USING INFINITE WORDS TO MODEL NASH EQUILIBRIA IN INFINITELY REPEATED GAMES

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ABSTRACT

In this paper, we deal with modelling infinitely repeated games by using infinite words. These games are supposed noncooperative and with a perfect knowledge of the previous moves. In this context, we give a general definition of a Nash equilibrium, that we illustrate with a famous example.

***Index Terms*—** Modelling, infinite words, formal languages, game theory, strategy, Nash equilibrium.

1. INTRODUCTION

Game theory [5] is usually defined as a mathematical tool used to analyze strategical interaction, the game, between individuals which are called players. The games studied in this paper are supposed simultaneous, noncooperative, infinitely repeated and with a perfect knowledge of the previous moves. We will elucidate these ideas through a famous example.
In game theory, the distinction between the cooperative and noncoperative game is crucial. The Prisoner's Dilemma [2] is an interesting example to explain these notions. It is a game involving two players where each one has two possible actions : cooperate (c) or defect (d). The game consists of simultaneous actions of both players (called moves). It can be represented using the matrix :

π	c	d
c	(4,4)	(0, 5)
d	(5, 0)	(1, 1)

where each entry e_{ij} is an ordered pair of real numbers. The two players are referred to as the row player and the column player respectively. The actions of the first player are identified with the rows of the matrix and those of the second one with the columns. If the row player chooses action i and the second action j, the components of the ordered pair e_{ij} are the payoff received by the first and the second player respectively. It is clear that if they could play cooperatively and make a binding agreement, they would both play c. If the game is noncooperative, the best action for each player is d.

Suppose now that we consider now infinite repetitions of a noncooperative base game. This game is just as noncooperative as the base one, but it allows a certain form of interaction. Suppose that each player has a perfect knowledge of the previous moves of all the others. Then his strategy may depend on these previous moves and he may coordinate it with that of his opponents. For instance, if the base game is the Prisoner's Dilemma, grim-trigger is the strategy of cooperating in the first move and until your adversary defects, then of always defecting after the first defection of your opponent. Tit-for-tat is the strategy of playing at each step the action played by your adversary at the previous one ; the initial move is free.

In this paper, we make use of infinite words to analyze the kind of games we want to model. A match of such a game is represented as an infinite word on the alphabet A of moves. In this context, a strategy for player i can be viewed as a relation from the set of finite words on A to that of the actions of this player. The whole strategy of the game is defined as the vector composed by using the strategies of all players. We can associate to each strategy vector a language L of infinite words on A, defined as the set of all matches that the players may make if everyone follows the strategy he decided to apply.

Nash equilibrium is one of the most important notions in games theory. The whole strategy of the game is defined as the vector composed by using the strategies of all players. Intuitively, a strategy vector is a Nash equilibrium if one player's departure from it while the others remain faithful to it results in punishment. The idea is that once the players start playing according to such a strategy vector, then they all have a good reason to stay with it.

More precisely, our study will be organized as follows. Sections 1 contains some basic notions on game theory. In Section 2, we introduce the notion of strategy and give the definition of language generated by a strategy. Section 3 is devoted to the formal definition of a Nash equilibrium with some examples.

2. MATHEMATICAL MODEL FOR GAMES

Non cooperative games in which moves consist of simultaneous actions of n players, can be represented by a collection of n utility functions. The values of these functions define the expected amount paid to the players. A game is a tuple $G = (P, A, \pi)$ where

- $P = \{1, \cdots, n\}$, $n \in \mathbb{N}$, is the set of players.
- A_i is the set of the actions for player i.
- $A = A_1 \times \ldots \times A_n$ is the alphabet of the moves.
- $\pi_i : A \longrightarrow \mathbb{R}$. is the utility function for player i.
- $\pi = (\pi_1, \ldots, \pi_n) : A \longrightarrow \mathbb{R}^n$ is the utility vector.

We consider in this paper the δ-discounted infinitely repeated game of G, which we note by G^ω. In such a game, we model a match h as an infinite sequence of moves which can be represented by an infinite word on the alphabet of the moves A : $h = h_0 h_1 \cdots h_t \cdots \in A^\omega$. We denote by $h_{i,j}$ the j^{th} component of move $h_i \in A$.
The utility with discounting factor $\delta \in (0, 1)$ of a match h for player i is defined as :

$$\pi_i^\delta(h) = (1 - \delta) \sum_{k=0}^{\infty} \pi_i(h_k) \delta^k.$$

Example 2.1 *As concerns the Prisoner's Dilem ma, we have $P = \{1, 2\}$, $A_1 = A_2 = \{c, d\}$, $A = \{c, d\} \times \{c, d\}$ and the utility function is defined by the matrix given in the Introduction. The infinite word $h = (c, c)^\omega$ is an example of a match in which the two players cooperate infinitely. The value of the utility function with discounting factor $\delta \in (0, 1)$ of h for player i is :*

$$\begin{aligned} \pi_i^\delta(h) &= (1 - \delta) \textstyle\sum_{k=0}^{\infty} \pi_i((c, c)) \delta^k \\ &= 4(1 - \delta) \textstyle\sum_{k=0}^{\infty} \delta^k \\ &= 4. \end{aligned}$$

3. STRATEGIES AND LANGUAGES

A nondeterministic strategy, called also quasi-strategy, σ_i is a relation from A^* into A_i that describes the behaviour of player i during the game. A strategy vector on A is the relation

$$\sigma = (\sigma_1, \ldots, \sigma_n) : A^* \longrightarrow A$$

defined by :

$$(a_1, \ldots, a_n) \in \sigma(w) \iff a_i \in \sigma_i(w),$$

$\forall w \in A^*, \forall a_i \in A_i,\ 1 \leq i \leq n$.
Let Σ be the set of all strategy vectors on A. We consider the map : $\gamma : \Sigma \longrightarrow \mathcal{P}(A^\omega)$, where $\mathcal{P}(A^\omega)$ denotes the set of all languages in A^ω, which associates to each strategy $\sigma \in \Sigma$, the language of infinite words $\gamma(\sigma)$ given by :

$\gamma(\sigma) = \{h \in A^\omega |\ h_0 \in \sigma(\epsilon)\ and\ h_{t+1} \in \sigma(h_0$

$\cdots h_t), \forall t \geq 0\}$.
The language $\gamma(\sigma)$ represents the set of all matches that the players can play according to σ.

Example 3.1 *We give a strategy for the Prisoner's Dilemma game.*

$$\sigma(w) = \begin{cases} \{(c,c),(c,d)\} & if w \in (c,c)^* \\ \{(d,c),(d,d)\} & if w \in (c,c)^*(c,d) \\ & ((d,c)+(d,d))^* \\ \emptyset & \text{otherwise} \end{cases}$$

It is usually called the "grim-trigger" strategy. The language $L = \gamma(\sigma)$ is described by the ω-rational expression

$$(c,c)^\omega + (c,c)^*(c,d)((d,c)+(d,d))^\omega.$$

Example 3.2 *Consider the following strategy σ on the alphabet $A = \{a,b\}$:*

$$\sigma(w) = \begin{cases} \{a,b\} & \text{if } |w|_a < |w|_b \\ b & \text{otherwise} \end{cases}$$

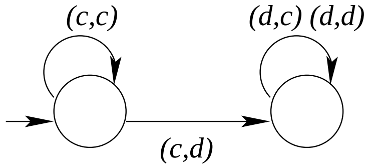
The language $L = \gamma(\sigma)$ associated is

$$\{h \in A^\omega \mid Pref(h) \in \{w \in A^* \mid |w|_a \leq |w|_b\}\}.$$

We note that this language is not ω-rational, in the sense of language theory.

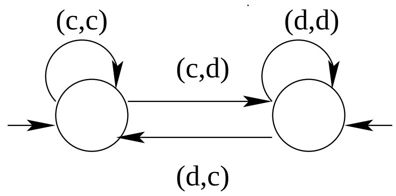
In the following examples, we deal with strategies in which the players need only a finite memory to store the past moves. Such strategies can be represented by finite Büchi automata [4]. The languages associated to these strategies are ω-rationals. We suppose that all states of the automata considered in this paper are final.

Example 3.3 Grim-trigger strategy for player 1
The grim-trigger strategy in Example 3.1 is given by the following automaton. Notice that this



automaton describes a strategy function for the first player. Indeed, all the arrows starting from each state are labelled with the same first component.

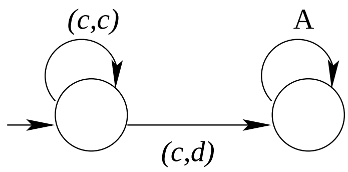
Example 3.4 Tit-for-tat strategy for player 1
The strategy described by this automaton is in



fact composed of two elementary strategy functions for the first player, which depend on the initial state chosen at the beginning of the match. We denote by L_c (resp. L_d) the set of matches played if the first player chooses action c (resp. d) to start. The language recognized by this automaton is $L = L_c \bigcup L_d$, where :

$L_c = ((c,c)+(c,d)(d,d)^*(d,c))(d,d)^\omega+((c,c)$
$+ (c,d)(d,d)^*(d,c))^\omega$,
$L_d = ((d,d)+(d,c)(c,c)^\omega(c,d))^*(d,c)(c,c)^\omega$
$+ ((d,d)+(d,c)(c,c)(c,d))^\omega$.

Example 3.5 Weak grim-trigger strategy for player 1 *Here we have a deterministic automa-*



ton, the strategy of which is a pure relation. The behaviour of the first player becomes unpredictable after the defection of player 2. The set of matches recognized by this automaton is the language :

$$L = (c,c)^\omega + (c,c)^*(c,d)A^\omega.$$

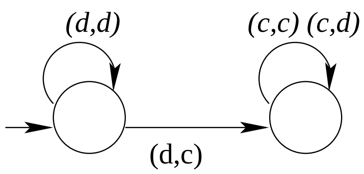
4. NASH EQUILIBRIUM

Intuitively, a strategy vector is a Nash equilibrium if no player has any interest in leaving his strategy, while his opponents remain faithful to theirs. Let us first introduce some basic notions, in order to give a formal definition of a Nash equilibrium.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$. We call *i-variation* of α every $\beta \in A$ such that $\alpha_i \neq \beta_i$ and $\alpha_j = \beta_j, \forall j \neq i$.
Let X be a language of A^ω. We call *i-variation* of a match $h = h_0 h_1 \ldots h_t \ldots$ in X every match $\overline{h} \in X$ for which there exists $t \geq 0$, an i-variation α of h_t and a word $w \in A^\omega$ such that $\overline{h} = h_0 \ldots h_{t-1} \alpha w \in X$.
A *good match* for player i in X is a match $h \in X$ verifying $\pi_i(h) \geq \pi_i(\overline{h})$ for every i-variation $\overline{h}$ of h. Denote by $GM_i(X)$ the set of all good matches for player i in X.

Example 4.1 *Consider the language $L = (c,c)^\omega + (d,d)^\omega$. It is obvious that the words $(c,c)^\omega$ and $(d,d)^\omega$ do not admit any i-variation in L. So we have $GM_i(L) = L,\ \forall i = 1, 2$.*

Example 4.2 *Let L be the language of infinite words recognized by the following automaton. This language involves the Prisoner's Dilemma*



strategy in which the first player defects as far as his adversary defects and cooperates infinitely as soon as his opponent cooperates. It is clear that $L = (d,d)^\omega + (d,d)^*(d,c)((c,c)+(c,d))^\omega$. We claim that $h = (d,c)(c,d)^\omega \in GM_2(L)$ if $\delta > 1/5$, otherwise $(d,d)^\omega \in GM_2(L)$. Indeed, let $\overline{h}$ be a 2-variation of h. Then $\overline{h} \in (d,d)^\omega + (d,d)^*((c,c) + (c,d))^\omega$. But, at the sight of the payment matrix given in the Introduction, we notice it pays more payful for the second player always to choose d instead of c after his first cooperation. Thus, we will only examine the 2-variations belonging to $(d,d)^\omega + (d,d)^*(d,c)(c,d)^\omega$. We obtain successively for $n \in \mathbb{N}$:
$\pi_2^\delta((d,d)^n(d,c)(c,d)^\omega) =$
$(1-\delta)(\sum_{k=0}^{n} \delta^k + \sum_{k=n+2}^{\infty} 5\delta^k) =$
$(1-\delta)[\sum_{k=0}^{n} \delta^k + 5(\sum_{k=0}^{\infty} \delta^k - \sum_{k=0}^{n+1} \delta^k)] =$
$1 - \delta^{n+1} + 5\delta^{n+2} =$
$1 + \delta^{n+1}(5\delta - 1)$.
The case of $\overline{h} = (d,d)^\omega$ can be dropped when $\delta > 1/5$, because we have $1 + \delta^{n+1}(5\delta - 1) > 1 = \pi_2^\delta((d,d)^\omega)$. Furthemore, one can easily verify that the maximum of the function $n \longmapsto 1 + \delta^{n+1}(5\delta - 1)$ is reached for $n = 0$. Hence, the word $(d,c)(d,c)^\omega$ belongs to $GM_2(L)$.

The notion of Nash equilibrium also requires the introduction of some basic strategy vectors. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a strategy vector and let $X = \gamma(\sigma)$ be the associated language. We denote by $\varpi_i : A^* \longrightarrow A_i$ the unpredictable strategy for player i, given by $\varpi_i(w) = A_i,\ \forall w \in A^*$.
We define for all $1 \leq i \leq n$, the following strategy vectors :

$$\begin{array}{lcl} \mu^{(i)} & = & (\varpi_1, \ldots, \varpi_{i-1}, \sigma_i, \varpi_{i+1}, \ldots, \varpi_n) \\ \nu^{(i)} & = & (\sigma_1, \ldots, \sigma_{i-1}, \varpi_i, \sigma_{i+1}, \ldots, \sigma_n). \end{array}$$

We denote by $X_i = \gamma(\mu^{(i)})$ and by $Y_i = \gamma(\nu^{(i)})$

Proposition 4.3 *We have :*

- $X = \bigcap_{1 \leq i \leq n} X_i$;
- $Y_i = \bigcap_{j \neq i} X_j \ \forall 1 \leq i \leq n$.

Proof. For the first part, we obtain the succession of equations :
$X =$
$\{h \in A^\omega |\ h_0 \in \sigma(\epsilon)\ and\ h_{t+1} \in \sigma(h_0 \ldots h_t),\ \forall t \geq 0\} =$
$\{h \in A^\omega |\ h_{0,i} \in \sigma_i(\epsilon)\ and\ h_{t+1,i} \in \sigma_i(h_0 \ldots h_t),\ \forall 1 \leq i \leq n,\ \forall t \geq 0\} =$
$\bigcap_{1 \leq i \leq n} \{h \in A^\omega |\ h_{0,i} \in \sigma_i(\epsilon)\ and\ h_{t+1,i} \in \sigma_i(h_0 \ldots h_t),\ \forall t \geq 0\} =$
$\bigcap_{1 \leq i \leq n} \gamma(\mu^{(i)}) =$
$\bigcap_{1 \leq i \leq n} X_i$.
The second part of the proof is easier, since we immediately obtain by using the lines above :
$Y_i =$
$\{h \in A^\omega |\ h_{0,j} \in \sigma_j(\epsilon)\ and\ h_{t+1,j} \in \sigma_j(h_0 \ldots h_t),\ \forall t \geq 0,\ \forall j \neq i\} =$
$\bigcap_{j \neq i} \gamma(\mu^{(j)}) =$
$\bigcap_{j \neq i} X_j$.

Definition 4.4 *A strategy vector*

$$\sigma = (\sigma_1, \ldots, \sigma_n)$$

is a Nash equilibrium if

$$\bigcap_{i=1}^{n} GM_i(Y_i) \neq \emptyset.$$

In other words, a strategy vector is a Nash equilibrium if there exists a match that represents a good match for each player in the set of matches of the others [2]. In particular, in the case of two players, the general definition becomes :

$$GM_1(X_2) \bigcap GM_2(X_1) \neq \emptyset.$$

Example 4.5 *We consider in the Prisoner's Dilemma game, the vector $\sigma = (\sigma_1, \sigma_2)$ in which both players follow the grim-trigger strategy. In this case we have :*

$$\begin{array}{lcl} X_1 & = & (c,c)^\omega + (c,c)^*(c,d)((c,c),(d,d))^\omega, \\ X_2 & = & (c,c)^\omega + (c,c)^*(d,c)((c,d),(d,d))^\omega. \end{array}$$

We claim that (σ_1, σ_2) is a Nash equilibrium if and only if the discounting factor $\delta \geq 1/4$. Indeed $(c,c)^\omega \in GM_1(X_2) \cap GM_2(X_1)$. Suppose $\overline{h} = (c,c)^{k-1}(c,d)(d,d)^\omega$ be a match with defection of the first player at the rank $k \geq 0$. We obtain after computations

$$\pi_1^\delta(h) - \pi_1^\delta(\overline{h}) = \delta^k(4\delta - 1).$$

Then $\pi_1^\delta(h) - \pi_1^\delta(\overline{h}) \leq$ iff $\delta \leq 1/4$, which proves that $h \in GM_1(X_2)$ whenever $\delta \geq 1/4$. In the same way, we can show that $h \in GM_2(X_1)$.

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