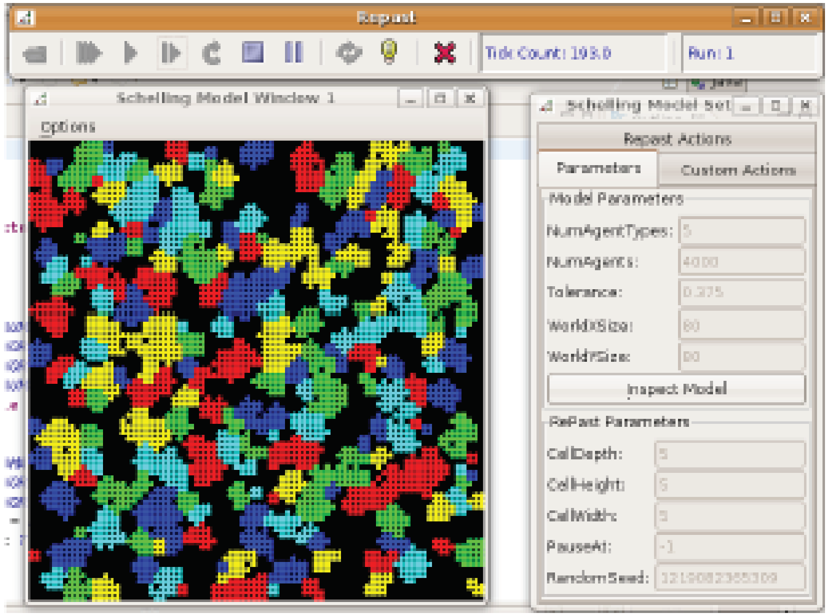


Figure 3: Singular situation for 5 populations seggregation Schelling model with density = 0.47 and with tolerance rate = 0.376

organization and structural interaction between the system and its components occur. In figure 4, we concentrate on the emergence of organizational systems from geographical systems. The continuous dynamical development of the organization feed-back on the geographical system which contains the organization components and their environment.



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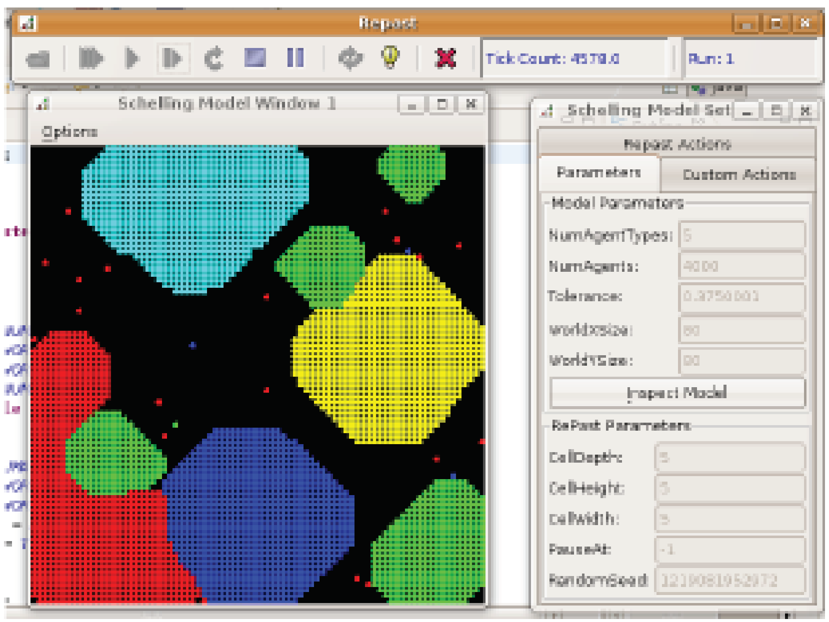


(c) Final Stable situation for a tolerence rate = 0.3750001

Critical Thresholds !

In order to modelize in an equidistributed way, one must use a « tree » representation. Here, the grading is the number of inhabitants and the « return » operator is the deletion of the image of « n » (if n is the number of inhabitants) as well as « n » itself (i. e., one withdraw (n,f(n)) to the graph of « f », the moving mapping).

The « moving mapping » is a generalized derangement (one never moves from a place to itself !). If m(n,k) is the number of moving mappings for n inhabitants into n+k lodgings, one gets

$$m(n,k)=m(n,k-1)+n.\ m(n-1,k)$$

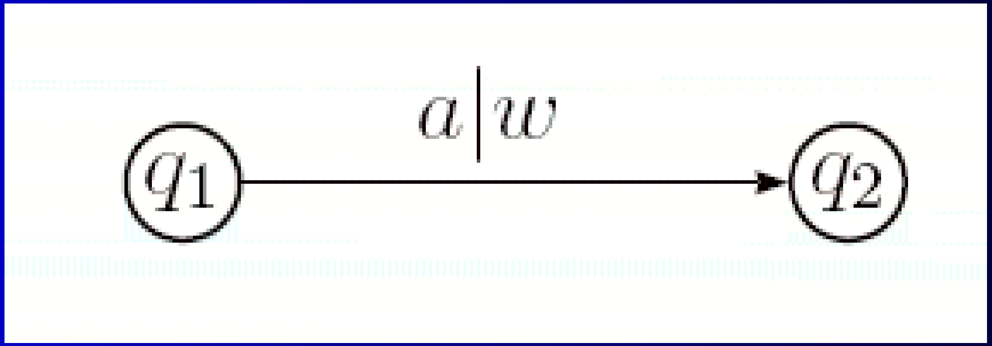
A third example . . .

Let us define in full

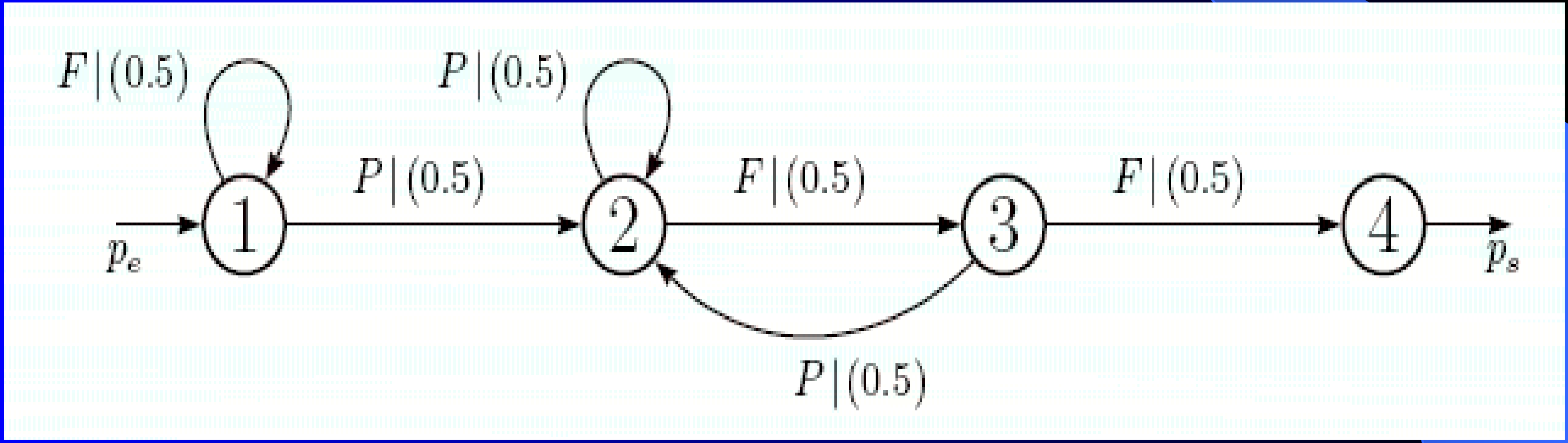
Definition (transition structure) : It is a graph (finite of infinite) with its arcs marked with pairs

(command letter | coefficient)

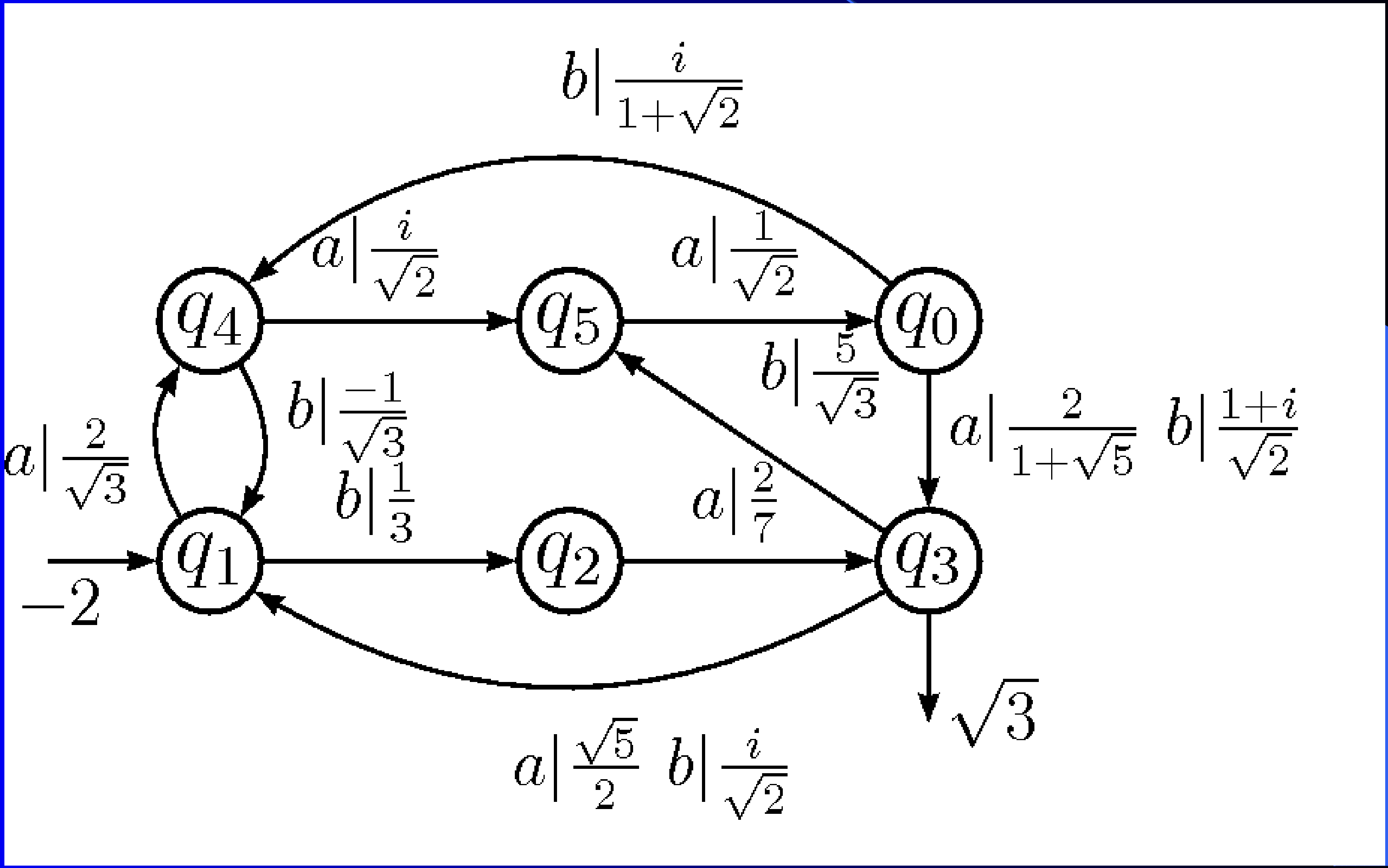
Examples : Prisoner's dilemma, Markov chains, classical engineering.



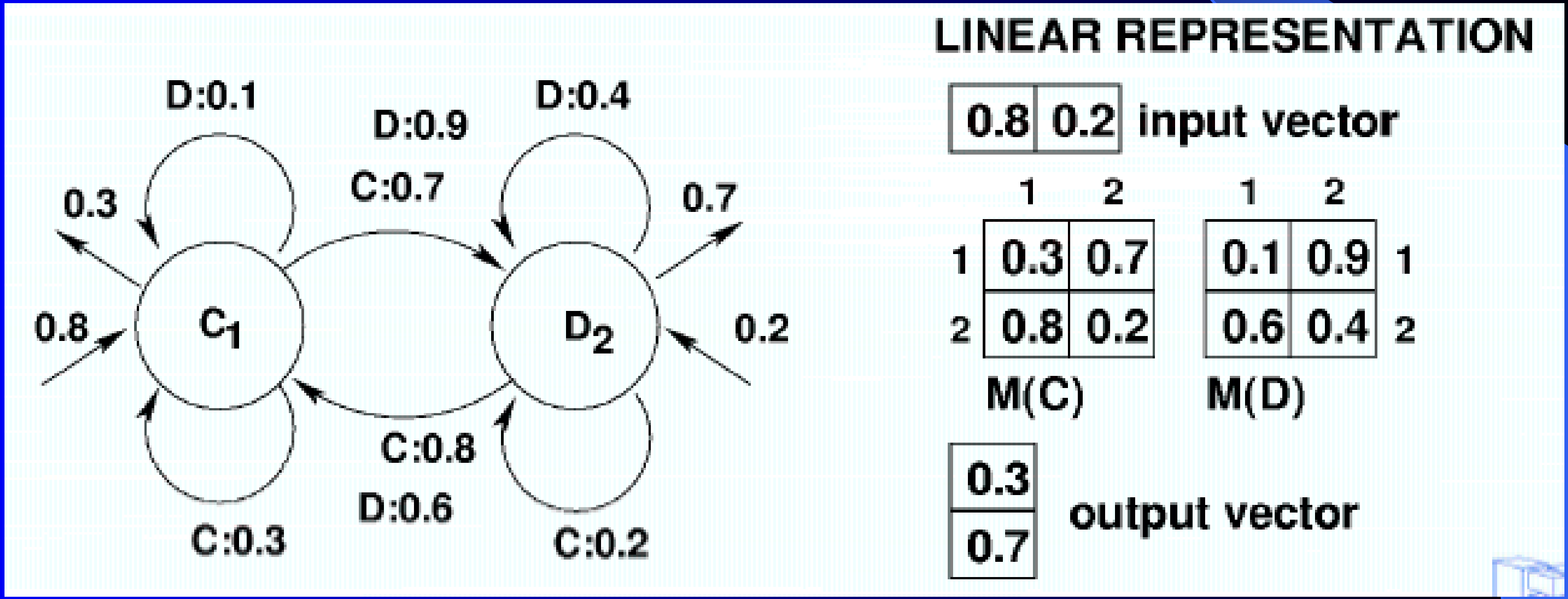
Example : A Markov chain generated by a game.



Example : An automaton generated by arbitrary transition coefficients (real, complex or taken in an arbitrary semiring).



Example of Probabilistic Automaton



Behaviour of an Automaton and how to compute it effectively

An automaton is a machine which takes a string (sequence of letters) and returns a value.

This value is computed as follows :

1) The weight of a path is the product of the weights (or coefficients) of its edges

2) The label of a path is the product (concatenation) of the labels of its edges

Behaviour ... (cont'd)

3) The behaviour between two states « r,s » w.r.t. A word « w » is the product of

3a) the ingoing coefficient of the first state (here « r ») by

3b) the sum of the weights of the paths going from « r » to « s » with label « w » by

3c) the outgoing coefficient of the second state (here « s »)

Behaviour ... (cont'd)

4) The behaviour of the automaton under consideration w.r.t. a word « w » is then the sum of all the behaviours of the automaton between two states « r,s » for all possible pairs of states.

Behaviour ... (cont'd)

There is a simple formula using the linear representation. The behaviour of an automaton with linear representation (I,M,T) is the product

$$IM(w)T$$

where M(w) is the canonical exention of M to the strings.

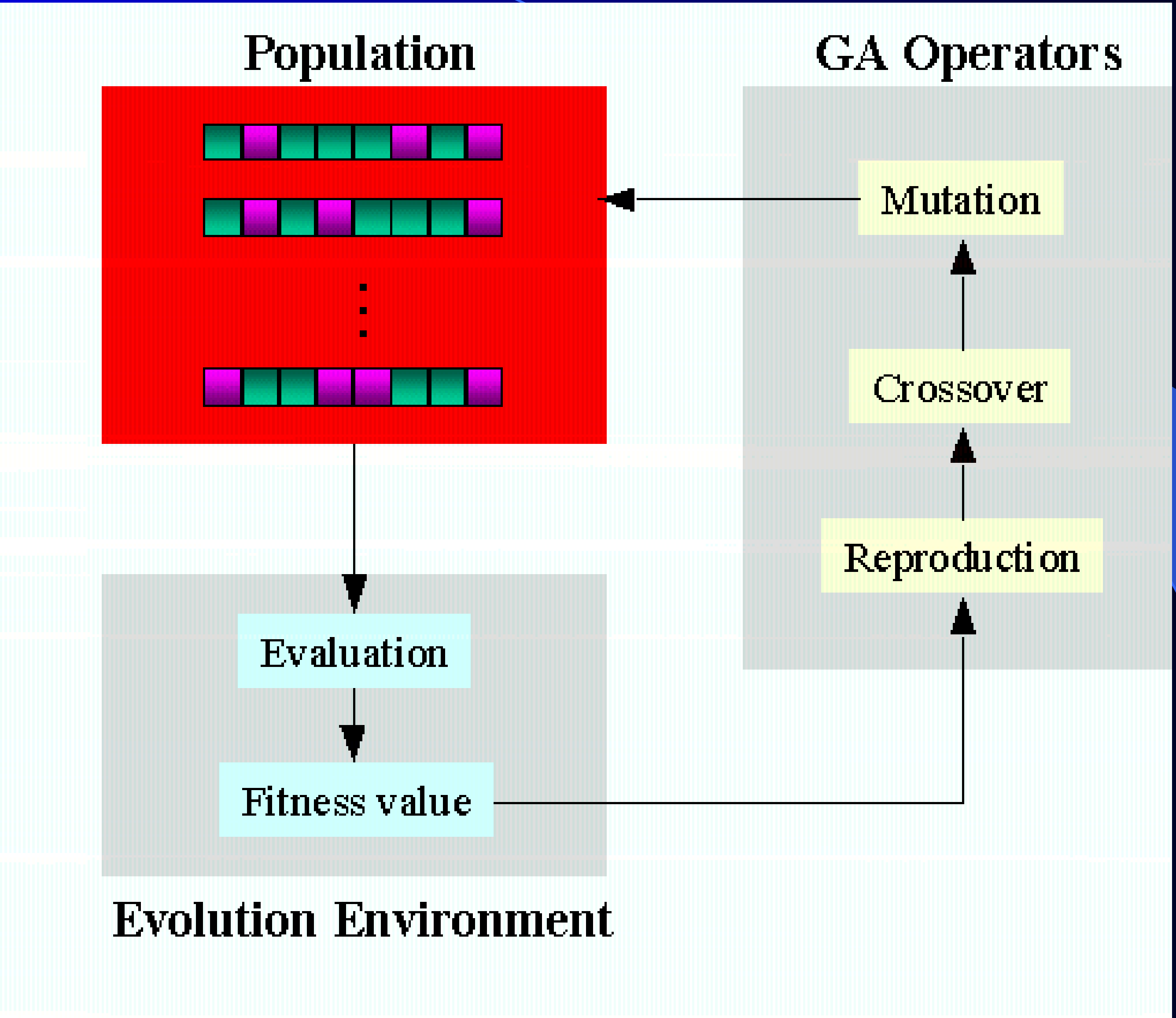
$$M(a_1 a_2 \ldots a_n) = M(a_1)M(a_2)\ldots M(a_n)$$

Behaviour … (end)

The behaviour, as a function on words belongs to the rational class. If time permits, we will return to its complete calculation as a rational expression and the problem of its algorithmic evaluation by means of special cancellation operators. Linear representations can also be used to compute distances between automata.

Example -> use of genetic algorithms to control indirect (set of) parameters : in what follows, the spectrum of a matrix.

Genetic algorithms : general pattern



Genetic Algorithm Evolution Flow

Genetic algorithms : implementation

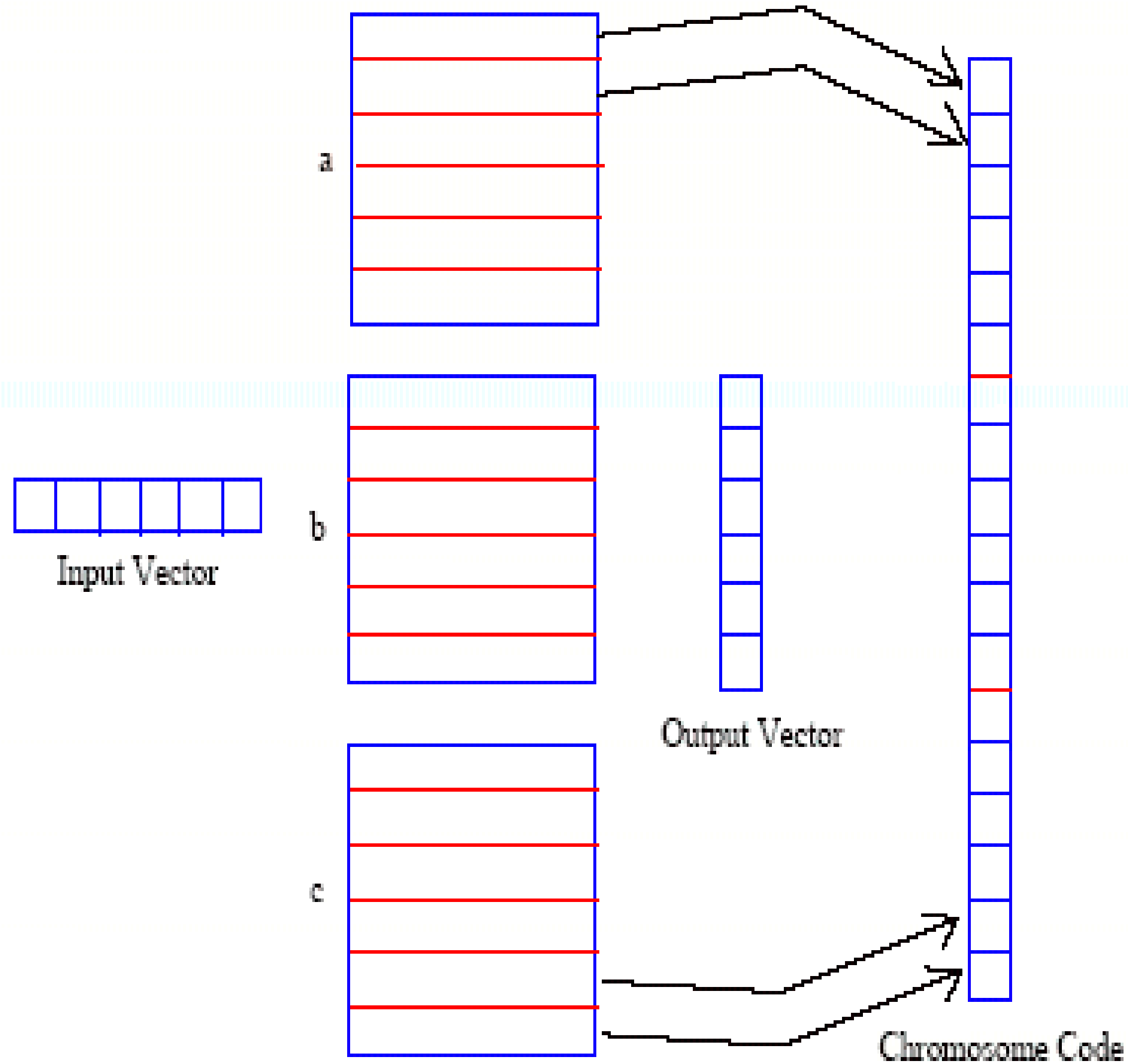


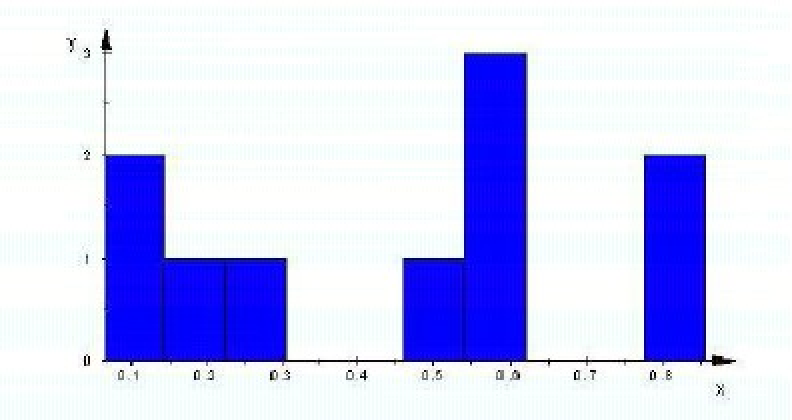
Figure 4.13: Chromosome code

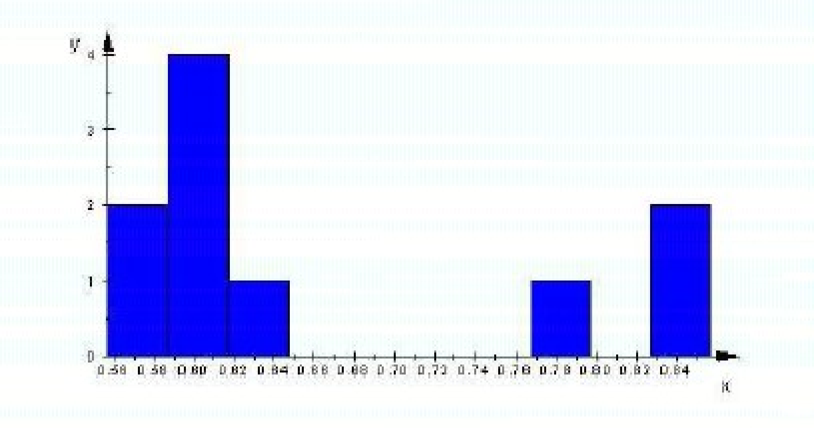
Genetic algorithms : implementation

Below, the results of an experiment aiming to control the second greatest eigenvalue of the transfer matrix of a population of probabilistic automata.

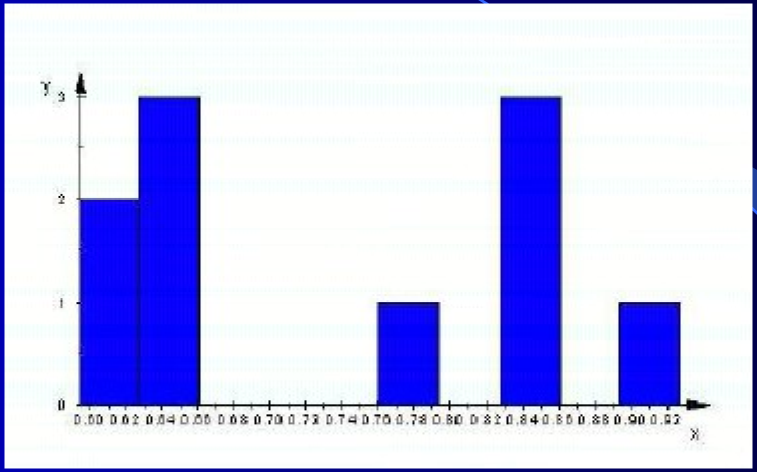
• The fitness function of each automaton corresponds to the second greatest eigenvalue (in module).
The first being, of course, of value 1.

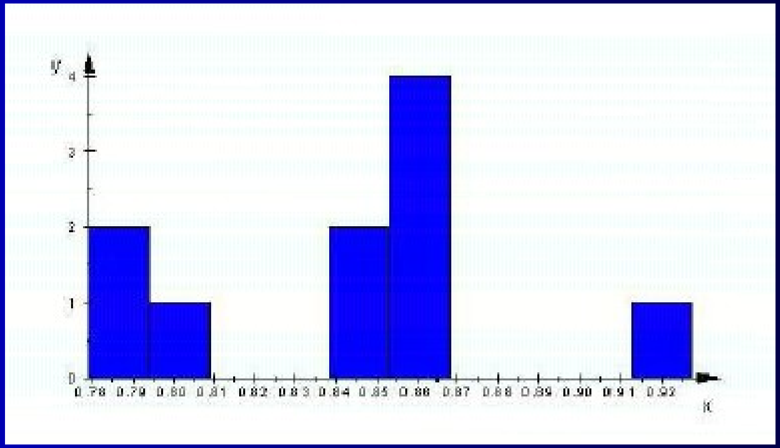
Genetic algorithms ; results



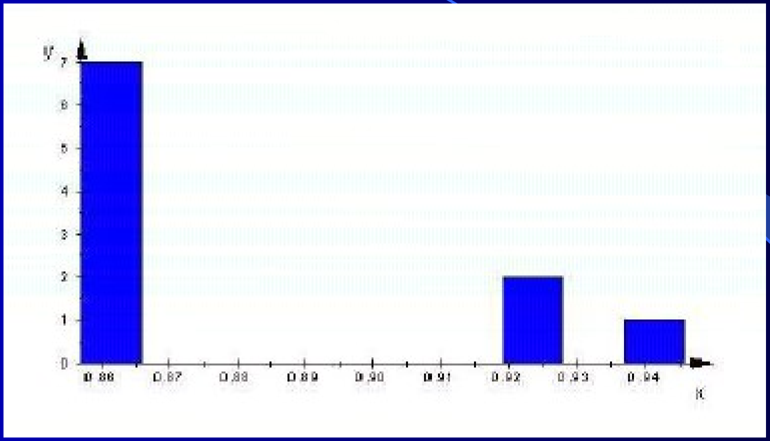


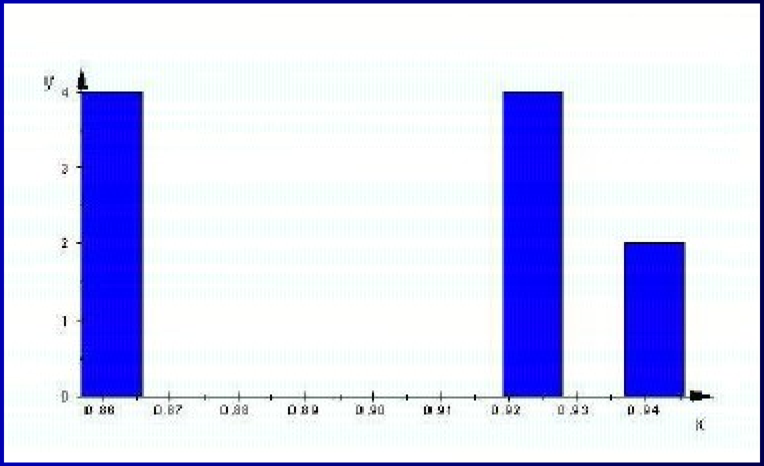
Genetic algorithms ; results



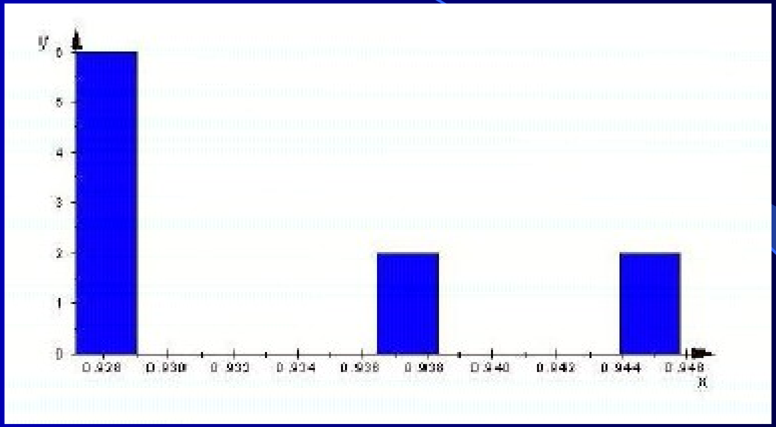


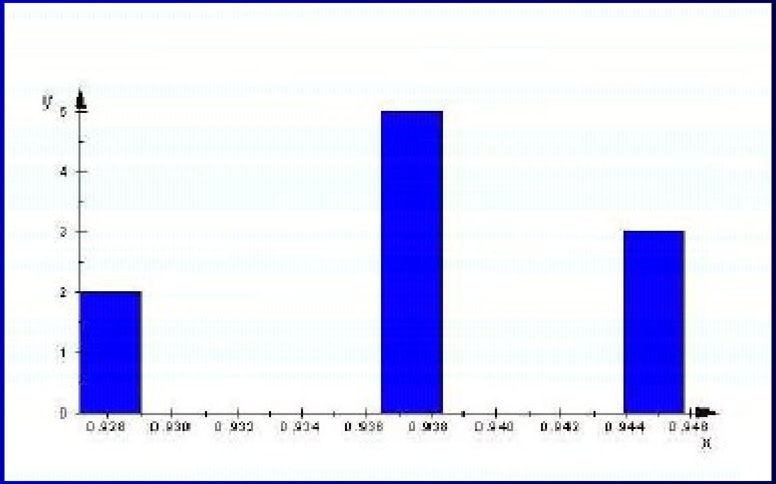
Genetic algorithms ; results



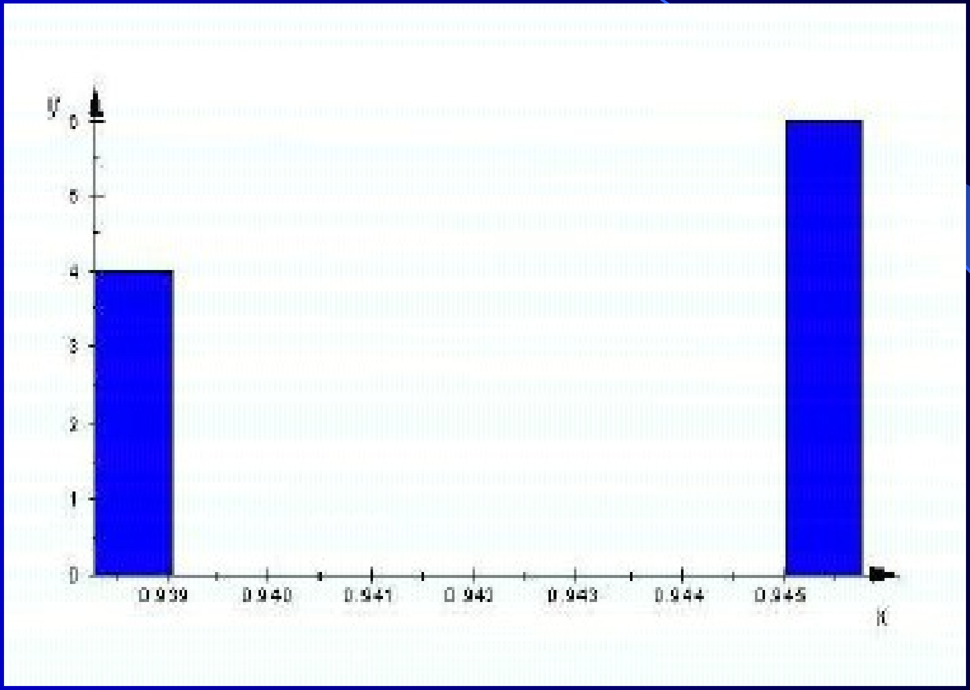


Genetic algorithms ; results



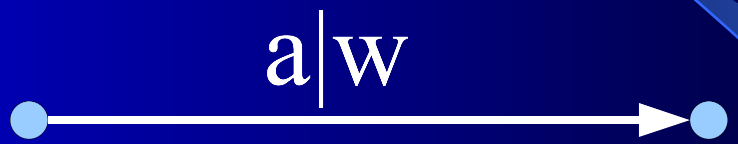


Genetic algorithms ; results



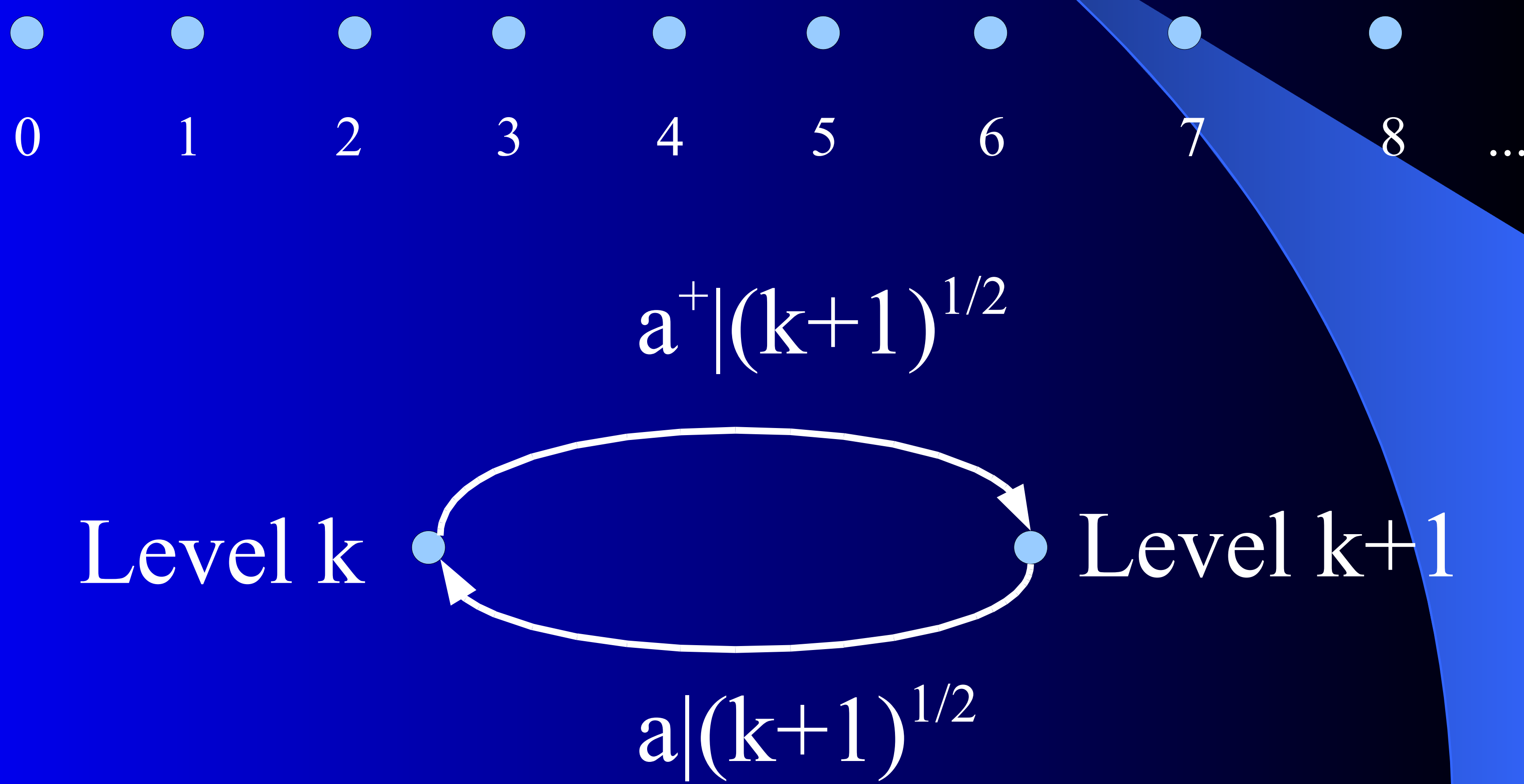
Final result : the population is rendered homogeneous

General transition systems



- Automata (finite number of edges)
- Sweedler's duals (physics, finite number of states)
- Representations
- Level systems (Quantum Physics)
- Markov chains (prob. automata when finite)

Example in Physics :



The (classical, for bosons) normal ordering problem goes as follows.

- Weyl (two-dimensional) algebra defined as

$$<a^+, a ; [a, a^+] = 1>$$

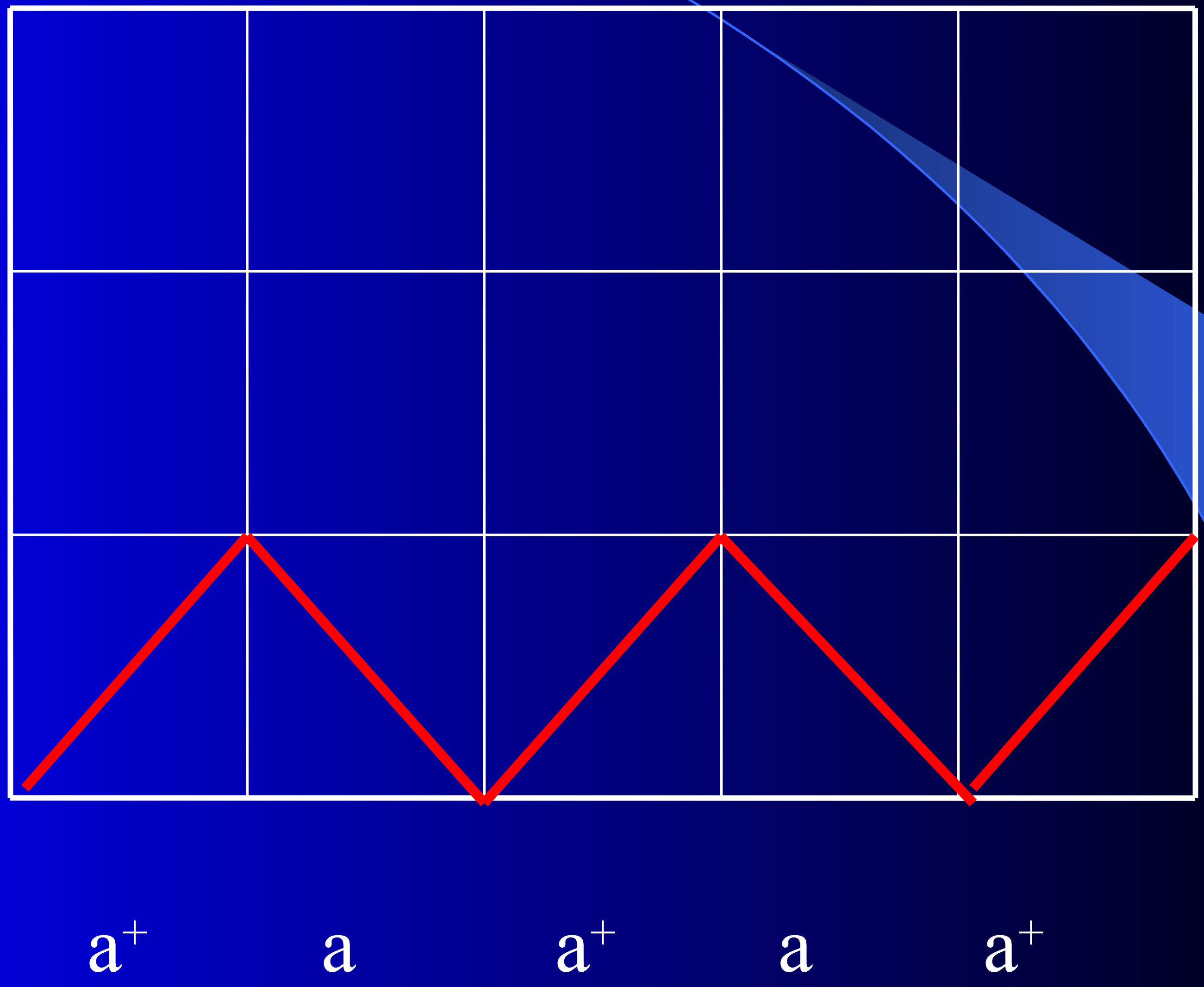
- Known to have no (faithful) representation by bounded operators in a Banach space.

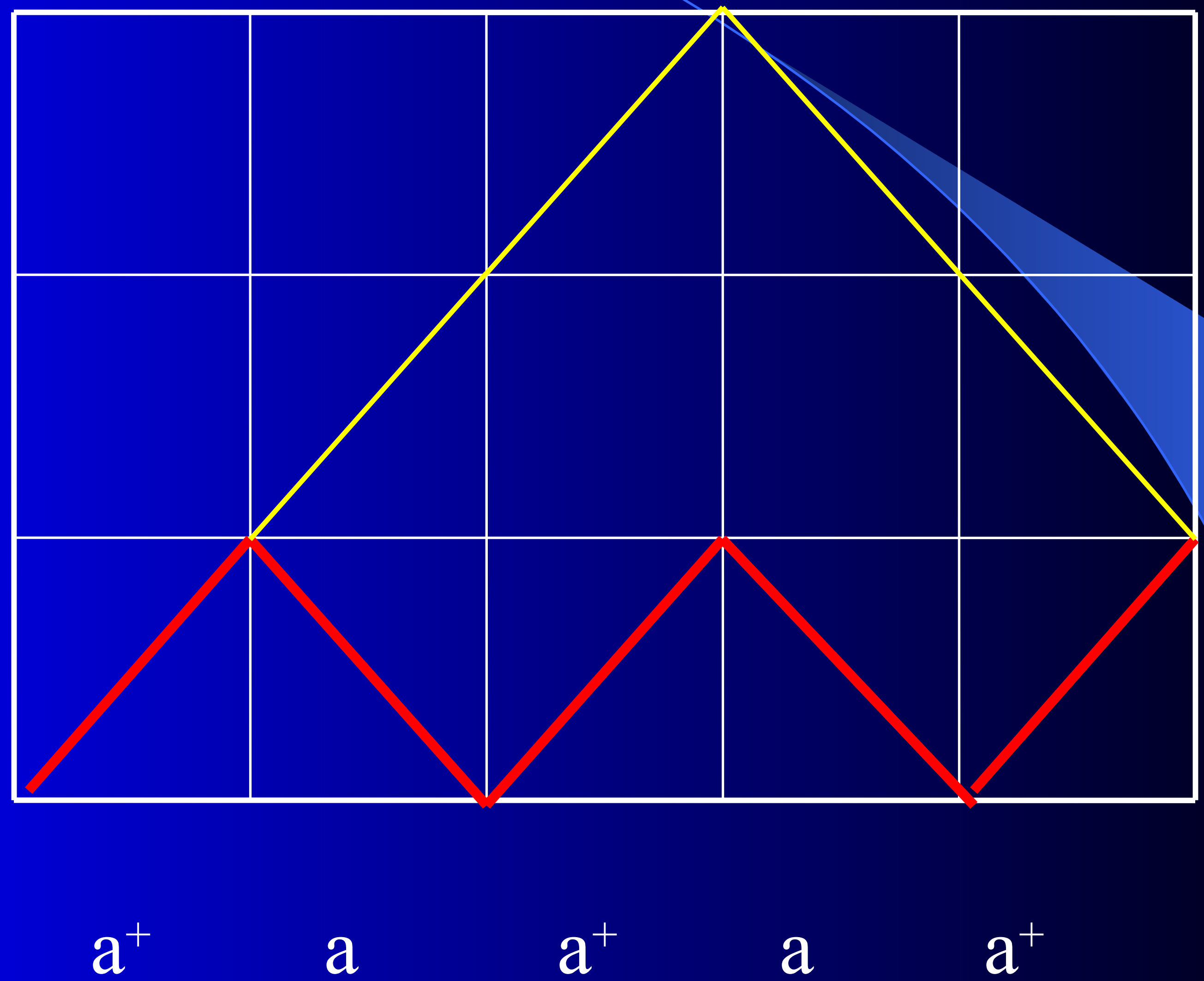
There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

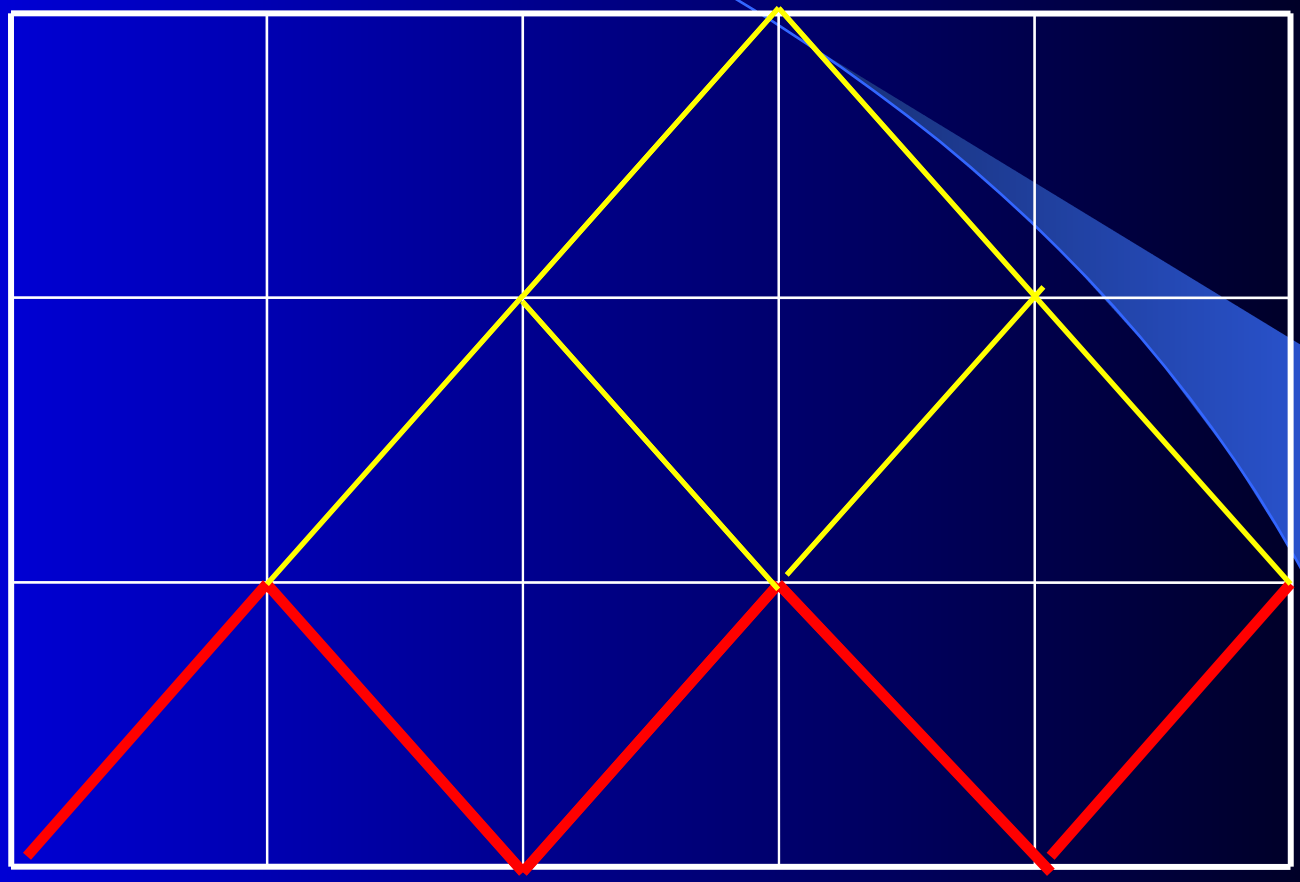
$a \rightarrow d/dx$; $a^+ \rightarrow x$

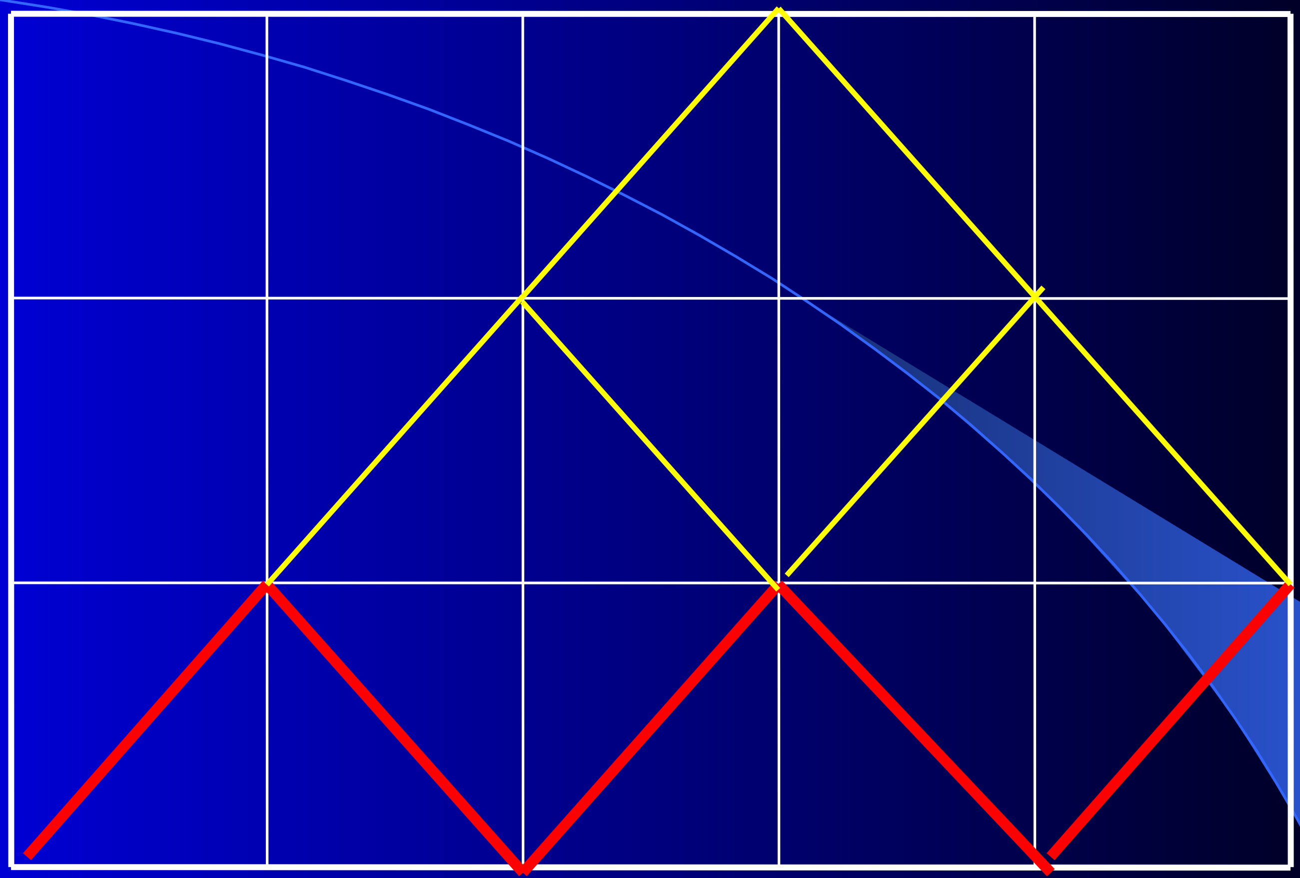
where a has degree -1 and a^+ has degree 1.

Example with $\Omega = a^{+} a \; a^{+} a \; a^{+}$




a^+
a
a^+
a
a^+


a^+
a
a^+
a
a^+



$a^+ \quad a \quad a^+ \quad a \quad a^+$

$$a^+aa^+aa^+ = 1\ a^+a^+a^+aa + 3\ a^+a^+a + 1\ a^+$$

Through Bargmann-Fock representation

$$a \to d/dx \; ; \; a^{+} \to x$$

Operators who have only one annihilation have exponentials who act as one-parameter groups of substitutions.

One can thus use computer algebra to determine their generating function.

For example, with

$$\Omega = a^{+2} a \, a^{+} + a^{+} a \, a^{+2}$$

the computation reads

Maple 9 - [Untitled (1) - [Server 1]]

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One parameter group by f(v(u(x)+λ)); v is reciprocal of u

```
> T1(lambda,x):=(-4*(-1/(4*x^2)+lambda))^(-1/2);
```

$$T1(\lambda, x) := \frac{1}{\sqrt{\frac{1}{x^2} - 4\lambda}}$$

We suppose x>0

```
> T1:=(lambda,x) -> x/((1-4*lambda*x^2)^(1/2));
```

$$T1 := (\lambda, x) \to \frac{x}{\sqrt{1 - 4\lambda x^2}}$$

Checking the tangent vector

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))  -T1(lambda1+lambda2,x)  );
```

$$0$$

Time: 0.2s Bytes: 1.94M Available: 1.33G

démarrer Microsoft PowerPoint ... Maple 9 - [Untitled (1... Le Robert & Collins FR 14:25

And the action of exp(λ ω) on [f(x)] is

$$U_\lambda(f) = x^{-\frac{3}{2}} f(s_\lambda(x)).(s_\lambda(x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

which explains the prefactor. Again, we can check by computation that the composition of (U_λ s) amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of Eq1 (slide 15) we must have

$$\exp(\lambda\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k \left(\frac{d}{dx}\right)^k$$

So, using this new technique, one can compute easily the coefficients of the matrix giving the normal forms.

Maple 9 - [calc1.mws - [Server 1]]

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```
> g1:=exp(y*x*exp(x));d1:=taylor(g1,x=0,7);
```

$$g1 := e^{(y\,x\,e^{x})}$$

$$d1 := 1 + y\,x + \left(y + \frac{1}{2}y^2\right)x^2 + \left(\frac{1}{2}y + y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{1}{6}y + y^2 + \frac{1}{2}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{24}y + \frac{2}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{6}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{1}{120}y + \frac{1}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{3}y^4 + \frac{1}{24}y^5 + \frac{1}{720}y^6\right)x^6 + O(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 \end{bmatrix}$$

<u>For these one-parameter groups and conjugates of vector fields</u>

G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,

One-parameter groups and combinatorial physics,

Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003. arXiv : quant-ph/0401126.

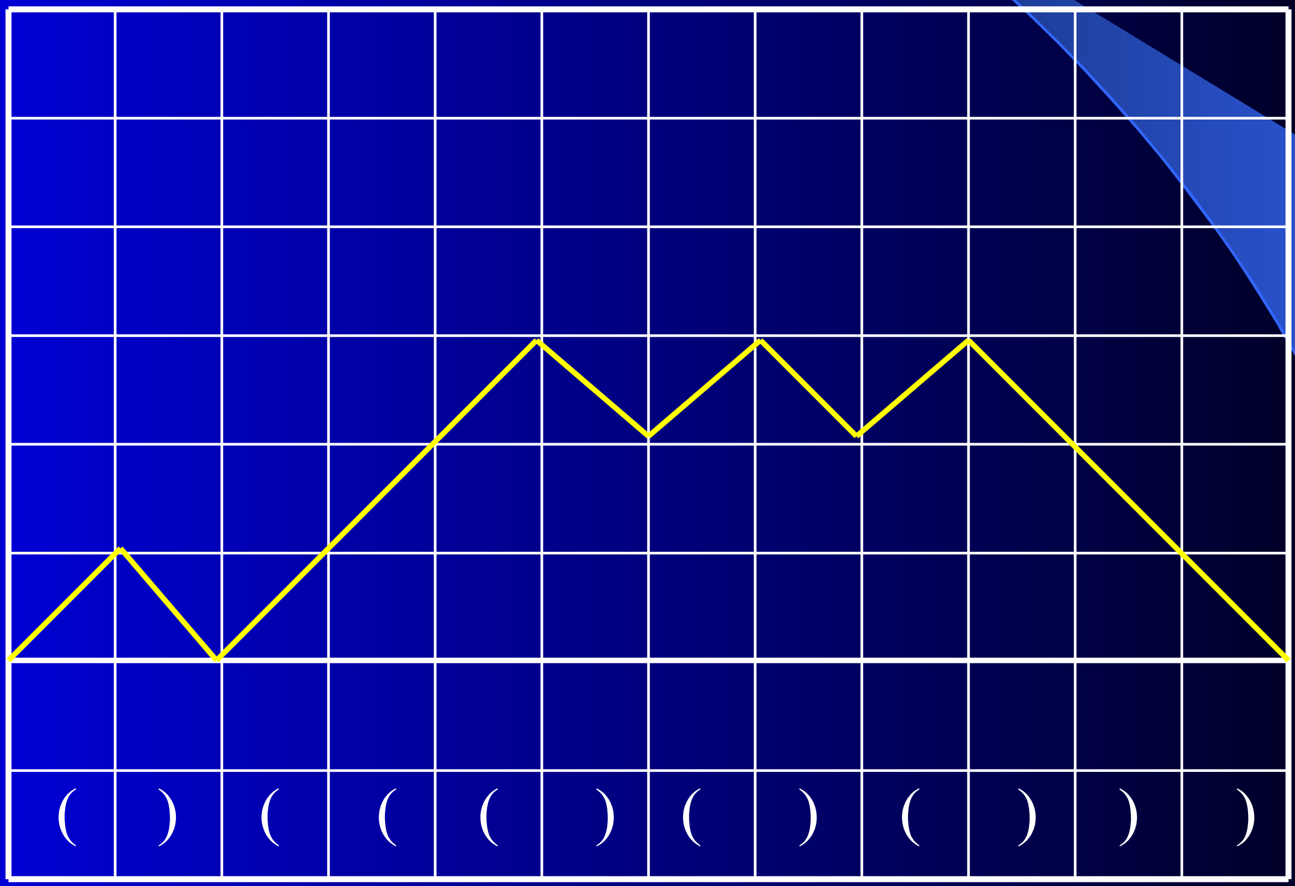
<u>For the Sheffer-type sequences and coherent states</u>

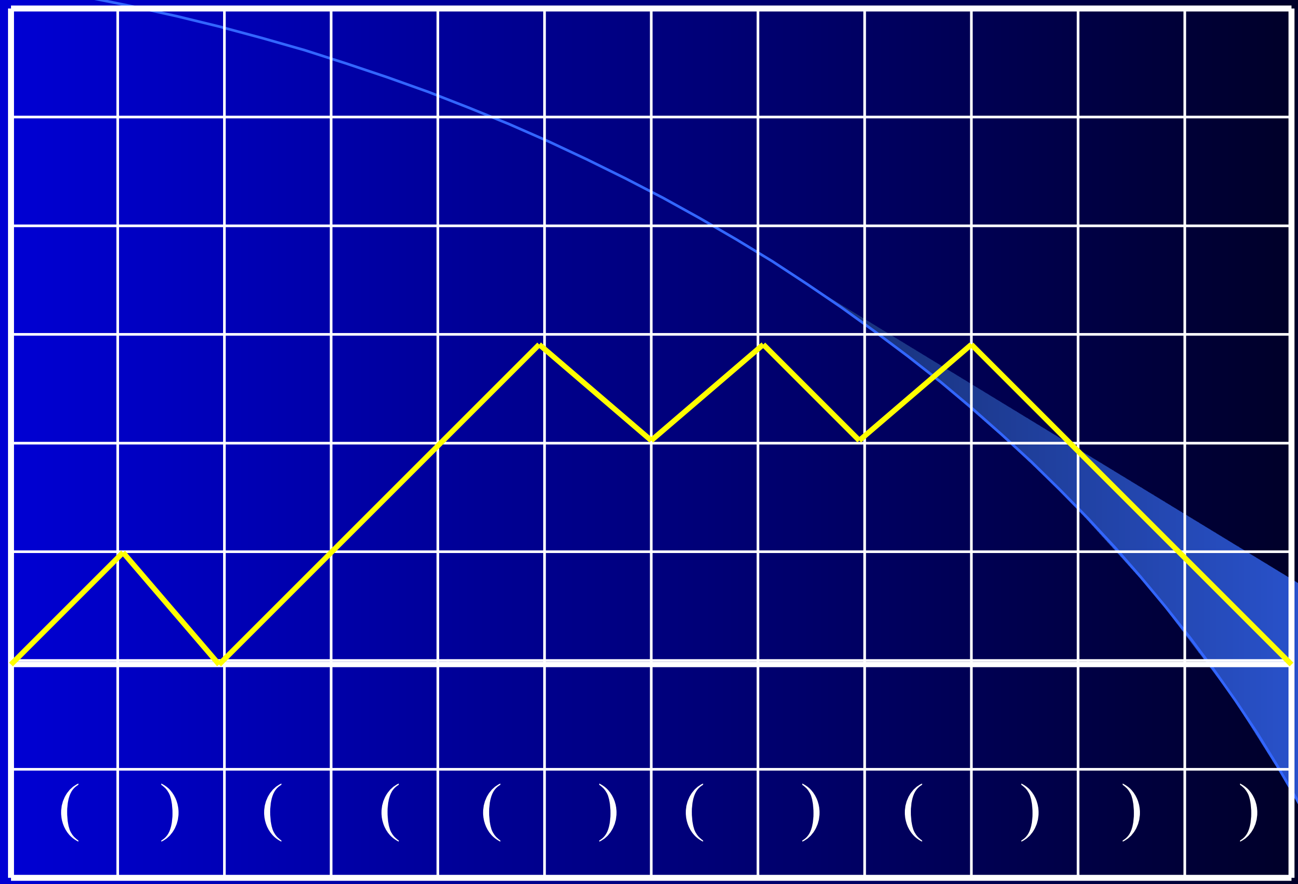
K A Penson, P Blasiak, G H E Duchamp, A Horzela and A I Solomon,

Hierarchical Dobinski-type relations via substitution and the moment problem,

J. Phys. A: Math. Gen. 37 3457 (2004) arXiv : quant-ph/0312202

A second application : Dyck paths
(systems of brackets, trees, physics, …)

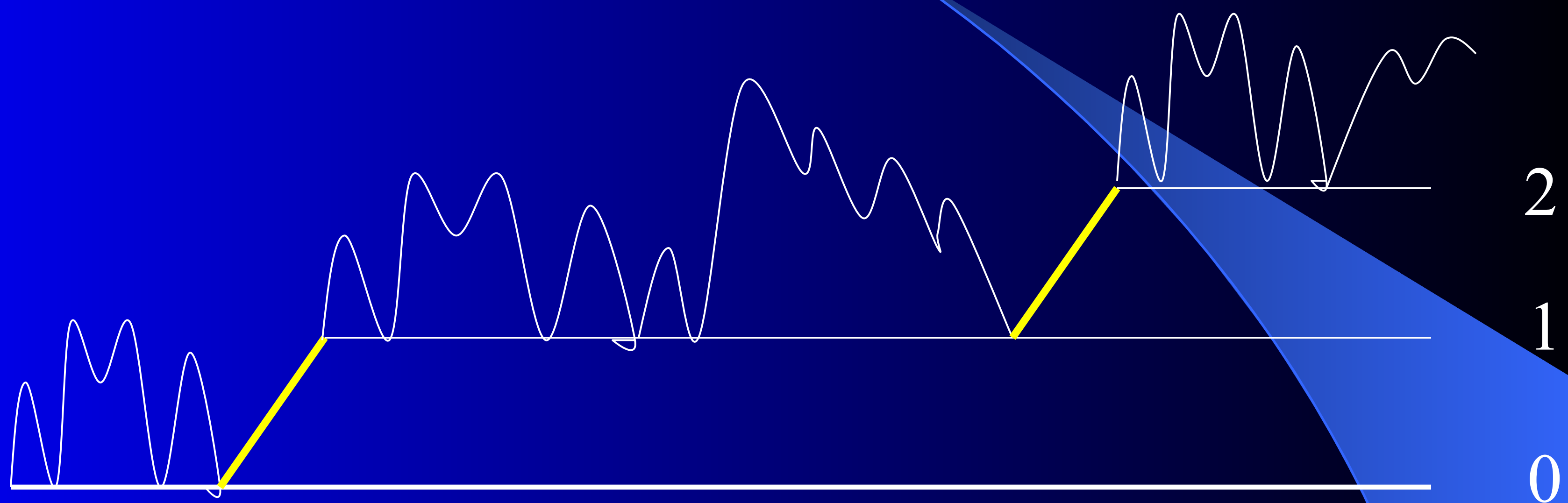




Equation : D = void + (D) D … one counts strings using an « x » by bracket and one finds $T(x)=x^0 + x^2\,T^2(x)$ which can be solved by elementary methods …

$x^2 T^2 - T + 1 = 0$ Variable : T Parameter : x

Change of level (physics)



Positifs = D(aD)*

$$Pos := \frac{Dyck}{1 - x\,Dyck}$$

```
> solve(x^2*T^2-T+1=0,T);
```

$$\frac{1+\sqrt{1-4x^2}}{2x^2},\ \frac{1-\sqrt{1-4x^2}}{2x^2}$$

```
> f:=1/(2*x^2)*(1-(1-4*x^2)^(1/2));
```

$$f := \frac{1-\sqrt{1-4x^2}}{2x^2}$$

```
> taylor(f,x=0,20);
```

$$1 + x^2 + 2x^4 + 5x^6 + 14x^8 + 42x^{10} + 132x^{12} + 429x^{14} + 1430x^{16} + O(x^{18})$$

```
> seq(binomial(2*k,k)/(k+1),k=1..8);
```

$$1, 2, 5, 14, 42, 132, 429, 1430$$

```
>
```

```
> Pos:=simplify(Dyck/(1-x*Dyck));
```

$$Pos := -\frac{2}{-1-\sqrt{1-4\,x\,y}+2\,x}$$

```
> coeftayl(Pos,[x,y]=[0,0],[6,4]);
```

$$90$$

```
> S:=0:for l from 0 to 6 do for k from 0 to 6 do
  S:=S+coeftayl(Pos,[x,y]=[0,0],[k,l])*x^k*y^l od
  od:S;
```

$$1+x+x\,y+20\,x^6\,y^2+14\,x^5\,y^2+5\,x^3\,y^3+2\,x^2\,y^2+x^3+28\,x^5\,y^3+x^4+x^5$$
$$+x^6+x^2+132\,x^6\,y^5+2\,x^2\,y+5\,x^3\,y^2+90\,x^6\,y^4+42\,x^5\,y^5+3\,x^3\,y$$
$$+132\,x^6\,y^6+4\,x^4\,y+14\,x^4\,y^4+14\,x^4\,y^3+5\,x^5\,y+9\,x^4\,y^2+48\,x^6\,y^3$$
$$+42\,x^5\,y^4+6\,x^6\,y$$

```
>
```

Automata and rationality

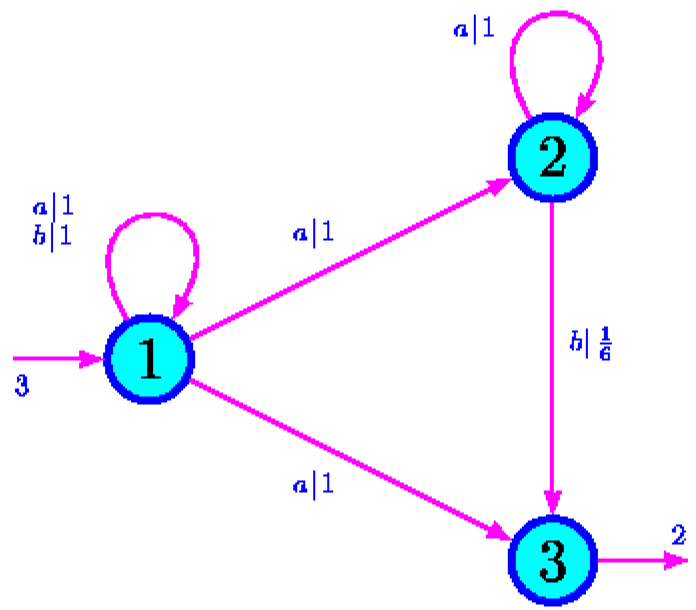


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6 + a^*b).$$

Un type particulier d'automate à multiplicités est constitué des automates à multiplicités avec des ε-transitions.

Un k-ε-automate $\mathcal{A}_\varepsilon$ est un k-automate sur l'alphabet $A_\varepsilon = A \cup \{\tilde{\varepsilon}\}$.

Exemple :

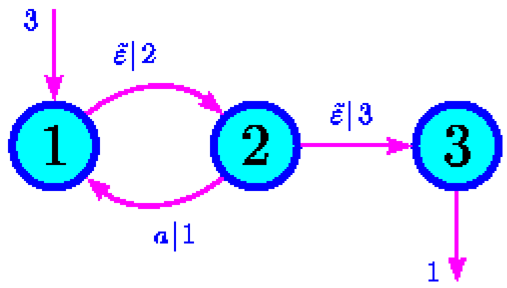
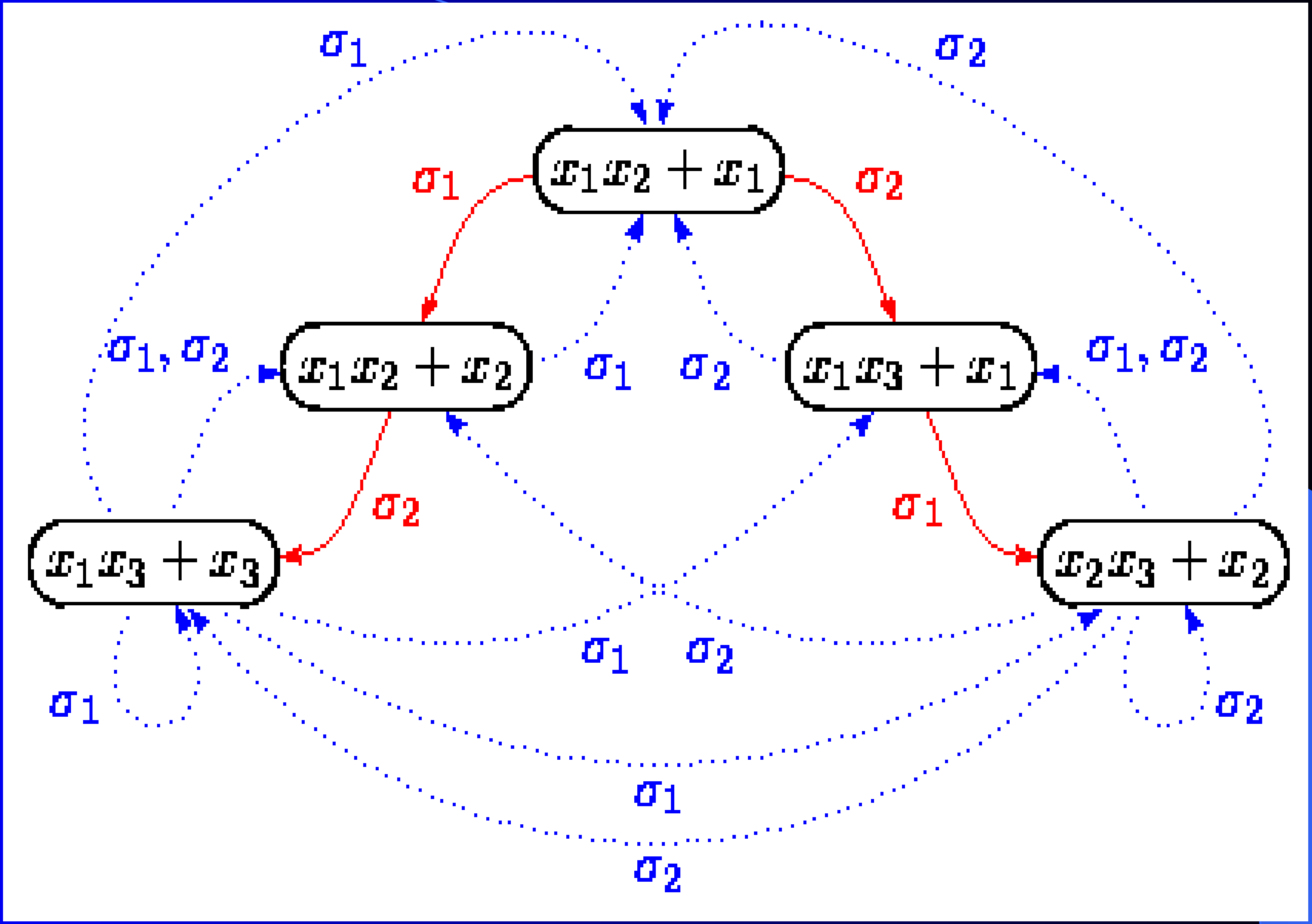


FIG. 2 – Un $\mathbb{N}$-ε-automate $\mathcal{A}_\varepsilon$

$$\text{comportement}(\mathcal{A}_\varepsilon) = 18\tilde{\varepsilon}\left(\sum_{i \in \mathbb{N}} 2^i (a\tilde{\varepsilon})^i\right)\tilde{\varepsilon}.$$



$x_1x_2+x_1$
$x_1x_2+x_2$
$x_1x_3+x_1$
$x_1x_3+x_3$
$x_2x_3+x_2$
σ_1
σ_2
σ_1
σ_2
σ_1, σ_2
σ_1
σ_2
σ_1, σ_2
σ_2
σ_1
σ_1
σ_2
σ_1
σ_2
σ_1
σ_2

A correct implementation of Schelling's model

Problem : If one scans the board, addressing the inhabitants one after one, result is sensitive to the order of scanning.

Solution : Invent a (combinatorial) data structure which adapted to the parallel structure of the moving intentions of the inhabitants.

Problem : If one scans the board, addressing the inhabitants one after one, result is sensitive to the order of scanning.

Solution : Invent a (combinatorial) data structure which adapted to the parallel structure of the moving intentions of the inhabitants --> this must be a global model.

Combinatorics (mathematics)

Complex
Systems

Information
(comp. sci.)

Physics
(class. quant.)

Thank You