Dynamic combinatorics, complex systems and applications to physics

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Mathematics

Chaos Theory

Continuous & Discrete Modelisation

Image Processing

Computer Science

Complex Systems Complexity

Business Banking

Computation Techniques

Decision Making

Physics

Artificial Intelligence

Mechatronics

Electronics

Adaptronics

Abstract ↔ Applied

Combinatorics
(discrete mathematics)

Information
(computer sci.)

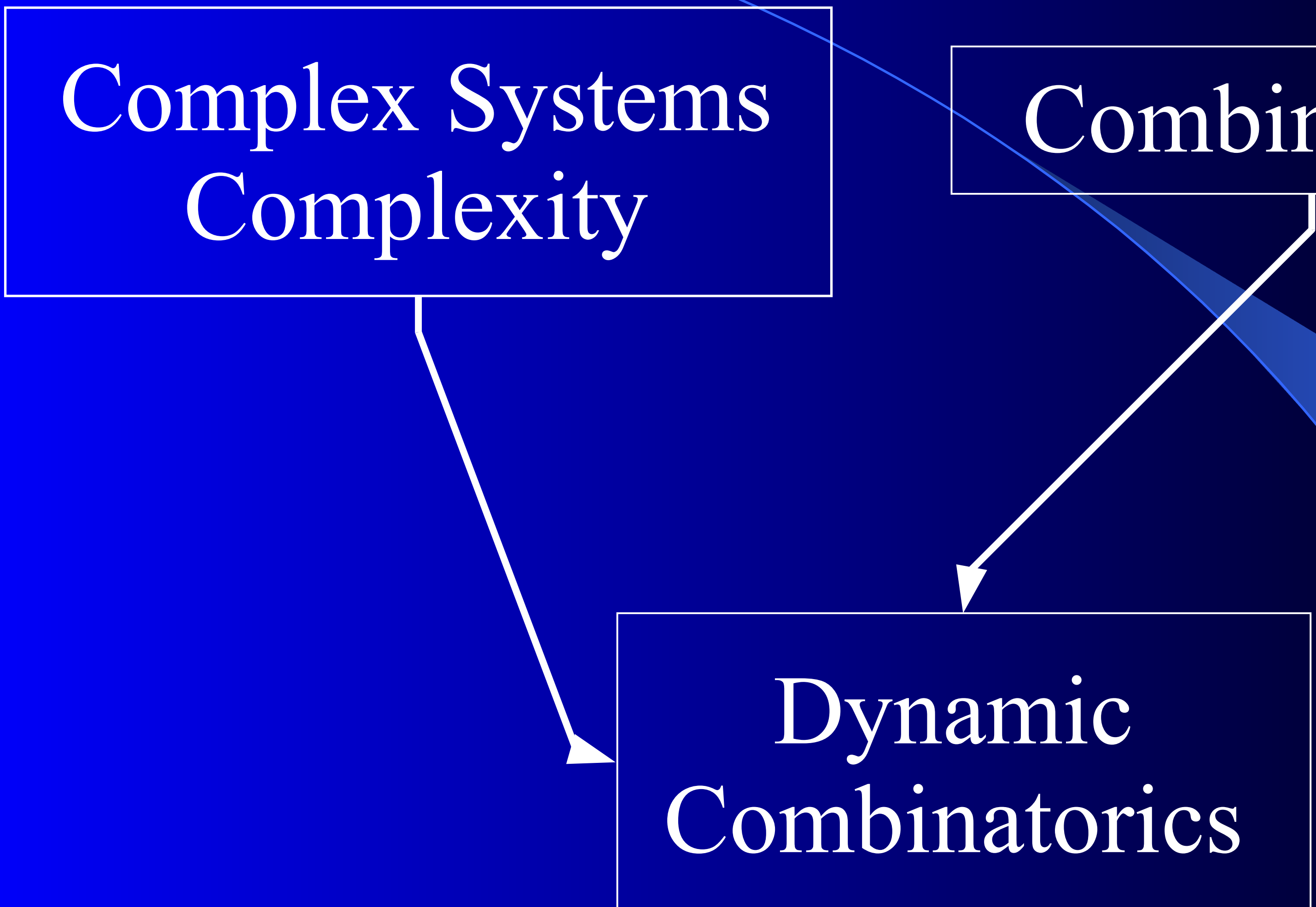
Physics
(classical/quant.)

Combinatorics
(discrete mathematics)

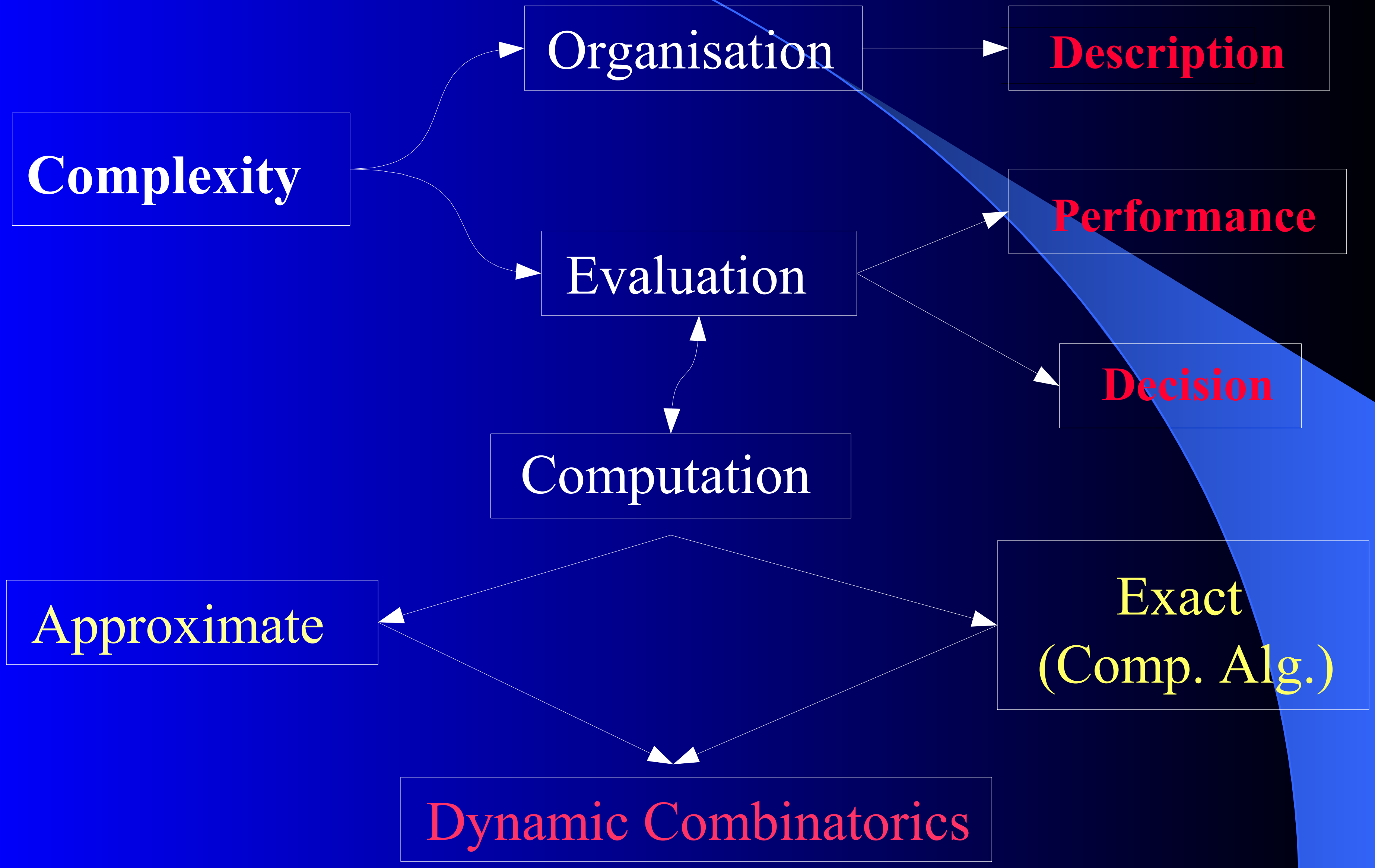
Complex
Systems

Information
(computer sci.)

Physics
(classical/quant.)


Complex Systems
Complexity
Combinatorics
Dynamic
Combinatorics

Problematics of Dynamic Combinatorics



What is the Legacy ?

Mainly:

- Data Structures
- Programs
- Theorems
- Computation rules
- Experiments

What is the Legacy ? (Cont'd)

Mathematics	Comp. Sciences	Physics
• Noncommutative	• Words	• Strings of operators
• Representations	• Automata Transition Structures	• Fields, Flows, Dynamic Systems (Chaos, Catastrophes)
• Formulas, Universal Algebra	• Trees with Operators	• Diagrams
• Deformations	• q-analogues	• Quantum Groups

Combinatorics

C o m b i n a t o r i c s

… on words

- Langages
- Theory of codes
- Automata
- Transition structures
- Grammars
- Transducers Rational and algebraic expressions
- …

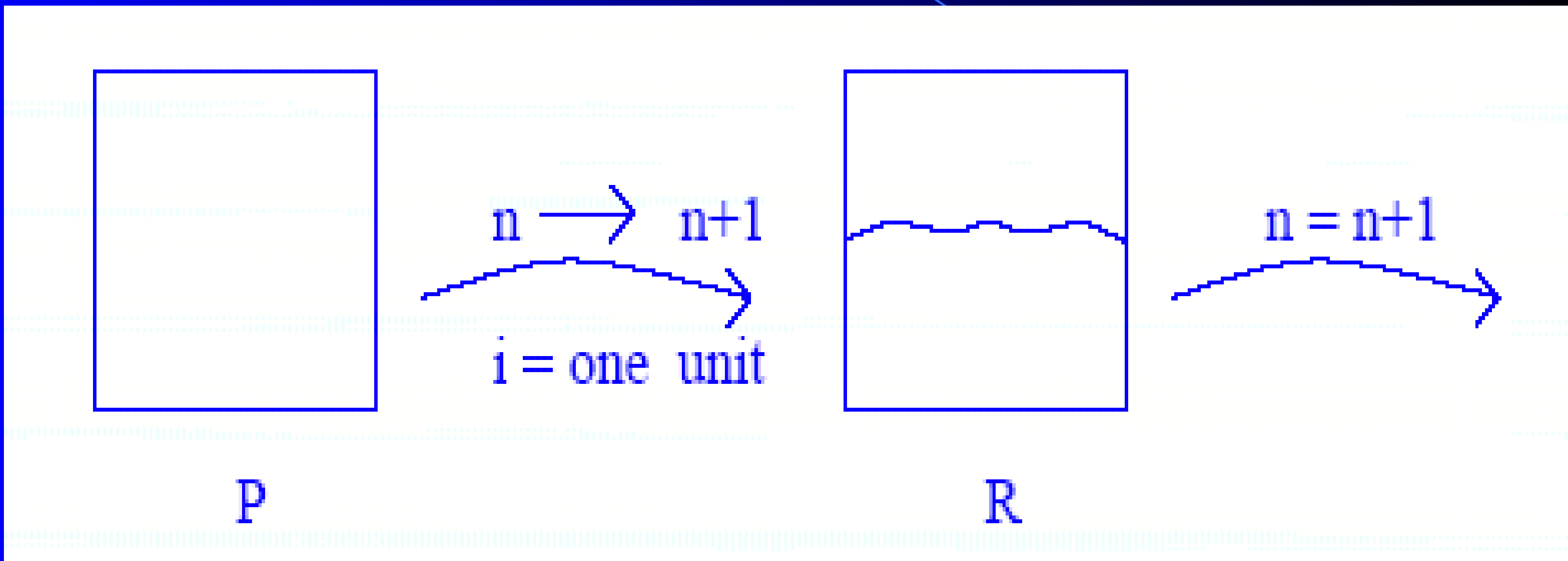
enumerative, analytic

- Polyominos
- Paths (Dyck,…)
- Configurations
- q-grammars
- Generating Functions
- Continued Fractions (mono, multivariate,.)
- Orthogonal Polynomials
- …

algebraic

- Non commutative Continued fractions
- Representations of groups and deformations
- Quantum Groups Functors
- Characters
- Special Functions
- …

A first example . . .


n
n+1
i = one unit
n = n+1
P
R

- $a_1 = 0$
- $a_{i+1} \leq a_i + 1$

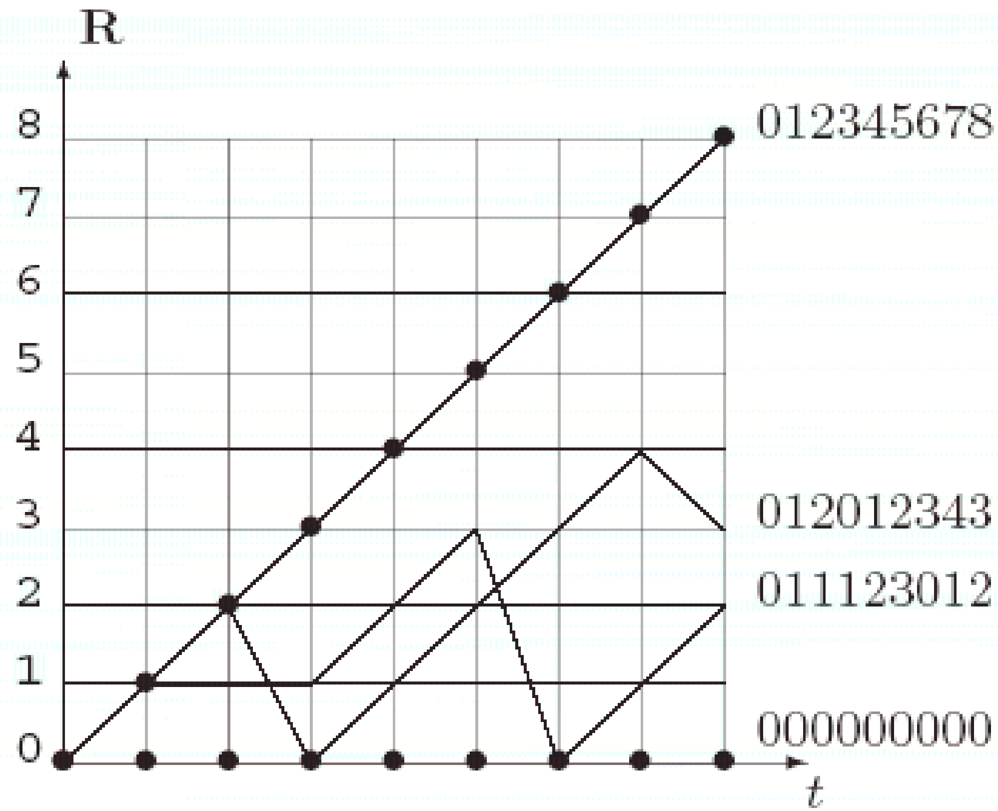
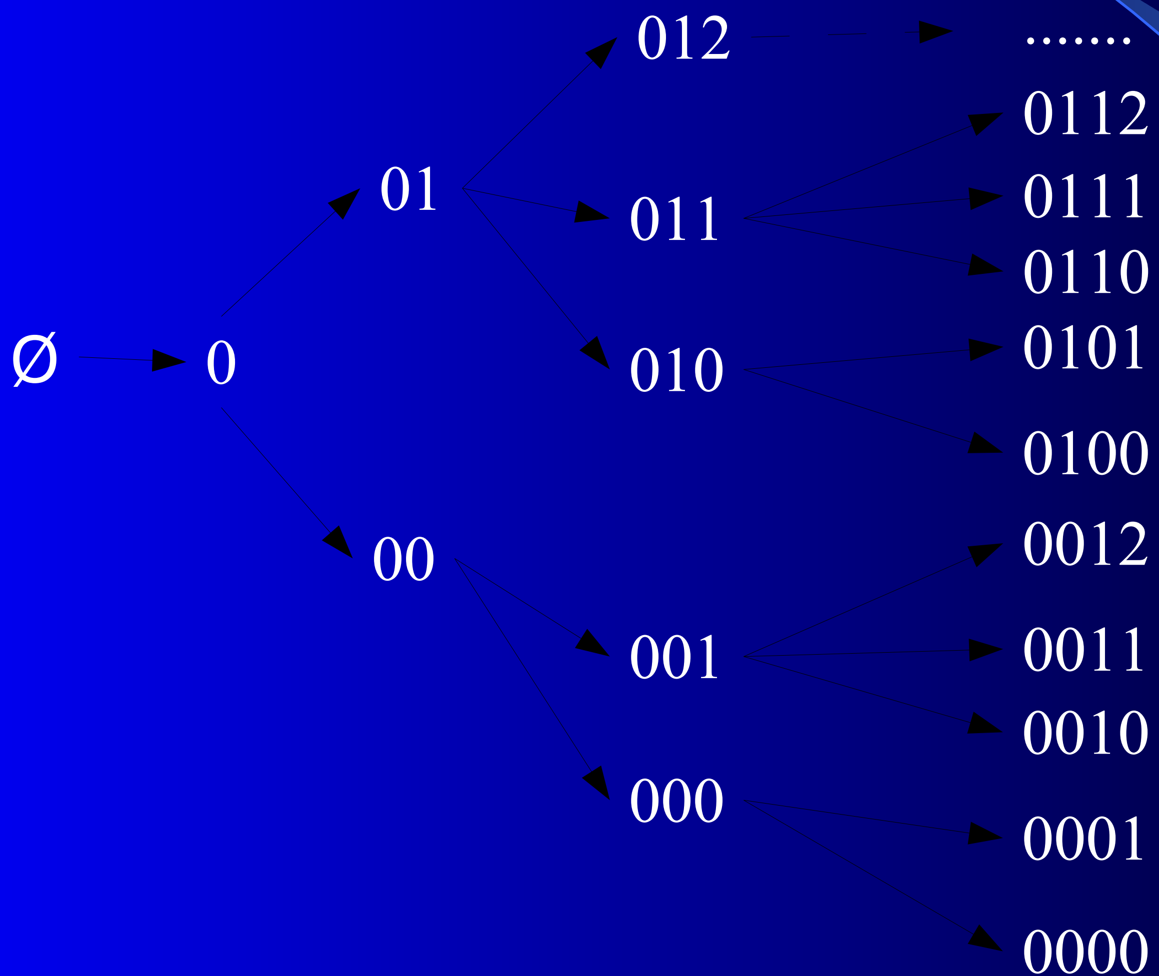


Figure 4.2: Maximal, minimal (dotted) and two intermediate trajectories. Their codes are on the right.

One can arrange all the trajectories on a dynamic graph



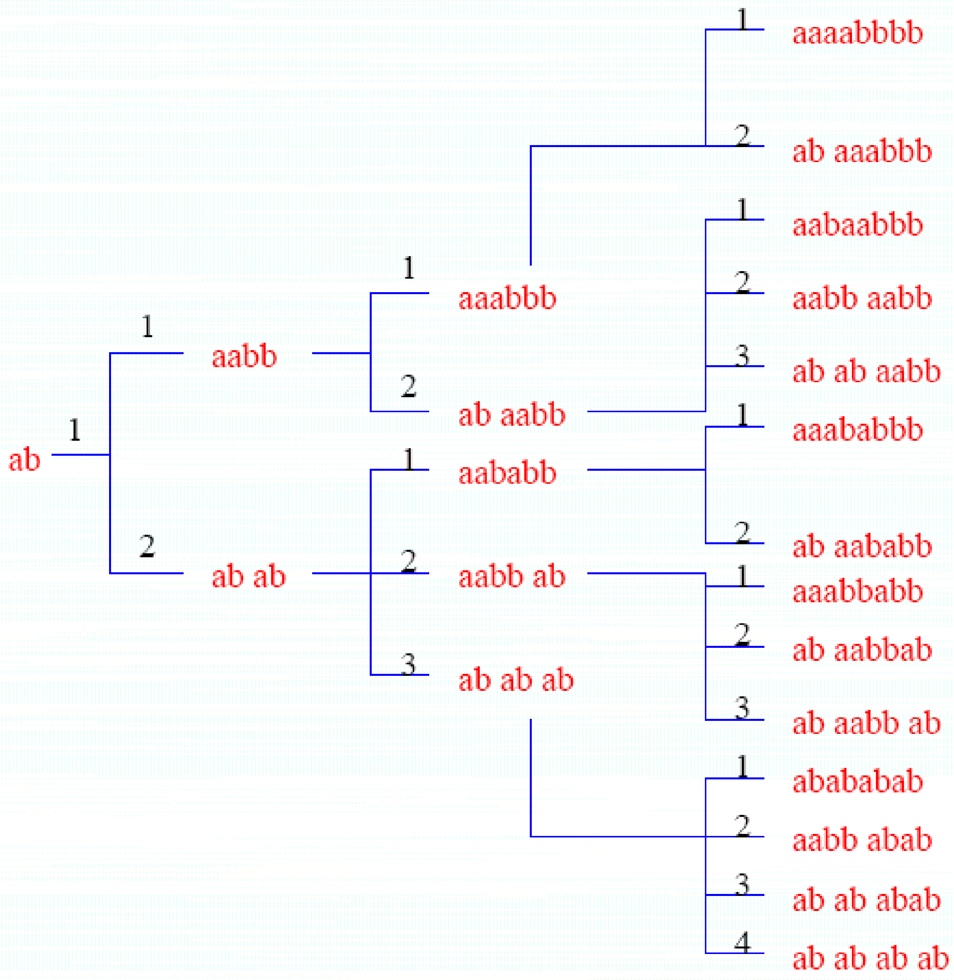


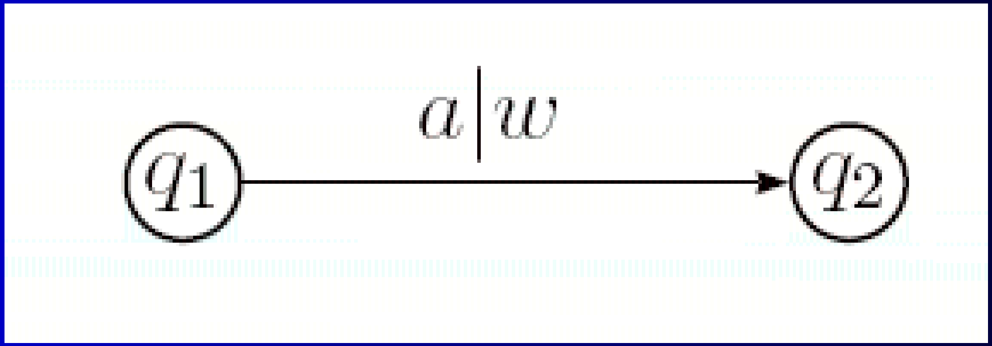
Figure 2.9: Dyck words of length 2n with k factors

We will return to the Dyck paths later on. For the moment let us define what is a transition structure.

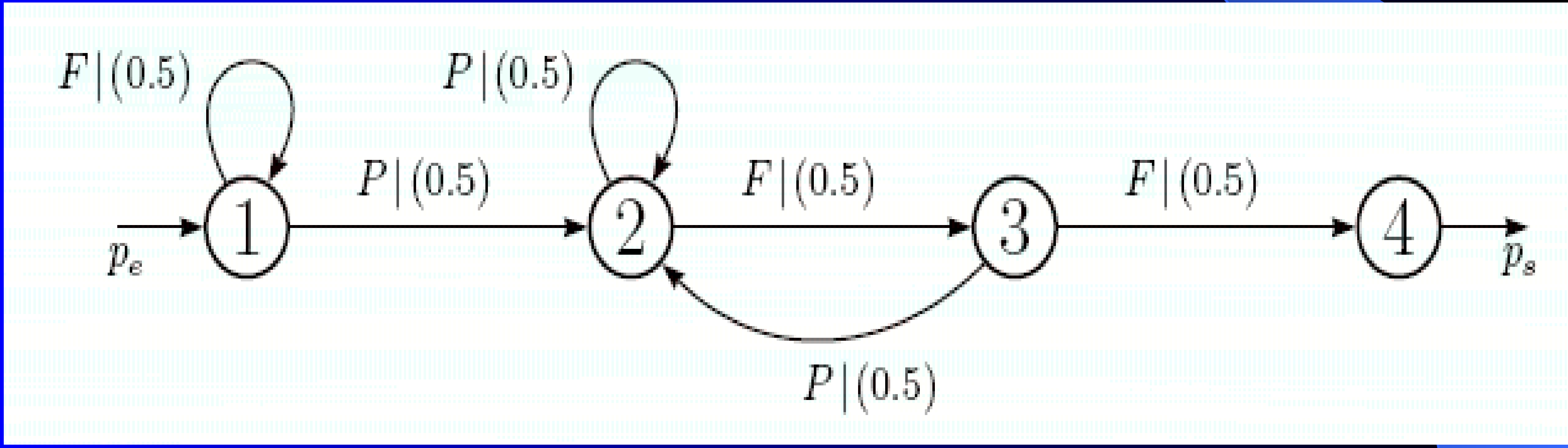
Definition (transition structure) : It is a graph (finite of infinite) with its arcs marked with pairs

(command letter | coefficient)

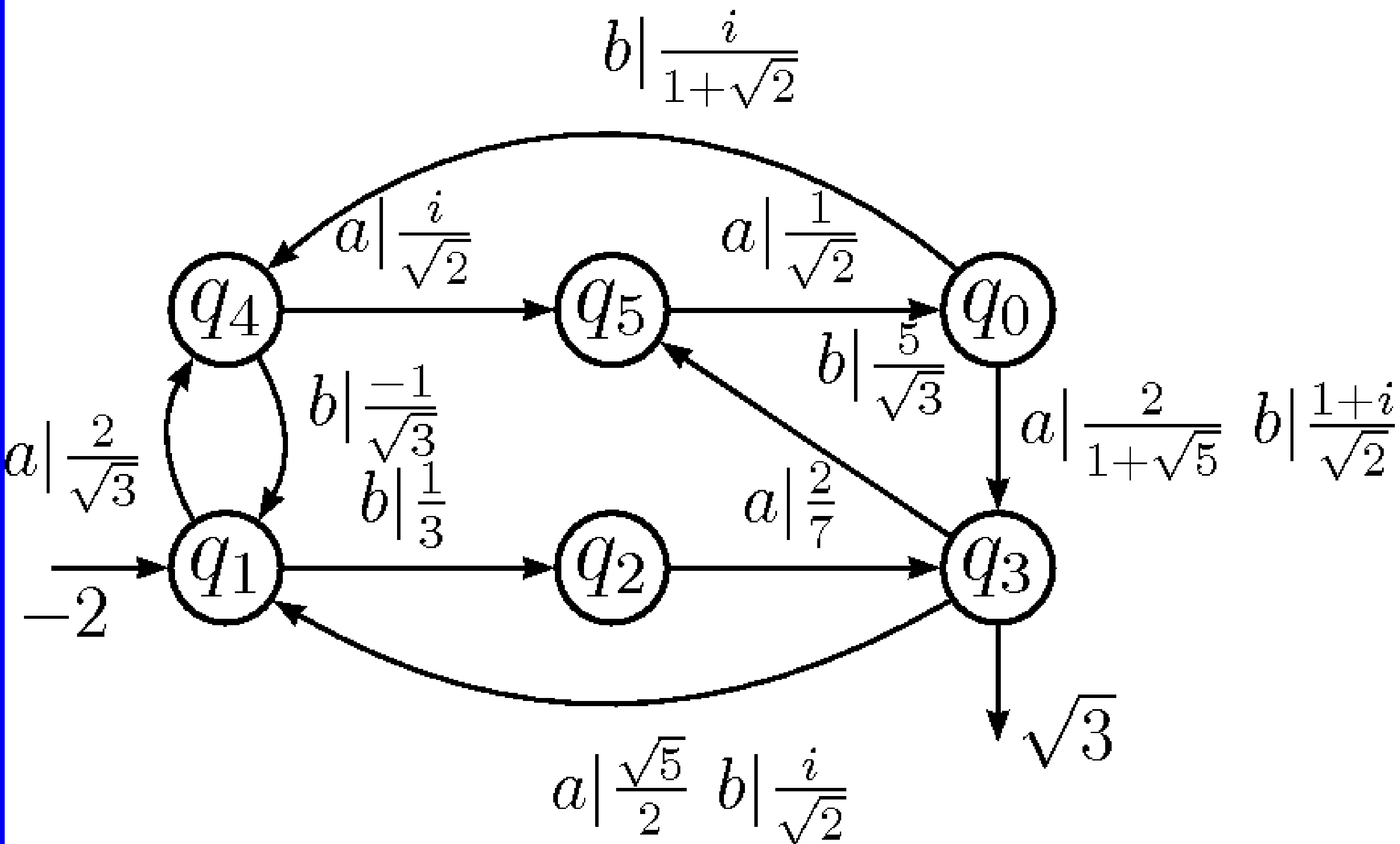
Examples : Prisoner's dilemma, Markov chains, classical engineering.



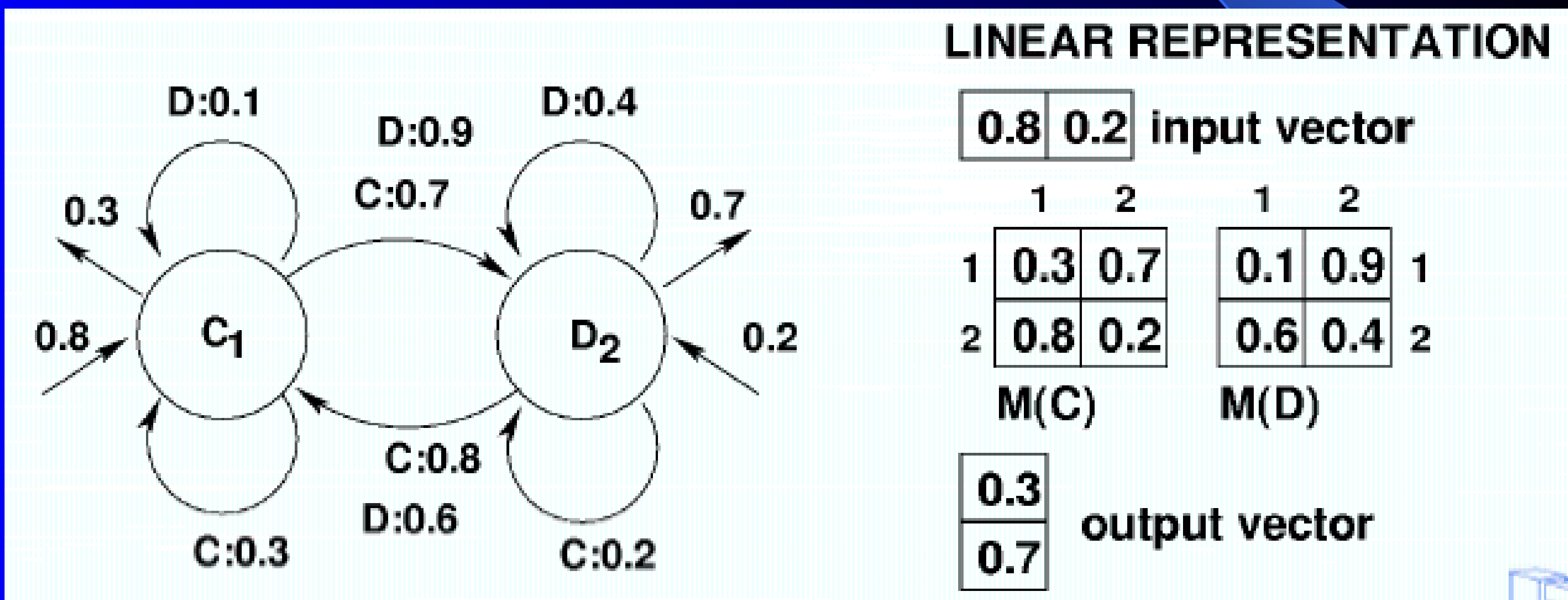
Example : A Markov chain generated by a game.



Example : An automaton generated by arbitrary transition coefficients.



Example of Probabilistic Automaton



Behaviour of an Automaton and how to compute it effectively

An automaton is a machine which takes a string (sequence of letters) and returns a value.

This value is computed as follows :

1) The weight of a path is the product of the weights (or coefficients) of its edges

2) The label of a path is the product (concatenation) of the labels of its edges

Behaviour ... (cont'd)

3) The behaviour between two states « r,s » w.r.t. A word « w » is the product of

3a) the ingoing coefficient of the first state (here « r ») by

3b) the sum of the weights of the paths going from « r » to « s » with label « w » by

3c) the outgoing coefficient of the second state (here « s »)

Behaviour ... (cont'd)

4) The behaviour of the automaton under consideration w.r.t. a word « w » is then the sum of all the behaviours of the automaton between two states « r,s » for all possible pairs of states.

Behaviour ... (cont'd)

There is a simple formula using the linear representation. The behaviour of an automaton with linear representation (I,M,T) is the product

$$IM(w)T$$

where M(w) is the canonical exention of M to the strings.

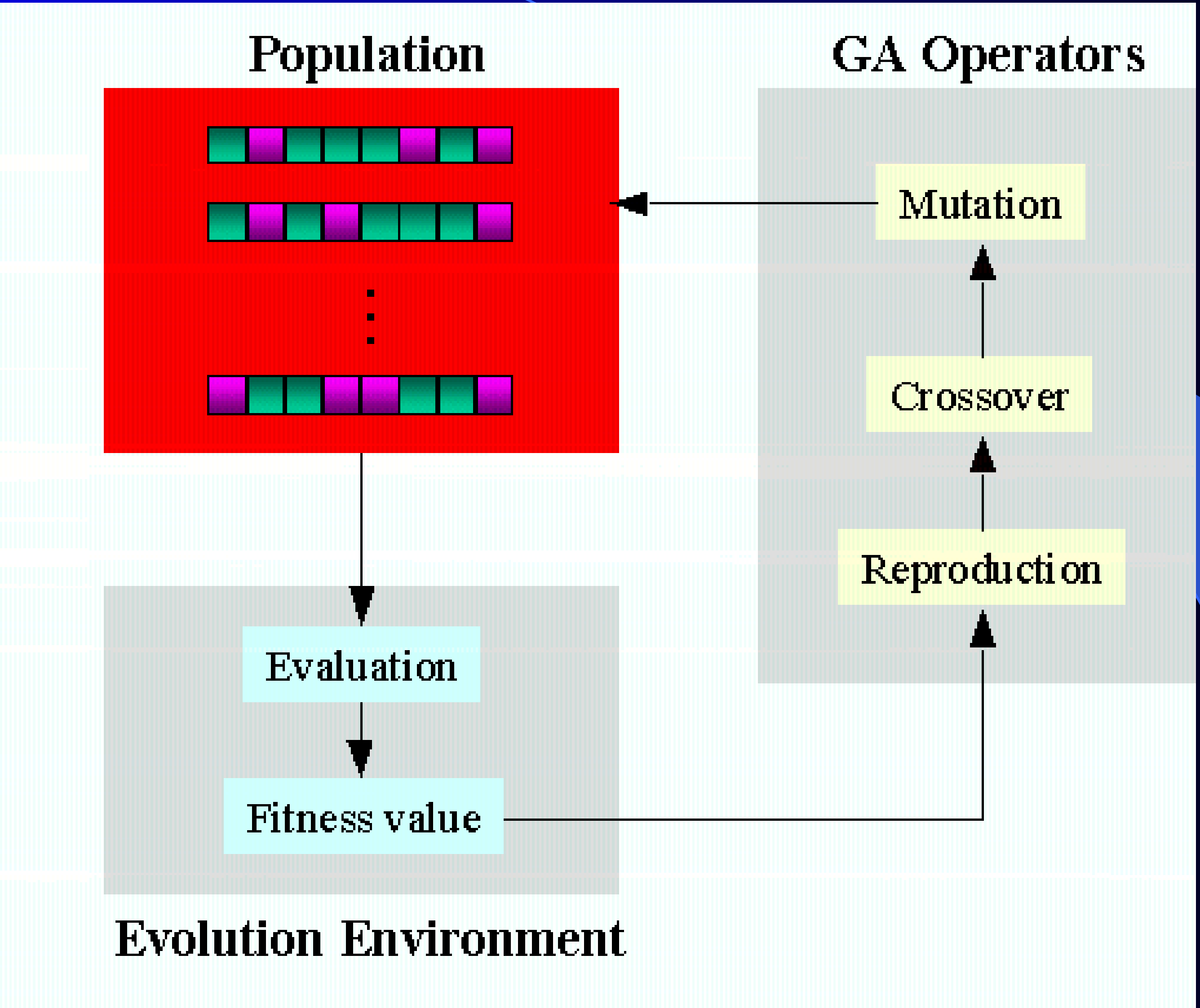
$$M(a_1a_2 \ldots a_n)=M(a_1)M(a_2)\ldots M(a_n)$$

Behaviour ... (end)

The behaviour, as a function on words belongs to the rational class. If time permits, we will return to its complete calculation as a rational expression and the problem of its algorithmic evaluation by means of special cancellation operators. Linear representations can also be used to compute distances between automata.

Example -> use of genetic algorithms to control indirect (set of) parameters : the spectrum of a matrix.

Genetic algorithms : general pattern



Genetic Algorithm Evolution Flow

Genetic algorithms : implementation

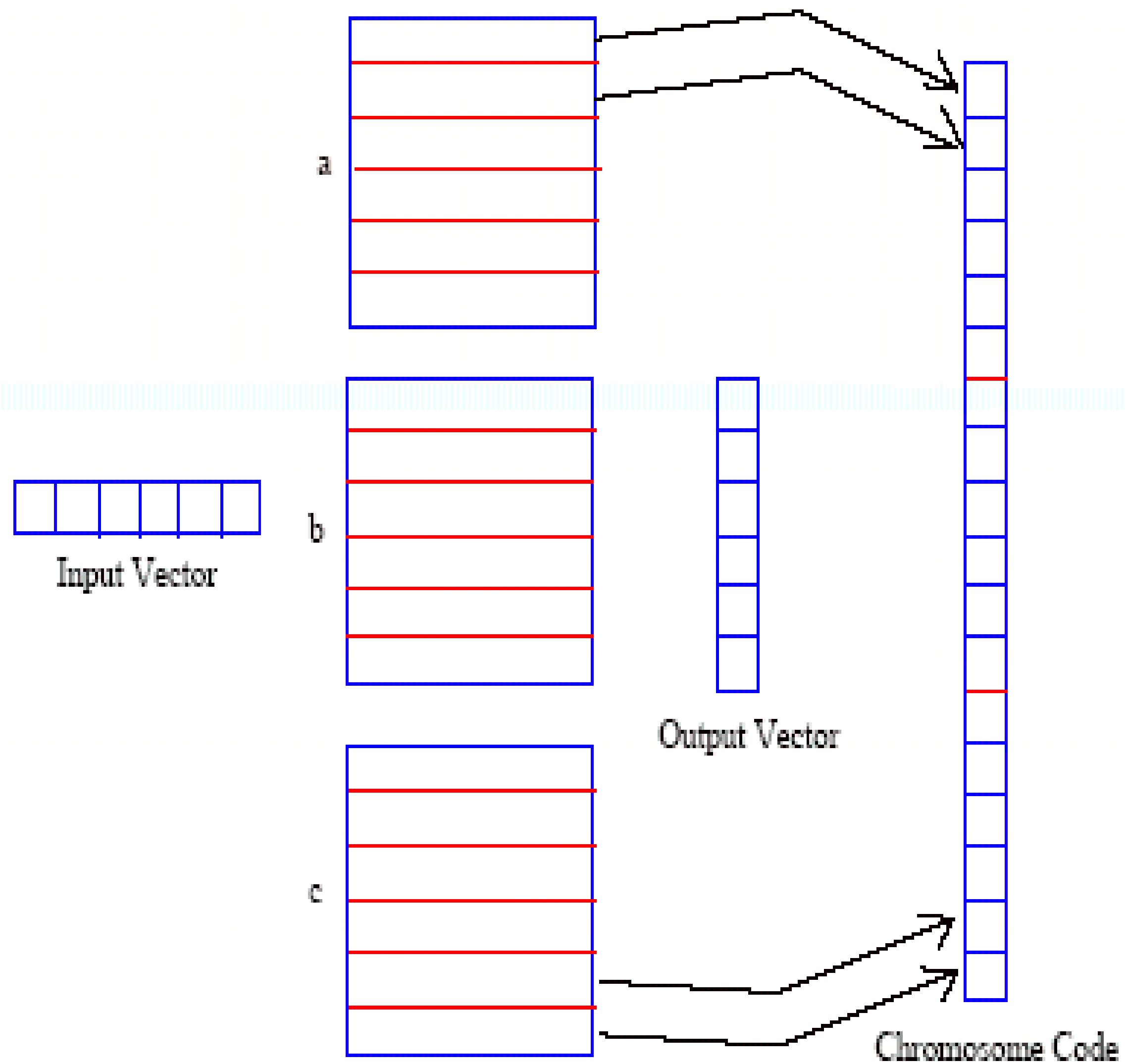


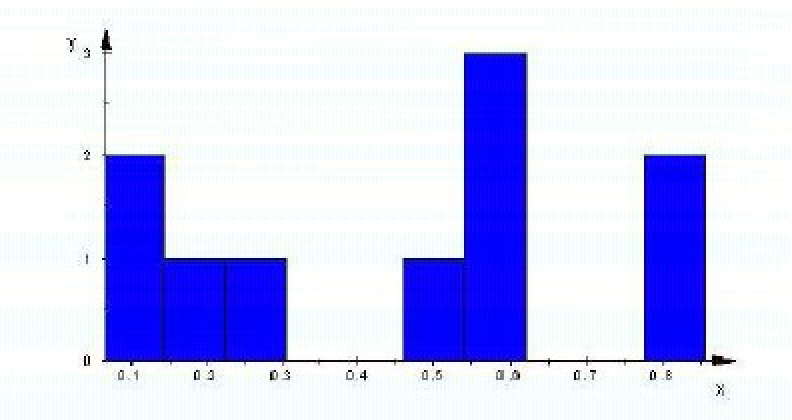
Figure 4.13: Chromosome code

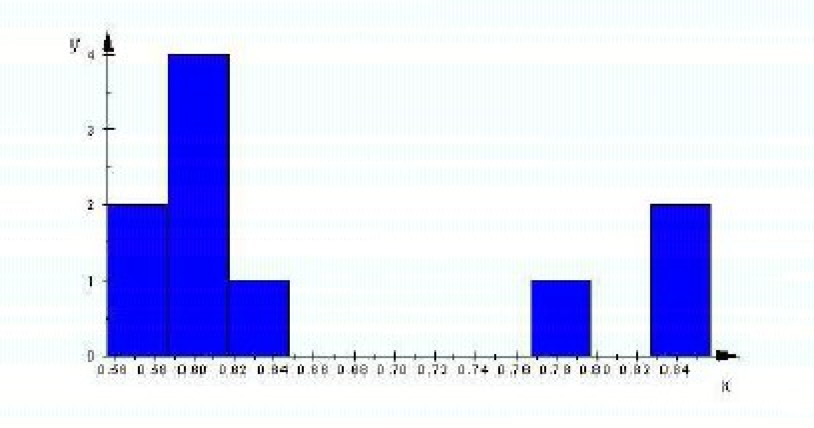
Genetic algorithms : implementation

Below, the results of an experiment aiming to control the second greatest eigenvalue of the transfer matrix of a population of probabilistic automata.

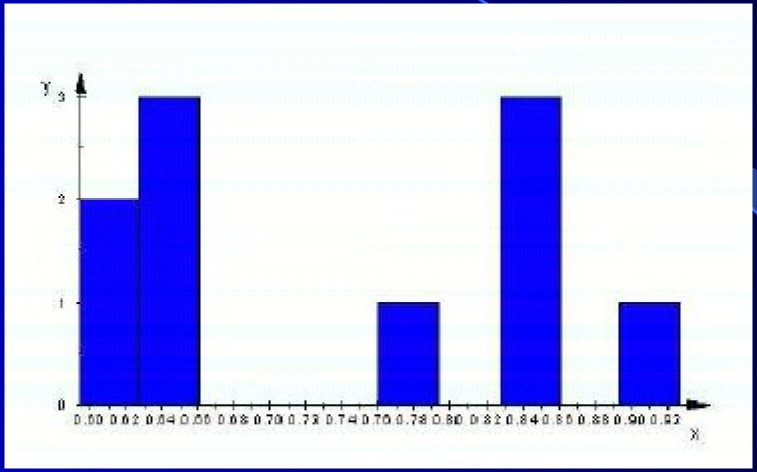
• The fitness function of each automaton corresponds to the second greatest eigenvalue (in module).
The first being, of course, of value 1.

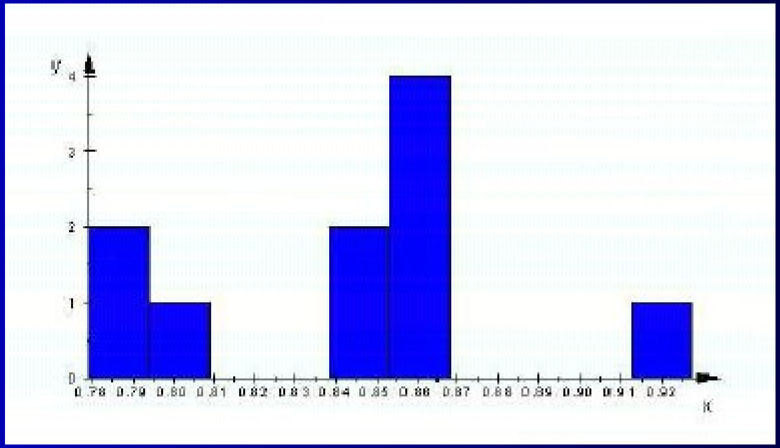
Genetic algorithms ; results



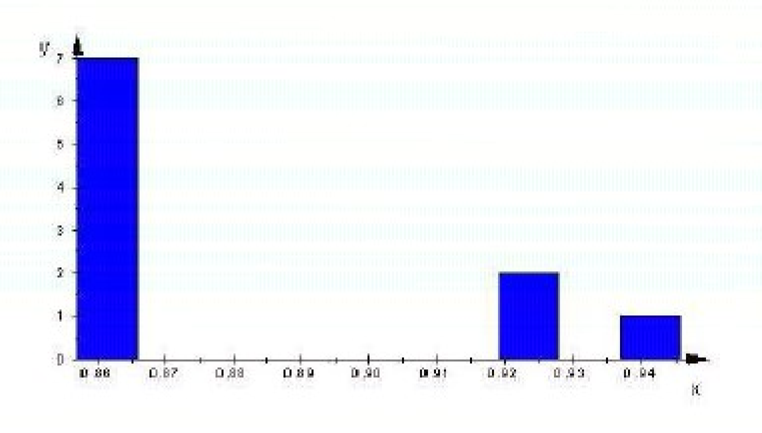


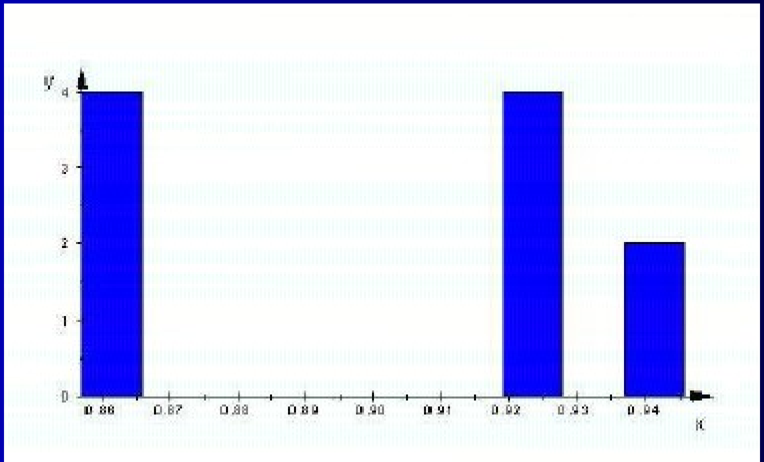
Genetic algorithms ; results



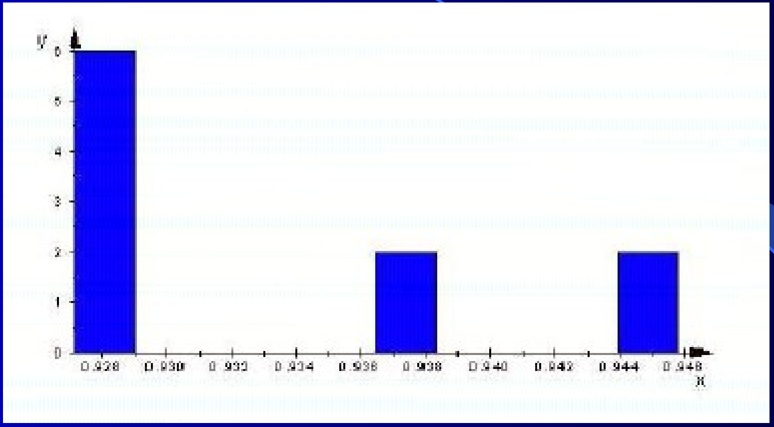


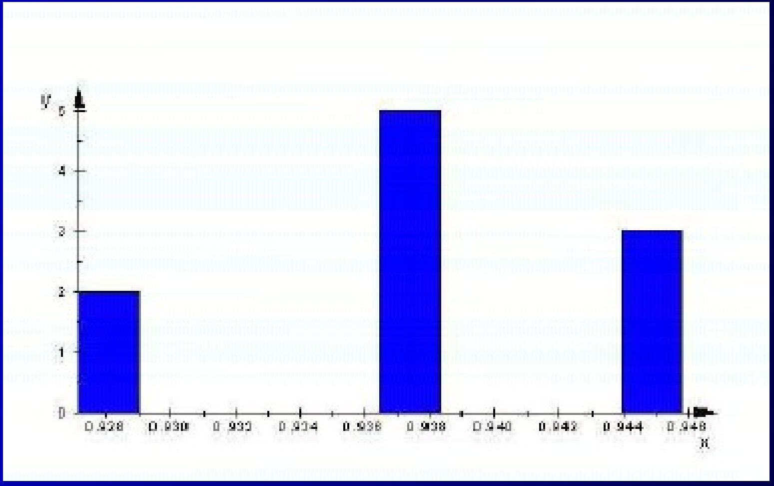
Genetic algorithms ; results



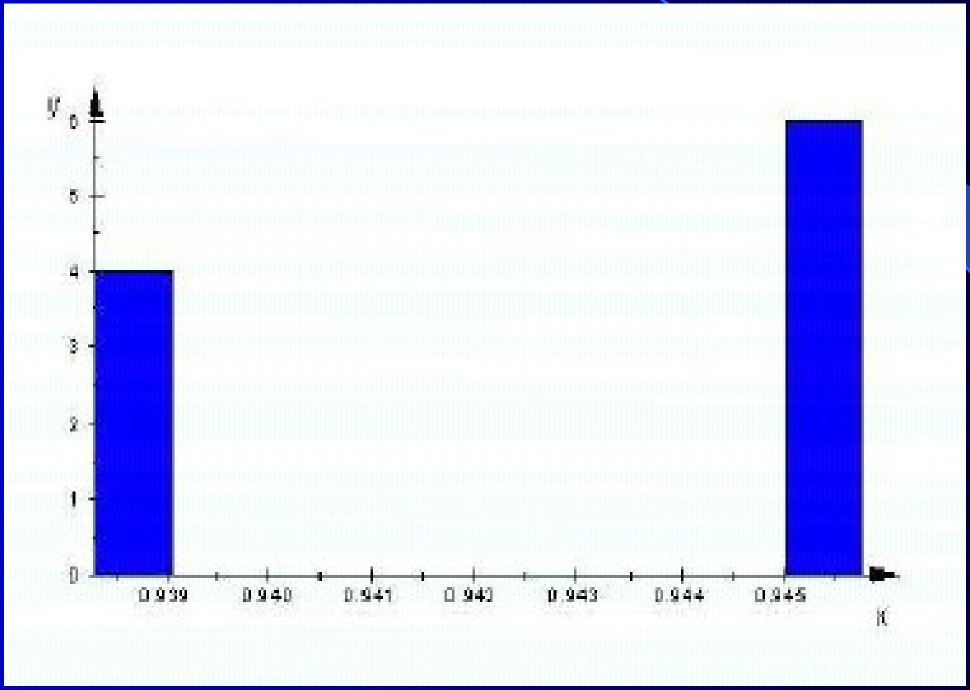


Genetic algorithms ; results



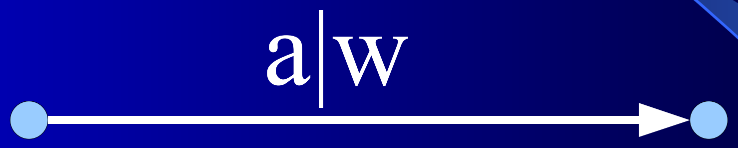


Genetic algorithms ; results



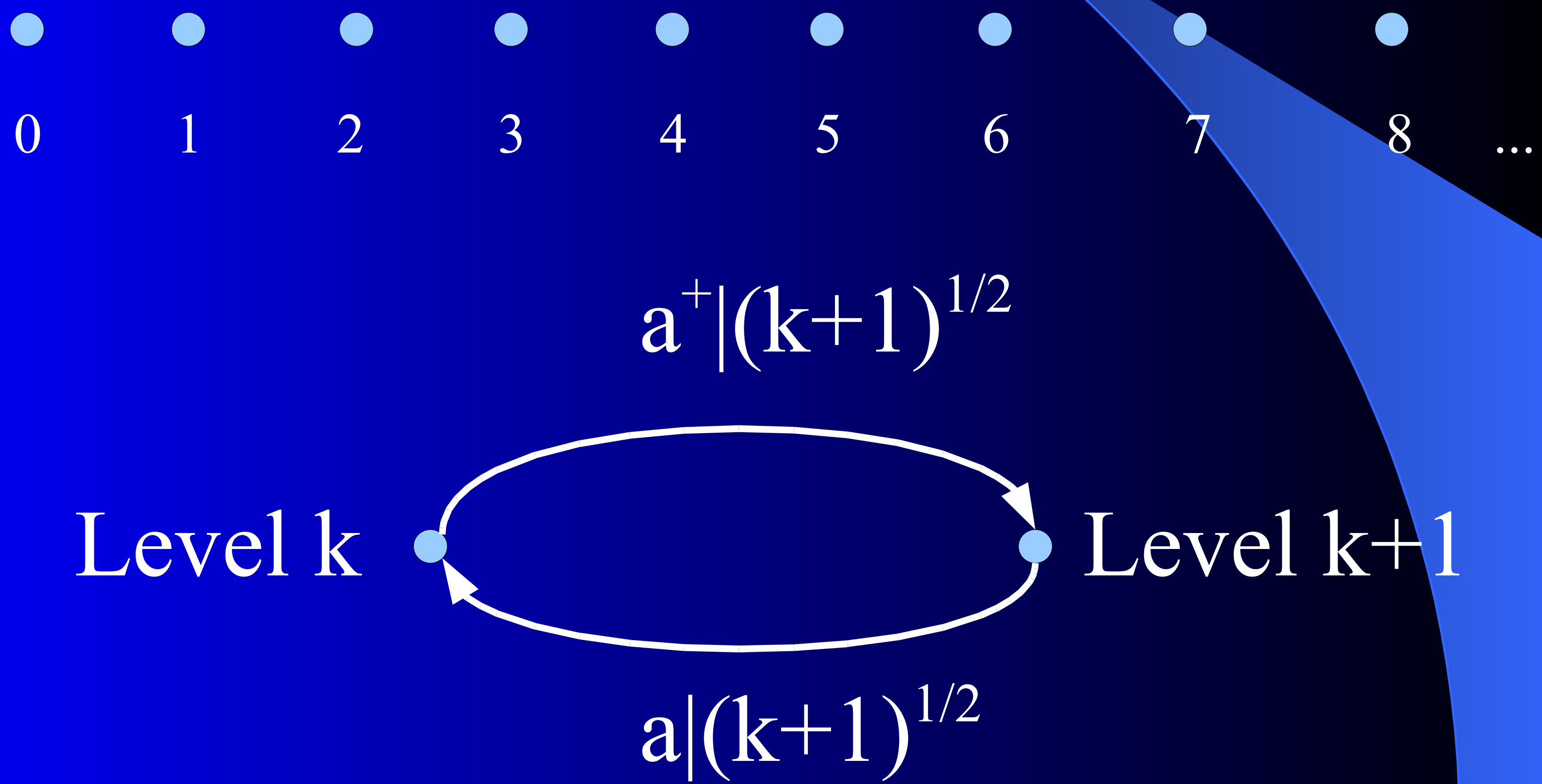
Final result : the population is rendered homogeneous

General transition systems



- Automata (finite number of edges)
- Sweedler's duals (physics, finite number of states)
- Representations
- Level systems (Quantum Physics)
- Markov chains (prob. automata when finite)

Example in Physics : annilhilation/creation operators



The (classical, for bosons) normal ordering problem goes as follows.

- Weyl (two-dimensional) algebra defined as

$$< a^+, a \; ; \; [a\, , a^+] = 1 >$$

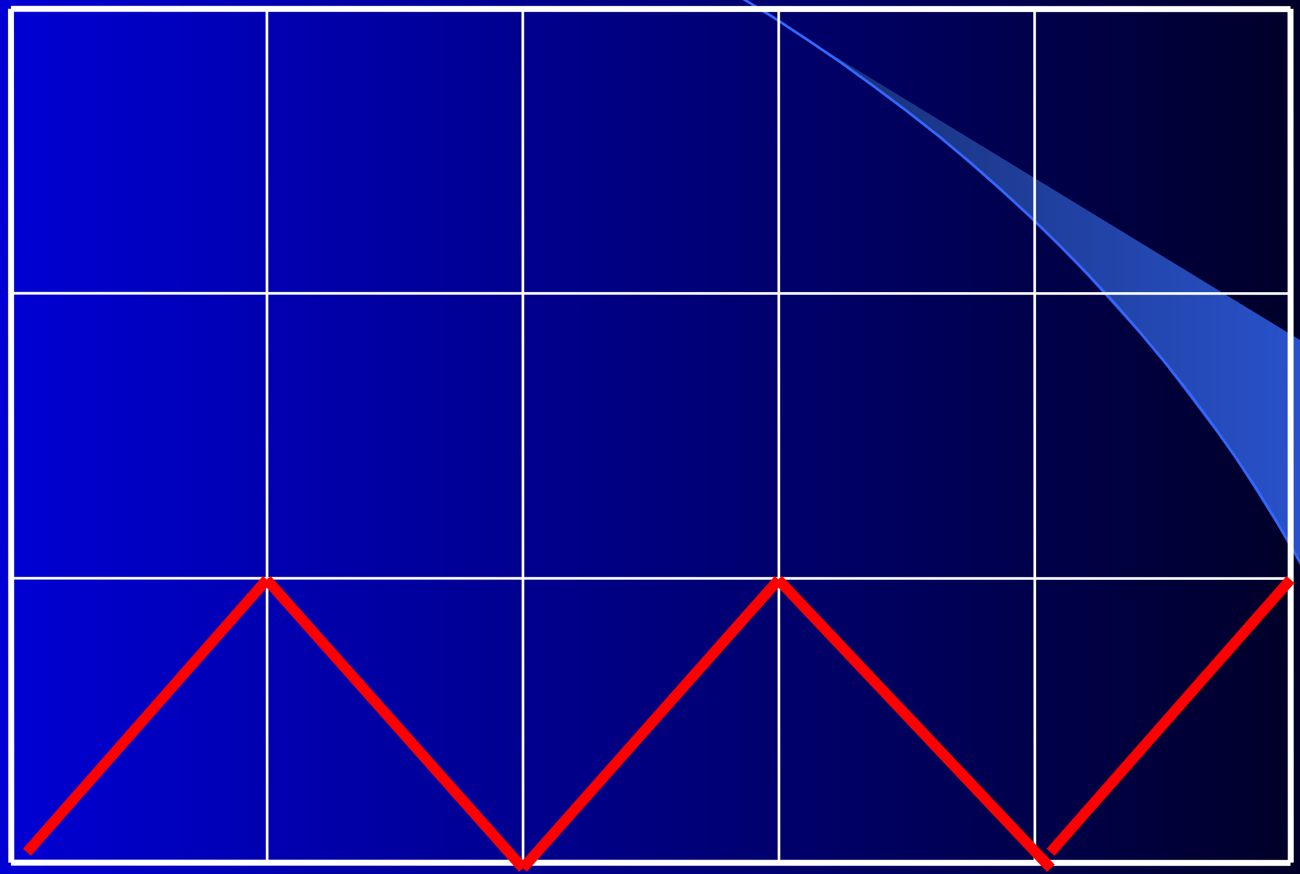
- Known to have no (faithful) representation by bounded operators in a Banach space.

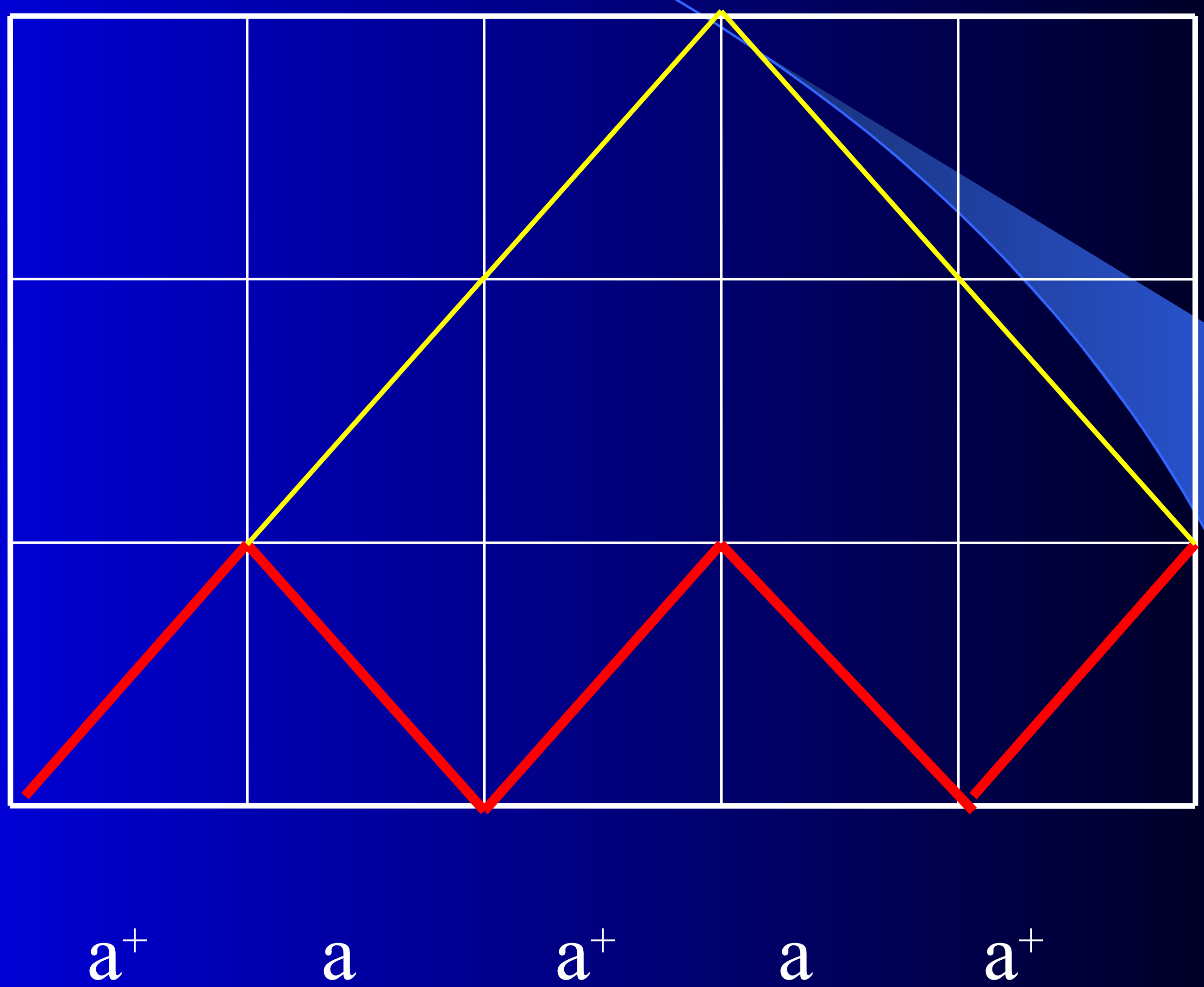
There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

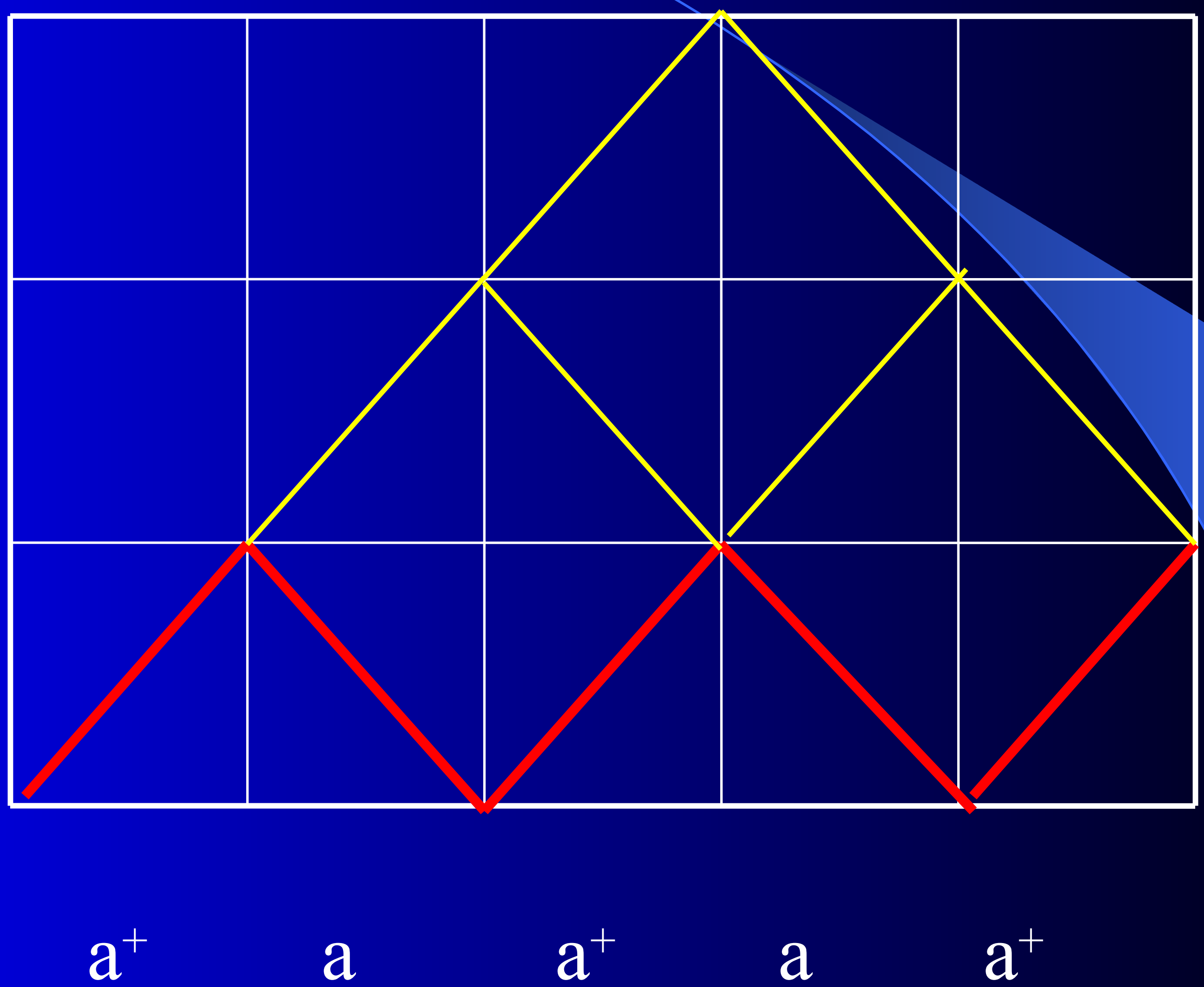
$$a \rightarrow d/dx \; ; \; a^+ \rightarrow x$$

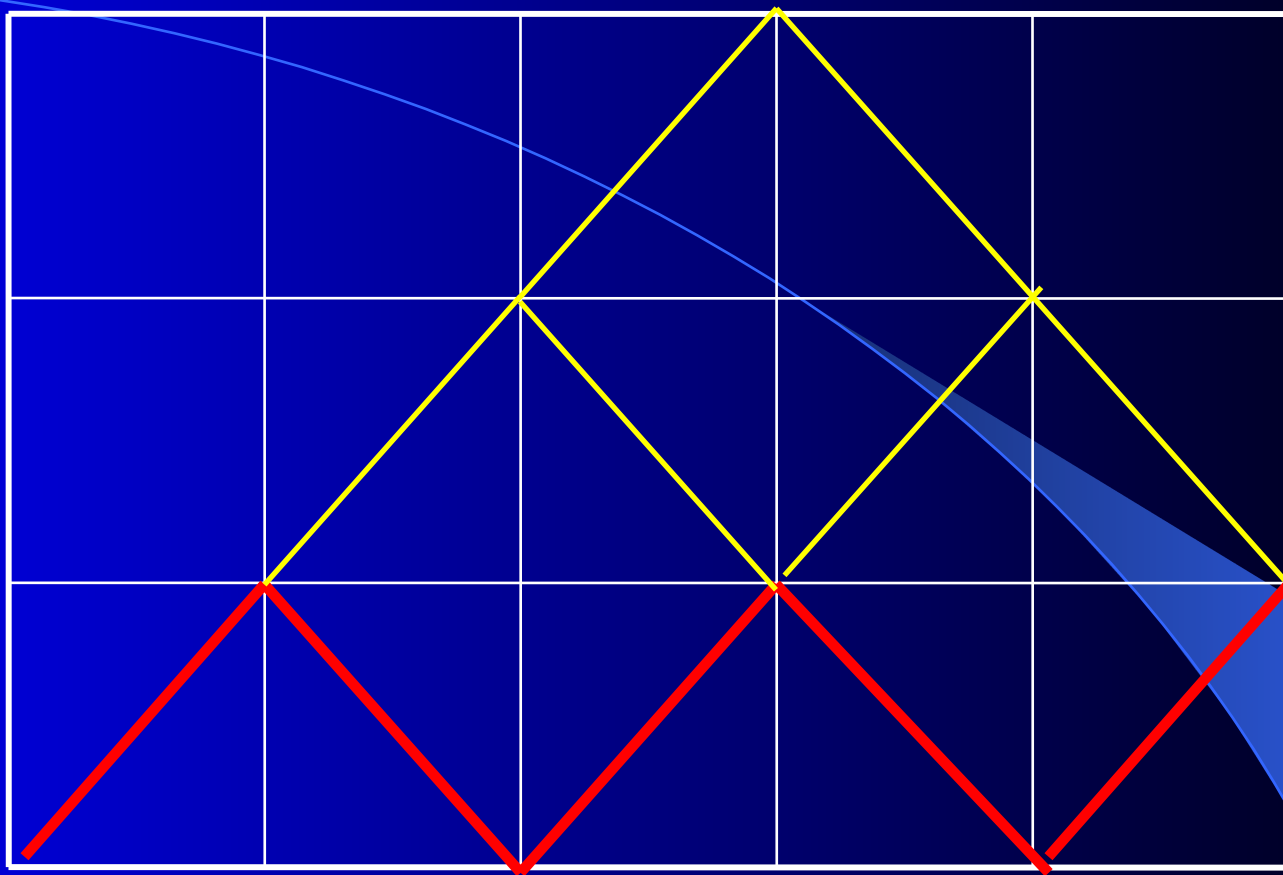
where a has degree -1 and a^+ has degree 1.

Example with $\Omega = a^+ a\ a^+ a\ a^+$




a^+
a
a^+
a
a^+


a⁺ a a⁺ a a⁺



a^+ a a^+ a a^+

$$a^+aa^+aa^+ = 1\, a^+a^+a^+aa + 3\, a^+a^+a + 1\, a^+$$

Through Bargmann-Fock representation

$$a \rightarrow d/dx \; ; \; a^{+} \rightarrow x$$

Operators who have only one annihilation have exponentials who act as one-parameter groups of substitutions.

One can thus use computer algebra to determine their generating function.

For example, with

$$\Omega = a^{+2} a\, a^{+} + a^{+} a\, a^{+2}$$

the computation reads

Maple 9 - [Untitled (1) - [Server 1]]

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One parameter group by f(v(u(x)+λ)); v is reciprocal of u

```
> T1(lambda,x):=(-4*(-1/(4*x^2)+lambda))^(-1/2);
```

$$T1(\lambda, x) := \frac{1}{\sqrt{\frac{1}{x^2} - 4\,\lambda}}$$

We suppose x>0

```
> T1:=(lambda,x) -> x/((1-4*lambda*x^2)^(1/2));
```

$$T1 := (\lambda, x) \to \frac{x}{\sqrt{1 - 4\,\lambda\,x^2}}$$

Checking the tangent vector

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2\,x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))  -T1(lambda1+lambda2,x)  );
```

$$0$$

Time: 0.2s Bytes: 1.94M Available: 1.33G

démarrer | Microsoft PowerPoint ... | Maple 9 - [Untitled (1... | Le Robert & Collins | FR 14:25

And the action of exp(λ ω) on [f(x)] is

$$U_\lambda(f) = x^{-\frac{3}{2}} f(s_\lambda(x)).(s_\lambda(x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

which explains the prefactor. Again we can check by computation that the composition of (U_λ)s amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of Eq1 (slide 15) we must have

$$\exp(\lambda\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k \left(\frac{d}{dx}\right)^k$$

So, using this new technique, one can compute easily the coefficients of the matrix giving the normal forms.

Maple 9 - [calc1.mws - [Server 1]]

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```
> g1:=exp(y*x*exp(x));d1:=taylor(g1,x=0,7);
```

$$g1 := e^{(y\,x\,e^x)}$$

$$d1 := 1 + y\,x + \left(y + \frac{1}{2}y^2\right)x^2 + \left(\frac{1}{2}y + y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{1}{6}y + y^2 + \frac{1}{2}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{24}y + \frac{2}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{6}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{1}{120}y + \frac{1}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{3}y^4 + \frac{1}{24}y^5 + \frac{1}{720}y^6\right)x^6 + O(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 \end{bmatrix}$$

__For these one-parameter groups and conjugates of vector fields__

G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,

One-parameter groups and combinatorial physics,

Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003. arXiv : quant-ph/0401126.

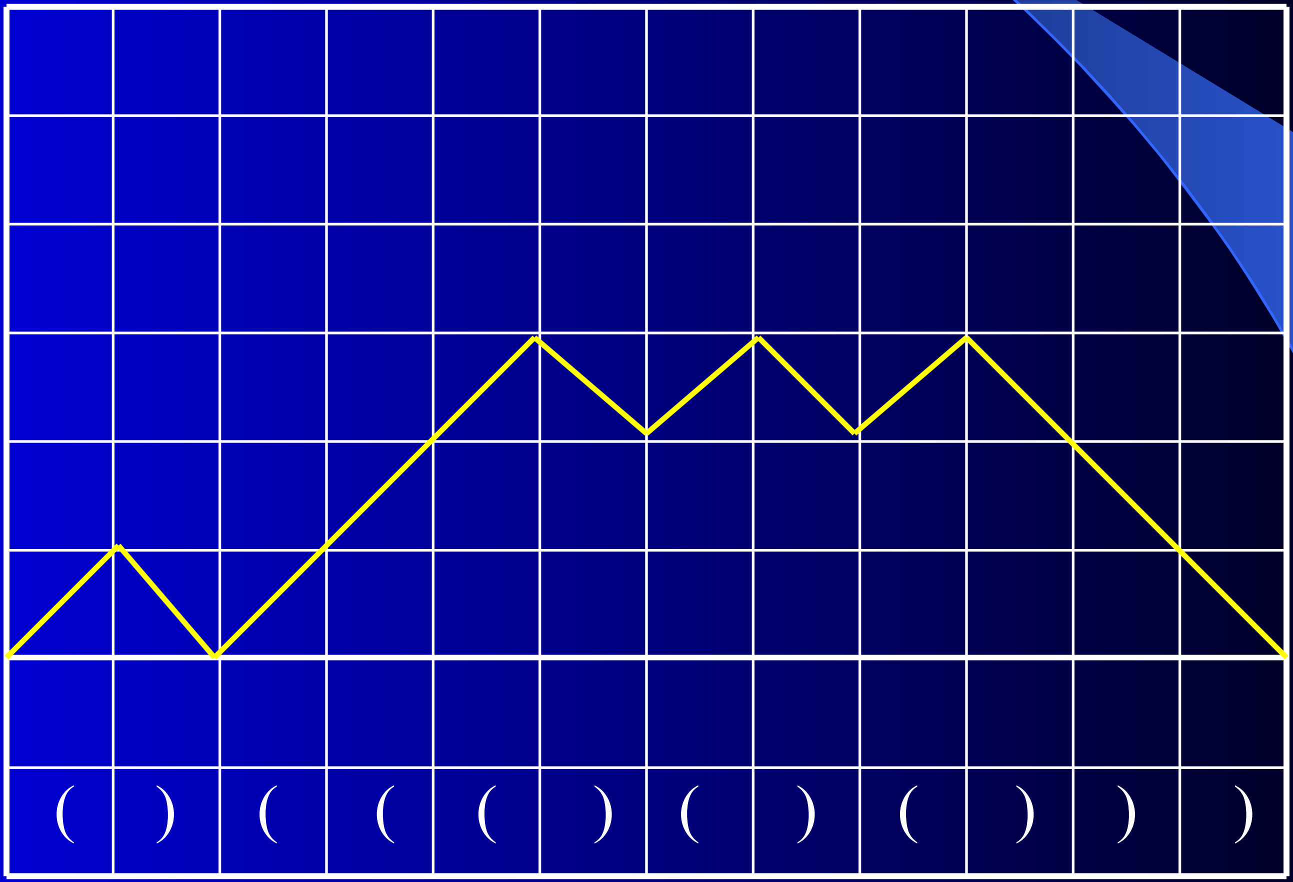
__For the Sheffer-type sequences and coherent states__

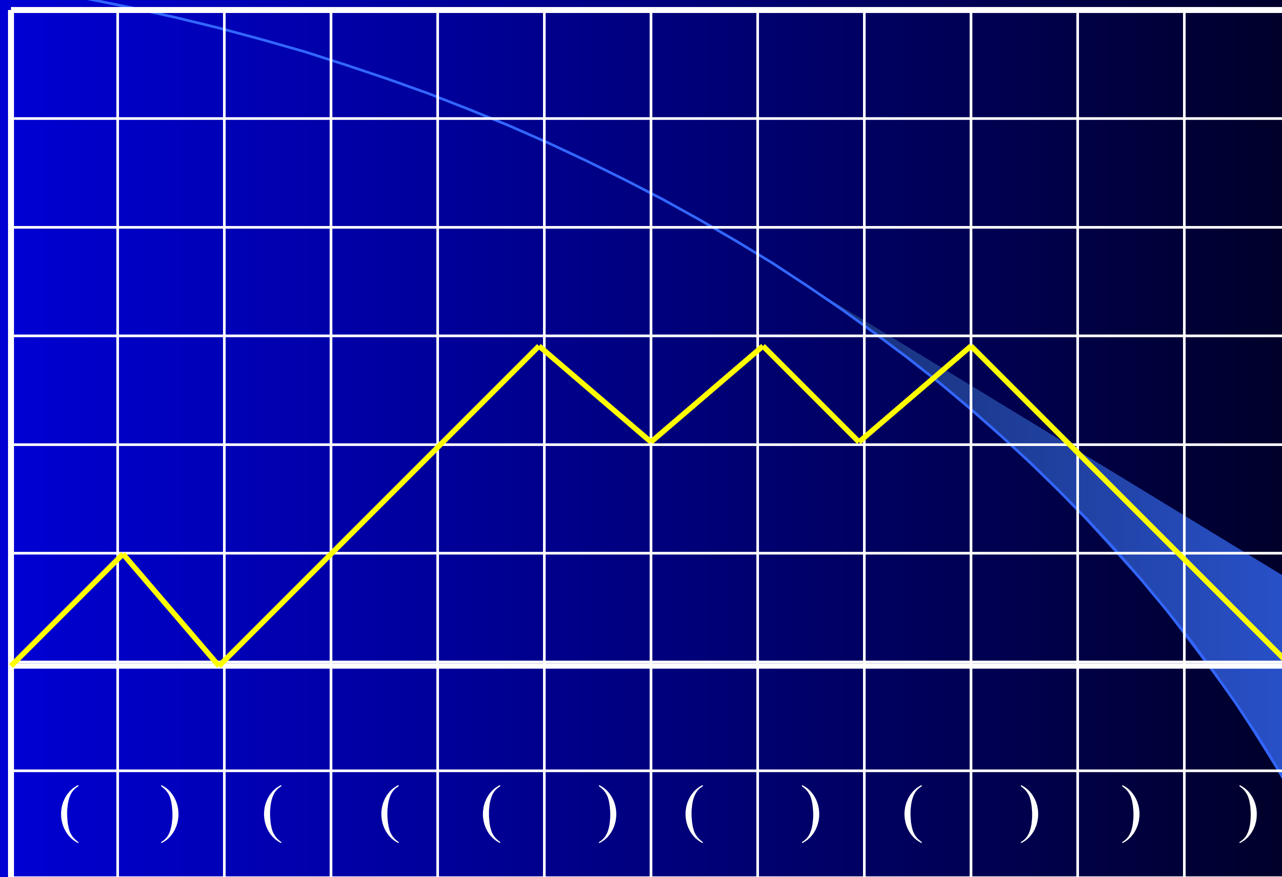
K A Penson, P Blasiak, G H E Duchamp, A Horzela and A I Solomon,

Hierarchical Dobinski-type relations via substitution and the moment problem,

J. Phys. A: Math. Gen. 37 3457 (2004) arXiv : quant-ph/0312202

A second application : Dyck paths
(systems of brackets, trees, physics, …)

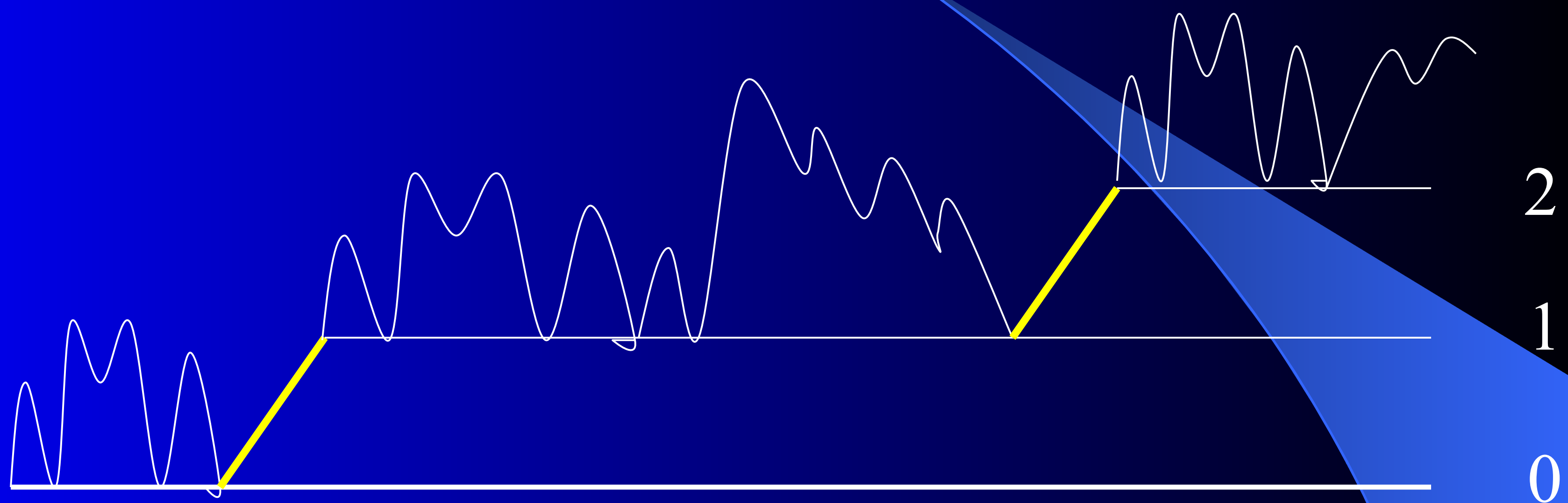




Equation : D = void + (D) D … one counts strings using an « x » by bracket and one finds $T(x)=x^0 + x^2\,T^2(x)$ which can be solved by elementary methods …

$x^2\,T^2 - T + 1 = 0$ Variable : T Parameter : x

Change of level (physics)



Positifs = D(aD)*

$$Pos := \frac{Dyck}{1 - x\,Dyck}$$

```
> solve(x^2*T^2-T+1=0,T);
```

$$\frac{1+\sqrt{1-4x^2}}{2x^2}, \frac{1-\sqrt{1-4x^2}}{2x^2}$$

```
> f:=1/(2*x^2)*(1-(1-4*x^2)^(1/2));
```

$$f := \frac{1-\sqrt{1-4x^2}}{2x^2}$$

```
> taylor(f,x=0,20);
```

$$1 + x^2 + 2x^4 + 5x^6 + 14x^8 + 42x^{10} + 132x^{12} + 429x^{14} + 1430x^{16} + O(x^{18})$$

```
> seq(binomial(2*k,k)/(k+1),k=1..8);
```

$$1, 2, 5, 14, 42, 132, 429, 1430$$

```
>
```

```
> Pos:=simplify(Dyck/(1-x*Dyck));
```

$$Pos := -\frac{2}{-1-\sqrt{1-4\,x\,y+2\,x}}$$

```
> coeftayl(Pos,[x,y]=[0,0],[6,4]);
```

$$90$$

```
> S:=0:for l from 0 to 6 do for k from 0 to 6 do
  S:=S+coeftayl(Pos,[x,y]=[0,0],[k,l])*x^k*y^l od
  od:S;
```

$$1+x+x\,y+20\,x^6\,y^2+14\,x^5\,y^2+5\,x^3\,y^3+2\,x^2\,y^2+x^3+28\,x^5\,y^3+x^4+x^5$$
$$+x^6+x^2+132\,x^6\,y^5+2\,x^2\,y+5\,x^3\,y^2+90\,x^6\,y^4+42\,x^5\,y^5+3\,x^3\,y$$
$$+132\,x^6\,y^6+4\,x^4\,y+14\,x^4\,y^4+14\,x^4\,y^3+5\,x^5\,y+9\,x^4\,y^2+48\,x^6\,y^3$$
$$+42\,x^5\,y^4+6\,x^6\,y$$

```
>
```

Automata and rationality

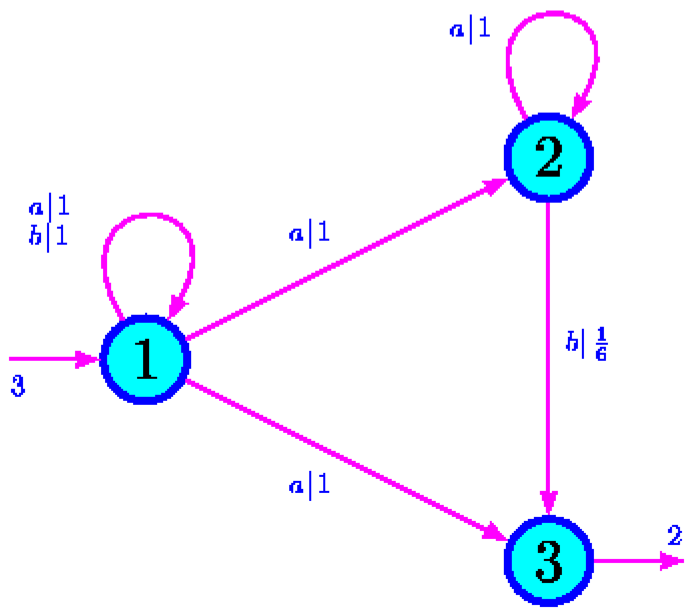


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6+a^*b).$$

Un type particulier d'automate à multiplicités est constitué des automates à multiplicités avec des ε-transitions.

Un k-ε-automate $\mathcal{A}_\varepsilon$ est un k-automate sur l'alphabet $A_\varepsilon = A \cup \{\tilde{\varepsilon}\}$.

Exemple :

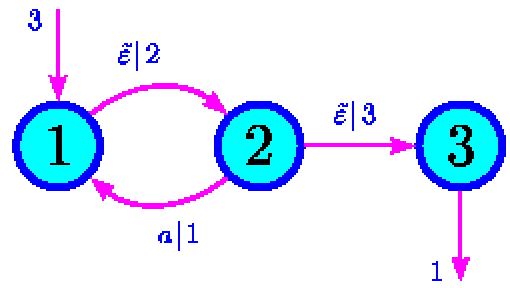
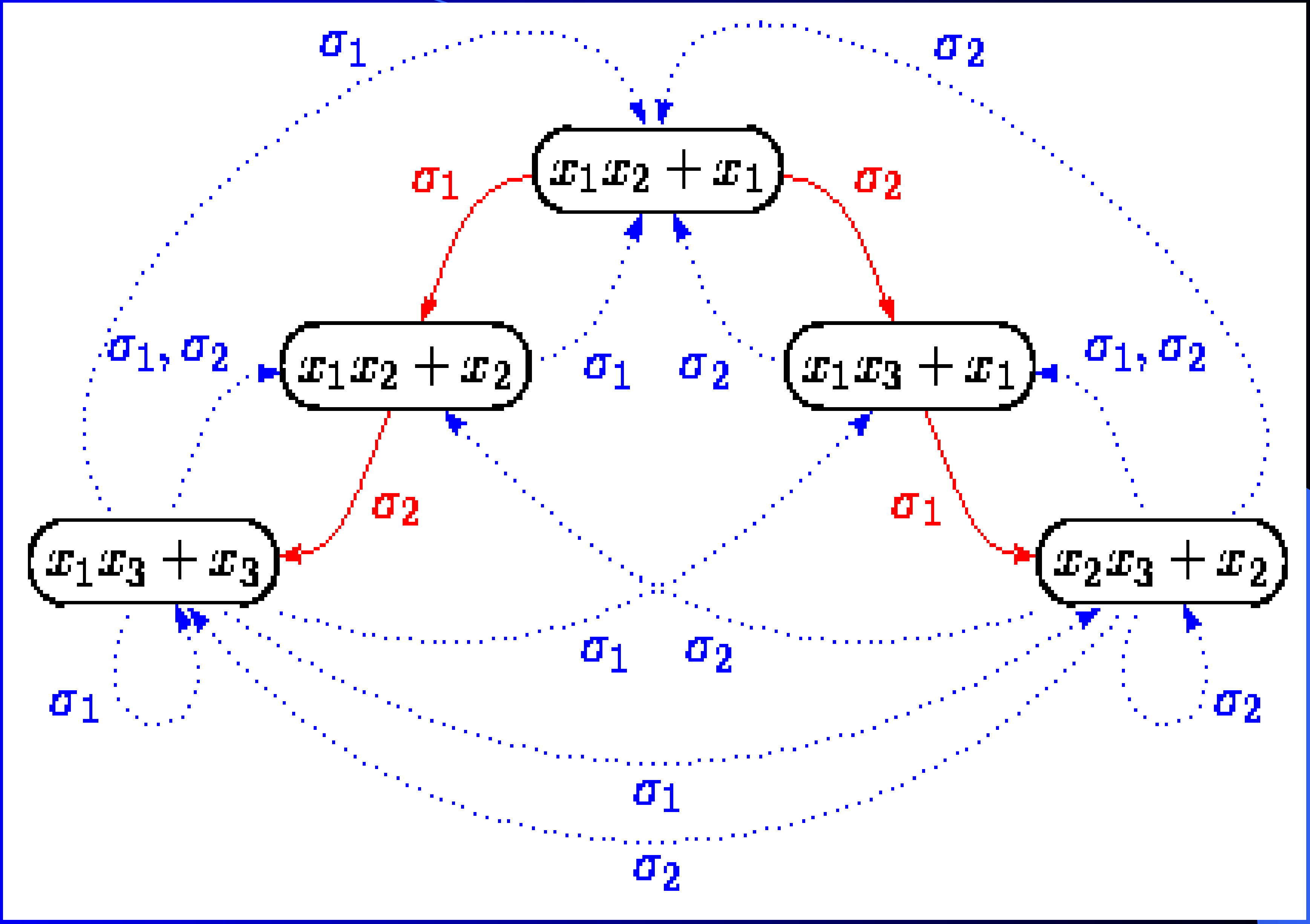


FIG. 2 – Un $\mathbb{N}$-ε-automate $\mathcal{A}_\varepsilon$

$$\text{comportement}(\mathcal{A}_\varepsilon) = 18\tilde{\varepsilon}\left(\sum_{i \in \mathbb{N}} 2^i (a\tilde{\varepsilon})^i\right)\tilde{\varepsilon}.$$


$x_1x_2 + x_1$
$x_1x_2 + x_2$
$x_1x_3 + x_1$
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A correct implementation of Schelling's model

<u>Problem</u> : If one scans the board, addressing the inhabitants one after one, result is sensitive to the order of scanning.

<u>Solution</u> : Invent a (combinatorial) data structure which adapted to the parallel structure of the moving intentions of the inhabitants.

Problem : If one scans the board, addressing the inhabitants one after one, result is sensitive to the order of scanning.

Solution : Invent a (combinatorial) data structure which adapted to the parallel structure of the moving intentions of the inhabitants --> this must be a global model.

Combinatorics (mathematics)

Complex Systems

Information
(comp. sci.)

Physics
(class. quant.)

Thank You