Heisenberg–Weyl algebra revisited: Combinatorics of words, paths and rook polynomials

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Abstract. The Heisenberg–Weyl algebra, underlying virtually all physical representations of Quantum Theory, is considered from the combinatorial point of view. We provide a concrete model of the algebra in terms of paths on a lattice with some decomposition rules. We also discuss the rook problem on the associated Ferrers board related to the calculus in the normally ordered basis. In this way we draw attention to a profound combinatorial underpinning of the Heisenberg–Weyl algebra offering novel perspectives, methods and applications.

1. Introduction

From a modern viewpoint the formalism and structure of Quantum Theory is founded on the theory of Hilbert space [1, 2]. The physical content of the theory consists of representing physical quantities as operators which satisfy some algebraic relations. Virtually all correspondence schemes come endowed with the Heisenberg–Weyl algebra structure, be it the canonical quantization scheme, the occupation number representation in quantum mechanics, or the second quantization formalism of quantum field theory. This derives from analogy to classical mechanics whose Poissonian structure is reflected in the commutator of position and momentum observables [3]. Ubiquitous and profound, the Heisenberg–Weyl algebra has become the hallmark of non-commutativity in Quantum Theory.

An exemplary model of the Heisenberg–Weyl algebra consists of derivative D, multiplication X and identity I operators acting on the space of polynomials. Physical examples are the position x and momentum p operators in the space of square integrable functions, or the annihilation a and creation $a^\dagger$ operators in Fock space. Here, without loss of generality, we conform to the notation $\{a, a^\dagger, I\}$ for generators of the algebra‡, satisfying

$$aa^\dagger = a^\dagger a + I. \tag{1}$$

We shall be interested in combinatorial aspects of this relation and discuss one of the ensuing models of the Heisenberg–Weyl algebra.

Indeed, the combinatorial properties of Eq.(1) have been recognized early and successfully applied in the domain of algebraic enumeration, principally concerning the action of the operators X an D on generating series. From this point of view these operators are auxiliary constructions facilitating enumeration of discrete structures [4, 5, 6, 7, 8, 9, 10].

However in this note we adopt another approach leading towards a combinatorial model of the algebra itself. Starting from the definition of the Heisenberg–Weyl algebra as the algebra of words supplemented by the relation of Eq. (1) we will recast it in the language of paths on a lattice with some decomposition rules. We shall also consider a convenient choice of basis, here taken to be normally ordered monomials, which permits a direct link to the combinatorics of words and the related algebra of paths. In this way algebraic problems may be expressed in the more concrete form of the decomposition and enumeration of paths. For illustration we consider the normal ordering procedure which reduces to the familiar rook problem on the Ferrers board and then derive the structure constants of the algebra by a simple combinatorial argument.

‡ We do not attach much weight to this particular realization, however, as we shall study algebraic properties only, for which the underlying Fock space plays no role. Our considerations hold true for any representation of the Heisenberg–Weyl algebra.

2. Heisenberg–Weyl algebra

By definition an *algebra* $\mathcal{A}$ is a linear vector space over a field $\mathbb{K}$ with bilinear multiplication law

$$\mathcal{A} \times \mathcal{A} \ni (b, c) \longrightarrow b\, c \in \mathcal{A} \tag{2}$$

which is associative and possesses a unit element I. Precisely, it is called an *associative algebra with unit* in distinction to algebraic structures lacking associativity or a unit (*e.g.* Lie algebras). A *Basis* of an algebra is a basis for its vector space structure. Each basis $(b_i)_{i\in I}$ defines a unique family $\gamma_{ij}^k \in \mathbb{K}$ such that for every ordered pair $(i,j) \in I \times I$ the set of $k \in I$ such that $\gamma_{ij}^k \neq 0$ is finite and

$$b_i\, b_j = \sum_{k\in I} \gamma_{ij}^k\, b_k\,. \tag{3}$$

The γ_{ij}^k are called the *structure constants* of the algebra $\mathcal{A}$ with respect to the basis $(b_i)_{i\in I}$ from which the multiplication law can be uniquely recovered.

The *Heisenberg–Weyl algebra*, denoted by $\mathcal{H}$, is the algebra generated by a, $a^\dagger$ and I, satisfying the relation of Eq. (1). Elements of the algebra $A \in \mathcal{H}$ are linear combinations of finite products of a and $a^\dagger$ of the form

$$A = \sum_{\mathbf{r},\mathbf{s}} \quad \alpha_{\mathbf{r},\mathbf{s}} \quad a^{\dagger\, r_1} a^{s_1} a^{\dagger\, r_2} a^{s_2} \ldots\, a^{\dagger\, r_k} a^{s_k}, \tag{4}$$

where $\mathbf{r} = (r_1, ..., r_k)$ and $\mathbf{s} = (s_1, ..., s_k)$ are nonnegative integer multi-indexes (with the convention $a^0 = a^{\dagger\, 0} = I$). This representation is ambiguous, however, due to the commutation relation Eq. (1) which yields different representations of the same element of the algebra, *e.g.* $aa^\dagger = a^\dagger a + I$. The problem can be resolved by fixing the preferred order of generators a and $a^\dagger$. Conventionally it is done by choosing the *normally ordered* form in which all annihilators stand to the right of creators. Consequently, each element of the algebra $\mathcal{H}$ can be uniquely written in the normally ordered form as

$$A = \sum_{r,s} \alpha_{rs}\; a^{\dagger\, r} a^s\,. \tag{5}$$

Hence the normally ordered monomials $a^{\dagger\, r} a^s$ constitute a legitimate basis for the Heisenberg–Weyl algebra

$$\text{Basis of } \mathcal{H}\ : \qquad b_{(r,s)} = a^{\dagger\, r} a^s\,, \tag{6}$$

indexed by pairs of integers $r, s = 0, 1, 2, ...$, and Eq. (5) is the expansion of element A in this basis. We should note that the normally ordered representation of the elements of the algebra suggests itself as the simplest one [11]. It is important and commonly used in practical applications in quantum optics [12, 13, 14] or quantum field theory [15, 16]. Working in this basis entails the reshuffling of a and $a^\dagger$ to the normally ordered form, which in general is a nontrivial task [17]. This brings up the issue of efficient calculation methods and intuitive schemes providing insight into the ordering procedure itself. Below, we provide a combinatorial model for the Heisenberg–Weyl algebra and then propose a resolution the problem from this starting-point.

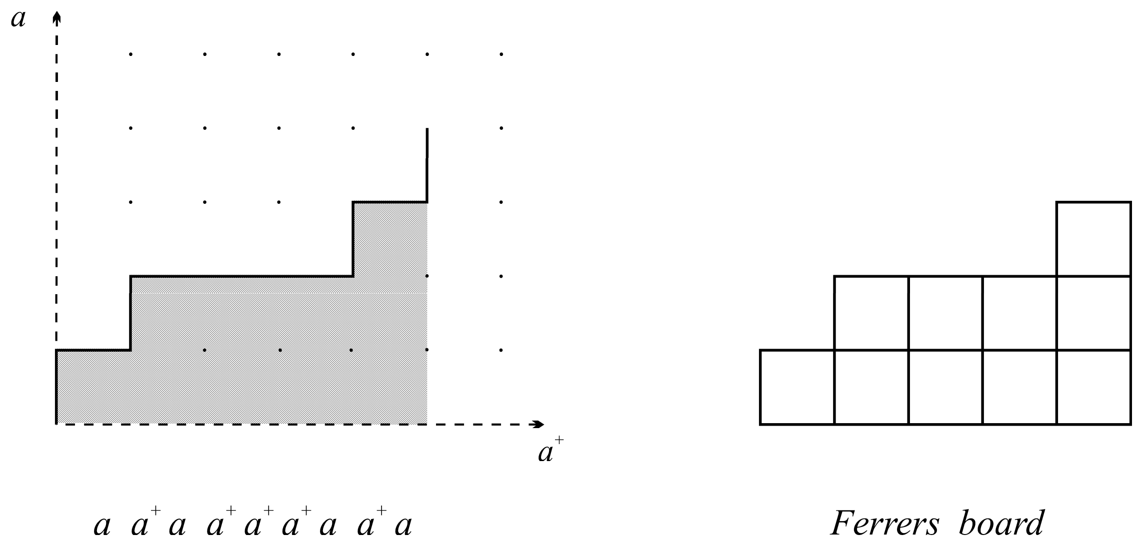

Figure 1. Path defined by the word $\mathtt{w}$ and the associated Ferrers board $\mathcal{B}_{\mathtt{w}}$.

3. Combinatorics of the Heisenberg–Weyl algebra

3.1. Words, paths and rook problems

We start by showing that each word in a two-letter alphabet can be uniquely encoded as a staircase path on a plane rectangular lattice. This observation will lead us to considering the associated Ferrers board and the rook problem. Rather than giving a formal construction, we shall illustrate it by an example from which the general scheme can be straightforwardly recovered.

Suppose we consider a word, say

$$\mathtt{w} = a\,a^\dagger a\,a^\dagger a^\dagger a^\dagger a\,a^\dagger a \tag{7}$$

to which we assign a staircase path on a rectangular lattice. Starting from the point $(0,0)$, it is constructed by reading the word $\mathtt{w}$ from the left and drawing a line to the right if the letter is $a^\dagger$ and up if the letter is a, as shown in Fig. 1 on the left. We observe that this scheme provides a unique encoding of words.

To each path (or word) one can associate a *Ferrers board* $\mathcal{B}_{\mathtt{w}}$ by retaining rectangular cells below the path, see Fig. 1. Note that this is many-one procedure as paths differing by a horizontal line at the beginning and a vertical line at the end yield the same board [5, 10].

The *rook problem* for the given board $\mathcal{B}$ consists in enumerating non–capturing arrangements of k rooks on the board which defines a finite sequence $R_{\mathcal{B}}(k)$, $k = 0, 1, 2, ...$, called the *rook numbers*. In our example one has

$$R_{\mathcal{B}}(k) = 1, 10, 23, 9, 0, 0, ... \qquad k = 0, 1, 2, 3, \ldots . \tag{8}$$

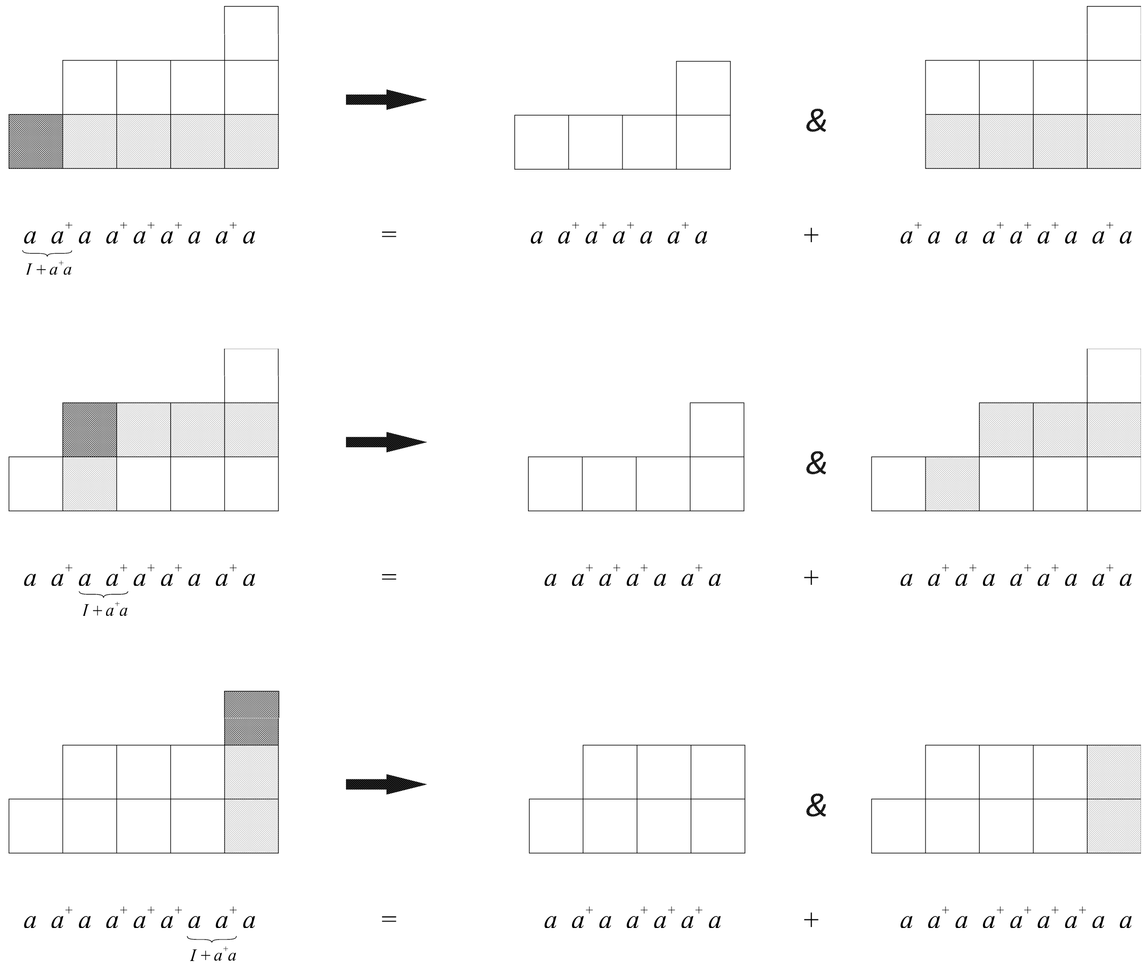

Figure 2. Three possible decompositions of the Ferrers board $\mathcal{B}_{\mathtt{w}}$ and the corresponding reduction of the word $\mathtt{w}$.

A rook sequence can be conventionally encoded in a *rook polynomial* defined by

$$R_{\mathcal{B}}(x) = \sum_k R_{\mathcal{B}}(k)\, x^k. \tag{9}$$

It is straightforward to show that these polynomials satisfy the recursion [4, 5, 10]

$$R_{\mathcal{B}}(x) = R_{\mathcal{B}'}(x) + x\, R_{\mathcal{B}''}(x), \tag{10}$$

$$R_{\varnothing}(x) = 1, \tag{11}$$

obtained by choosing a cell making a step in the diagram $\mathcal{B}$ and considering two cases: a rook is placed in the cell or not. This reduces the problem to the boards $\mathcal{B}''$ (with the row and column pertaining to the cell removed) and $\mathcal{B}'$ (with the chosen cell removed only). We note that there are many possible choices of a cell which give different decompositions of the board yielding various recursive patterns, see Fig. 2.

3.2. Normal ordering procedure

It was shown [18, 19, 20] that the normal ordering of a word $\mathtt{w}$ in a and $a^\dagger$ satisfying Eq. (1) reduces to the rook problem on the associated Ferrers board $\mathcal{B}_{\mathtt{w}}$. Namely,

$$\mathtt{w} = \sum_{k=0} R_{\mathcal{B}_{\mathtt{w}}}(k)\, \mathtt{w}^{(k)} \tag{12}$$

where $\mathtt{w}^{(k)}$ are normally ordered monomials $a^{\dagger\, r} a^s$ obtained from $\mathtt{w}$ by crossing out k pairs of a and $a^\dagger$ and then reshuffling the rest as if they were commuting variables (called the double dot operation used in quantum field theory). For example for a word in Eq. (7) we have $\mathtt{w}^{(0)} = a^{\dagger\,5}a^4$, $\mathtt{w}^{(1)} = a^{\dagger\,4}a^3$, $\mathtt{w}^{(2)} = a^{\dagger\,3}a^2$, *etc.*, and hence its normally ordered form reads (see Eq. (8))

$$\mathtt{w} = a^{\dagger\,5}a^4 + 10\, a^{\dagger\,4}a^3 + 23\, a^{\dagger\,3}a^2 + 9\, a^{\dagger\,2}a\ . \tag{13}$$

A rigorous proof of Eq. (12) relies on the observation that each word can be reduced to the sum of two simpler ones by choosing the places in which a precedes $a^\dagger$ and reshuffling them according to Eq. (1), *i.e.* $aa^\dagger \to I + a^\dagger a$. For example, for a word of Eq. (7) there are three choices which exactly correspond to possible decompositions of the associated Ferrers diagram $\mathcal{B}_{\mathtt{w}}$ in Fig. 2. We note that although there are various possible decomposition schemes the result is unique.

In short, the normal ordering of a word reduces to the enumeration of possible non-capturing rook arrangements on the associated Ferrers board. The problem can be systematically handled by successive decompositions of the board. Moreover, one can devise simple algorithms based on this recursive rule.

3.3. Combinatorial realization of the Heisenberg–Weyl algebra

We have observed in Sect. 3.1 that each word in two letters, here taken as a and $a^\dagger$, can be encoded as a path, see Fig. 1. This establishes an isomorphism between algebras of words $\mathcal{W} = \{a, a^\dagger\}^*$ and the algebra of paths $\mathcal{P}$ in which multiplication is given by simple concatenation (with the unit being the void $\emptyset$ word or path respectively). Both algebras are free. The Heisenberg–Weyl algebra arises by imposing on $\mathcal{W}$ the relation of Eq. (1), *i.e.* $\mathcal{H} = \mathcal{W}/_{aa^\dagger = a^\dagger a + I}$. This relation taken in $\mathcal{P}$ takes the form

$$\ulcorner = \lrcorner + \emptyset \tag{14}$$

which implies reducing the number of steps in a path either by removing a step or the whole row and column. Note that this reduction is exactly equivalent to the decomposition of the associated Ferrers board induced by the rook problem. Actually, any path can be uniquely decomposed into the finite sum of paths without steps (*i.e.* paths pertaining to monomials $a^{\dagger\, r}a^s$). The latter constitute the basis in $\mathcal{P}/_{\ulcorner = \lrcorner + \emptyset}$ corresponding to the normally ordered basis in $\mathcal{H}$.

In this way, we obtain the combinatorial model of the Heisenberg–Weyl algebra as the algebra of paths with the relation Eq. (14), *i.e.*

$$\mathcal{H} = \mathcal{W}/_{aa^\dagger = a^\dagger a + I} = \mathcal{P}/_{\ulcorner = \lrcorner + \emptyset}. \tag{15}$$

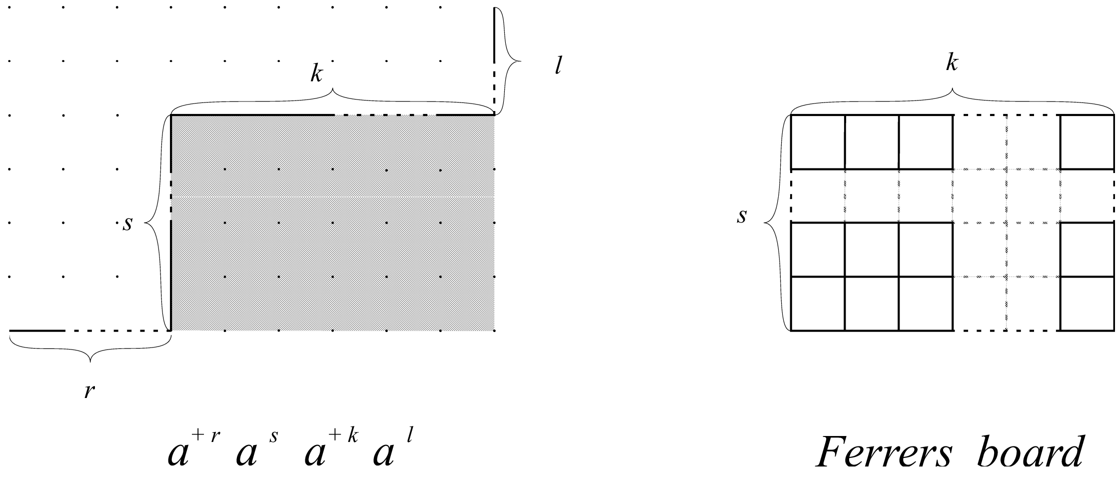

Figure 3. Path defined by the word $\mathtt{w} = a^{\dagger\,r} a^s\, a^{\dagger\,k} a^l$ and the associated Ferrers board.

With this equivalence the algebraic structure of $\mathcal{H}$ and its calculus is recast in combinatorial terms, thus providing different perspectives and making accessible intuitive combinatorial arguments.

3.4. Example: Structure constants

To illustrate the above equivalence we show how to calculate structure constants of the Heisenberg–Weyl algebra by simple combinatorial enumeration. Suppose, we multiply two elements of the basis of Eq. (6), say $b_{(r,s)}$ and $b_{(k,l)}$. Expansion in this basis of the product $b_{(r,s)}\, b_{(k,l)}$ essentially comes from the normal ordering of the word

$$\mathtt{w} = a^{\dagger\,r} a^s\, a^{\dagger\,k} a^l\ . \tag{16}$$

Following the scheme of Section 3.1 we draw the associated path and subsequently read out the Ferrers board $\mathcal{B}_{\mathtt{w}}$ which has a simple rectangular form, see Fig. 3. Enumeration of non–capturing rook arrangements on the board is a direct result of the observations: *(1)* the maximum number of rooks on the board is $\min\{s, k\}$, and *(2)* there are as many possible arrangements of i rooks as unordered choices of i columns and rows in the board $\mathcal{B}_{\mathtt{w}}$. Hence the rook numbers are

$$R_{\mathcal{B}_{\mathtt{w}}}(i) = i!\binom{s}{i}\binom{k}{i}\ , \tag{17}$$

for $i = 0\ ...\ \min\{s, k\}$ and zero otherwise. Consequently, Eq. (12) gives the normally ordered form of the word $\mathtt{w}$

$$a^{\dagger\,r} a^s\, a^{\dagger\,k} a^l = \sum_{i=0}^{\min\{k,s\}} i!\binom{s}{i}\binom{k}{i}\, a^{\dagger\,r+k-i} a^{s+l-i}\ . \tag{18}$$

Finally, from Eq. (3) one may read out the structure constants of the algebra $\mathcal{H}$ in the normally ordered basis. The only non-vanishing ones are

$$\gamma_{(r,s)(k,l)}^{(r+k\text{-}i,s+l\text{-}i)} = i!\binom{s}{i}\binom{k}{i} \quad \text{for} \quad i = 0 \ldots \min\{k,s\}\ . \tag{19}$$

4. Summary

We have considered the Heisenberg–Weyl algebra $\mathcal{H}$ starting from the free algebra of words $\mathcal{W}$ and imposing the relation of Eq. (1), *i.e.*

$$\mathcal{W} \xrightarrow{[a,a^\dagger]=I} \mathcal{H}$$

We have shown that words can be uniquely encoded as staircase paths on a plane, thus providing the realization of $\mathcal{W}$ as a combinatorial algebra of paths $\mathcal{P}$. This allowed us to construct a model of $\mathcal{H}$ in terms of paths with decomposition rule of Eq. (14) in $\mathcal{P}$ which reflects the defining relation of Eq. (1) in $\mathcal{W}$. The following diagram illustrates the whole scheme

$$\begin{array}{ccc} \mathcal{W} & \overset{1:1}{\longleftrightarrow} & \mathcal{P} \\ \downarrow & & \downarrow \\ \mathcal{W}/_{aa^\dagger=a^\dagger a+I} & \overset{1:1}{\longleftrightarrow} & \mathcal{P}/_{aa^\dagger=a^\dagger a+I} \end{array}$$

We have further looked at the normally ordered basis in $\mathcal{H}$ and the corresponding basis in $\mathcal{P}/_{aa^\dagger=a^\dagger a+I}$. It was pointed out that the decomposition rule of Eq. (14), reducing the number of steps in a path, is closely related to the familiar rook problem on the associated Ferrers board. This permits a graphical illustration of the calculus in this basis and, more generally, of the normal ordering problem.

In this note we have advocated a combinatorial approach to the Heisenberg–Weyl algebra by showing that it can be conceived as having a purely combinatorial origin. This gives a new perspective on the whole algebraic framework which is easily amenable to the sophisticated methods of discrete mathematics. We should mention other models of the algebra based on various discrete structures such as set partitions, graphs or urn models [21, 22, 23] as well as results deriving from combinatorial methodology, see [17] for reference.

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