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Heisenberg–Weyl algebra revisited: combinatorics of words and paths

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Abstract

The Heisenberg–Weyl algebra, which underlies virtually all physical representations of quantum theory, is considered from the combinatorial point of view. We provide a concrete model of the algebra in terms of paths on a lattice with some decomposition rules. We also discuss the rook problem on the associated Ferrers board; this is related to the calculus in the normally ordered basis. From this starting point we explore combinatorial underpinning of the Heisenberg–Weyl algebra, which offers novel perspectives, methods and applications.

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1. Introduction

From a modern viewpoint, the formalism and structure of quantum theory are founded on the theory of Hilbert space [1, 2]. The physical content of the theory consists of representing physical quantities as operators which satisfy some algebraic relations. Virtually all correspondence schemes come endowed with the Heisenberg–Weyl algebra structure, be it the canonical quantization scheme, the occupation number representation in quantum mechanics or the second quantization formalism of quantum field theory. This derives from the analogy with classical mechanics whose Poissonian structure is reflected in the commutator of position and momentum observables [3]. Ubiquitous and profound, the Heisenberg–Weyl algebra has become the hallmark of non-commutativity in quantum theory.

An exemplary model of the Heisenberg–Weyl algebra involves combinations of derivative D, multiplication X and identity I operators acting on the space of polynomials. Physical examples are the position $\hat{x}$ and momentum $\hat{p}$ operators in the space of square integrable functions, or the annihilation a and creation $a^\dagger$ operators in Fock space. Here, without loss of generality, we conform to the notation $\{a, a^\dagger\}$ for the generators of the (associative) algebra[5], satisfying

$$aa^\dagger = a^\dagger a + I, \tag{1}$$

where I is multiplicative identity. We shall be interested in combinatorial aspects of this relation and discuss one of the ensuing models of the Heisenberg–Weyl algebra.

Indeed, the combinatorial properties of equation (1) were recognized early and successfully applied to the domain of algebraic enumeration, principally concerning the action of the operators X and D on generating series. From this point of view, these operators are auxiliary constructions facilitating enumeration of discrete structures [4–10].

However, in this paper we adopt another approach, which leads towards a combinatorial model of the algebra itself. Starting from the definition of the Heisenberg–Weyl algebra as the algebra of words in a and $a^\dagger$ supplemented by the relation of equation (1), we will recast it in the language of paths on a lattice with some decomposition rules. We shall also consider a convenient choice of basis, here taken to be normally ordered monomials, which permits a direct link to the combinatorics of words and the related algebra of paths. In this way, algebraic problems may be expressed in the more concrete form of the decomposition and enumeration of paths. For illustration, we consider the normal ordering procedure which reduces to the familiar rook problem on the Ferrers board and then derive the structure constants of the algebra by a simple combinatorial argument.

2. Heisenberg–Weyl algebra

In this paper, we consider an *algebra* $\mathcal{A}$ to be a linear vector space over a field $\mathbb{K}$ with a bilinear multiplication law:

$$\mathcal{A} \times \mathcal{A} \ni (b, c) \longrightarrow b\, c \in \mathcal{A}, \tag{2}$$

which is associative and possesses a unit element I. More precisely, it is called an *associative algebra with the unit* as distinct from an algebraic structure lacking associativity or a unit (e.g. Lie algebra). A *basis* of an algebra is a basis for its vector space structure. Each basis $(b_i)_{i\in\Lambda}$ defines a unique family $\gamma_{ij}^k \in \mathbb{K}$ such that for every ordered pair $(i, j) \in \Lambda \times \Lambda$, the set of $k \in \Lambda$ is such that $\gamma_{ij}^k \neq 0$ is finite and

$$b_i b_j = \sum_{k\in\Lambda} \gamma_{ij}^k\, b_k. \tag{3}$$

γ_{ij}^k are called the *structure constants* of the algebra $\mathcal{A}$ with respect to the basis $(b_i)_{i\in\Lambda}$, from which the multiplication law can be uniquely recovered.

The *Heisenberg–Weyl algebra*, denoted by $\mathcal{H}$, is the algebra generated by $a, a^\dagger$, satisfying the relation of equation (1). Elements of the algebra $A \in \mathcal{H}$ are linear combinations of finite products of a and $a^\dagger$ of the form

$$A = \sum_{\mathbf{r},\mathbf{s}} \alpha_{\mathbf{r},\mathbf{s}}\, a^{\dagger r_1} a^{s_1} a^{\dagger r_2} a^{s_2} \ldots a^{\dagger r_k} a^{s_k}, \tag{4}$$

where $\mathbf{r} = (r_1, \ldots, r_k)$ and $\mathbf{s} = (s_1, \ldots, s_k)$ are non-negative integer multi-indices (with the convention $a^0 = a^{\dagger 0} = I$). This representation is ambiguous, however, due to the

[5] We do not attach much weight to this particular realization, however, as we shall study algebraic properties only, for which the underlying Fock space plays no role. Our considerations hold true for any representation of the Heisenberg–Weyl algebra.

commutation relation (1) which yields different representations of the same element of the algebra, e.g. $aa^\dagger = a^\dagger a + I$. The problem can be resolved by fixing the preferred order of the generators a and $a^\dagger$. Conventionally, it is done by choosing the *normally ordered* form in which all annihilators stand to the right of creators. In this case, each element of the algebra $\mathcal{H}$ is uniquely written in the normally ordered form as

$$A = \sum_{r,s} \beta_{rs}\, a^{\dagger r} a^s. \tag{5}$$

Hence, the normally ordered monomials $a^{\dagger r} a^s$ constitute a natural basis for the Heisenberg–Weyl algebra:

$$\text{Basis of } \mathcal{H}: \quad b_{(r,s)} = a^{\dagger r} a^s, \tag{6}$$

indexed by pairs of integers $r, s = 0, 1, 2, \ldots,$ and equation (5) is the expansion of the element A in this basis. We should note that the normally ordered representation of the elements of the algebra suggests itself as the simplest one [11]. It is important and commonly used in practical applications in quantum optics [12–14] or quantum field theory [15, 16]. Working in this basis entails the reshuffling of a and $a^\dagger$ to the normally ordered form, which in general is a nontrivial task [17]. This brings up the issue of efficient calculation methods and intuitive schemes providing insight into the ordering procedure itself. Below, we provide a combinatorial model for the Heisenberg–Weyl algebra and then propose a resolution of the problem from this starting point.

3. Combinatorics of the Heisenberg–Weyl algebra

3.1. Words, paths and rook problems

We start by showing that each word in a two-letter alphabet can be uniquely encoded as a *staircase path* on a plane rectangular lattice. This observation will lead us to considering the associated Ferrers board and the rook problem. Rather than giving a formal construction, we shall illustrate it by an example from which the general scheme can be straightforwardly recovered.

Suppose we consider a word, say

$$\mathtt{w} = aa^\dagger aa^\dagger a^\dagger a^\dagger aa^\dagger a \tag{7}$$

to which we assign a staircase path on a rectangular lattice. Starting from the point (0, 0), it is constructed by reading the word $\mathtt{w}$ from the left and drawing a line to the right if the letter is $a^\dagger$ and up if the letter is a, as shown in figure 1 on the left. We observe that this scheme provides a unique encoding of words.

With each path (or word), one can associate a *Ferrers board* $\mathcal{B}_{\mathtt{w}}$ by retaining rectangular cells below the path, see figure 1. Note that this is a many-one procedure as paths differing by a horizontal line at the beginning and a vertical line at the end yield the same board [5, 10].

The *rook problem* for the given board $\mathcal{B}$ consists in enumerating non-capturing arrangements of k rooks on the board which defines a finite sequence $r_\mathcal{B}(k)$, $k = 0, 1, 2, \ldots$, called the *rook numbers*. In our example, one has

$$r_\mathcal{B}(k) = 1, 10, 23, 9, 0, 0, \ldots \qquad k = 0, 1, 2, 3, \ldots. \tag{8}$$

A rook sequence can be conventionally encoded in a *rook polynomial* defined by

$$R_\mathcal{B}(x) = \sum_{k\geqslant 0} r_\mathcal{B}(k) x^k. \tag{9}$$

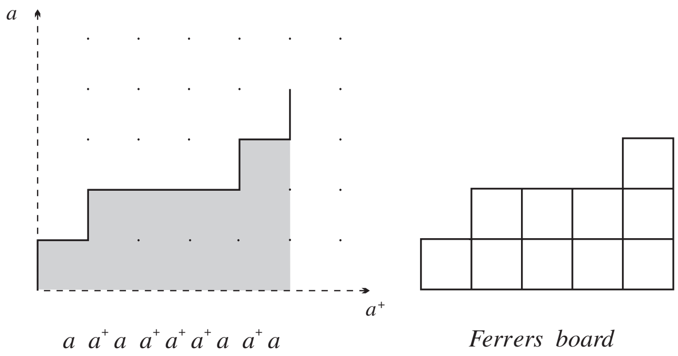

Figure 1. Path defined by the word $\mathtt{w}$ and the associated Ferrers board $\mathcal{B}_{\mathtt{w}}$.

It is straightforward to show that these polynomials satisfy the recursion [4, 5, 10]

$$\begin{aligned} R_{\mathcal{B}}(x) &= R_{\mathcal{B}'}(x) + x R_{\mathcal{B}''}(x), \\ R_{\emptyset}(x) &= 1, \end{aligned} \tag{10}$$

obtained by choosing a cell which forms a step in the diagram $\mathcal{B}$. This *step-forming* cell has no neighbouring cell to its left or above it. We now consider two cases: a rook is placed on the cell and a rook is not placed on the cell. This reduces the problem to the boards $\mathcal{B}''$ (with the row and column in which the cell is placed removed) and $\mathcal{B}'$ (with the chosen cell removed only). We note that there are many possible choices of such a cell, and these give different decompositions of the board yielding various recursive patterns; see figure 2.

3.2. Normal ordering procedure

It has been shown [18–20] that the normal ordering of a word $\mathtt{w}$ in a and $a^\dagger$ satisfying equation (1) reduces to the rook problem on the associated Ferrers board $\mathcal{B}_{\mathtt{w}}$. Namely,

$$\mathtt{w} = \sum_{k\geqslant 0} r_{\mathcal{B}_{\mathtt{w}}}(k)\mathtt{w}^{(k)}, \tag{11}$$

where $\mathtt{w}^{(k)}$ are normally ordered monomials $a^{\dagger r}a^s$ obtained from $\mathtt{w}$ by crossing out k pairs of a and $a^\dagger$ and then reshuffling the rest as if they were commuting variables (called the double dot operation used in quantum field theory). For example, for a word in equation (7) we have $\mathtt{w}^{(0)} = a^{\dagger 5}a^4$, $\mathtt{w}^{(1)} = a^{\dagger 4}a^3$, $\mathtt{w}^{(2)} = a^{\dagger 3}a^2$, etc, and hence its normally ordered form reads (see equation (8)) as

$$\mathtt{w} = a^{\dagger 5}a^4 + 10\,a^{\dagger 4}a^3 + 23\,a^{\dagger 3}a^2 + 9\,a^{\dagger 2}a. \tag{12}$$

A rigorous proof of equation (11) relies on the observation that each word can be reduced to the sum of two simpler ones by choosing the places in which a precedes $a^\dagger$ (which correspond to step-forming cells of the previous section) and reshuffling them according to equation (1), i.e. $aa^\dagger \to I + a^\dagger a$. For example, for a word of equation (7) there are three choices which exactly correspond to possible decompositions of the associated Ferrers diagram $\mathcal{B}_{\mathtt{w}}$ in figure 2. We note that although there are various possible decomposition schemes, it can be shown that the result is unique.

In short, the normal ordering of a word reduces to the enumeration of possible non-capturing rook arrangements on the associated Ferrers board. The problem can be

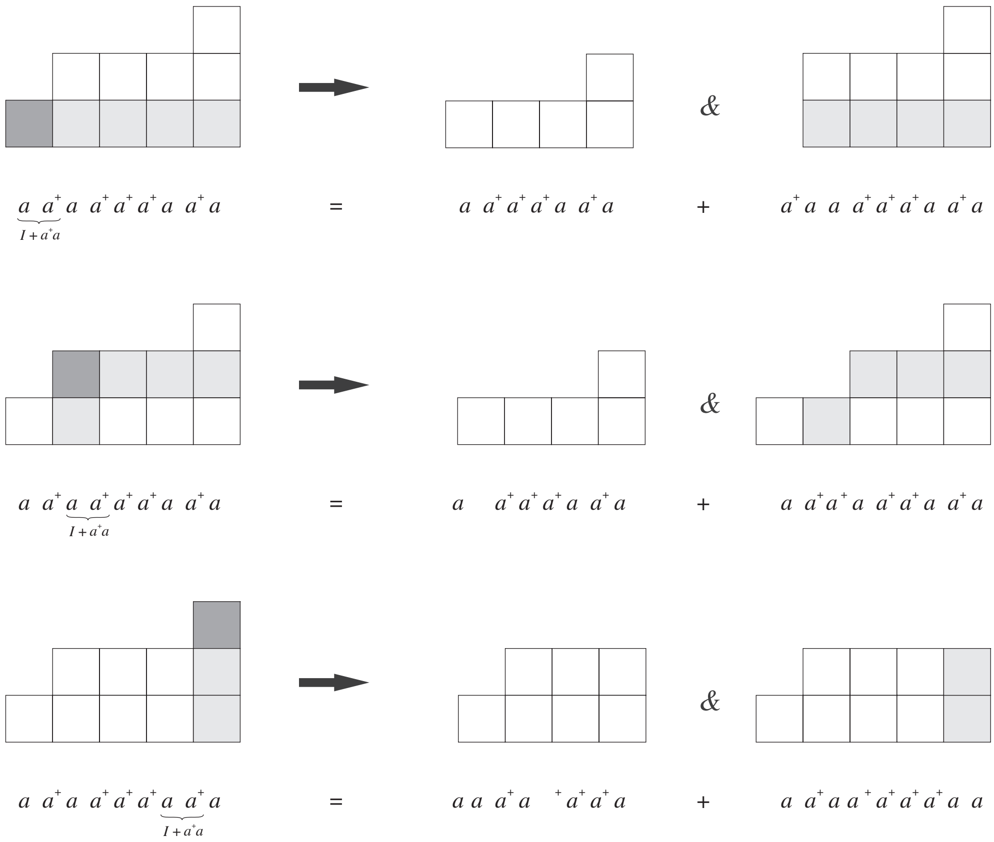

Figure 2. Three possible decompositions of the Ferrers board $\mathcal{B}_{\mathtt{w}}$ and the corresponding reduction of the word $\mathtt{w}$.

systematically handled by successive decompositions of the board. Moreover, one can devise simple algorithms based on the recursive rule given in equation (10). The methods described in this paper may be extended to the q-deformed case (see e.g. [20, 21]).

3.3. *Combinatorial realization of the Heisenberg–Weyl algebra*

We observed in section 3.1 that each word in two letters, here taken as a and $a^\dagger$, can be encoded as a path; see figure 1. This establishes an isomorphism between the algebra $\mathcal{W}$ of words in two letters and the algebra $\mathcal{P}$ of paths. In $\mathcal{W}$ multiplication is given by simple concatenation of words with the unit being the void word, while in $\mathcal{P}$ multiplication is given by concatenation of paths. In both cases we shall indicate the unit, which is the void word or path respectively, by the symbol $\oslash$. Both algebras are free. The Heisenberg–Weyl algebra arises by imposing on $\mathcal{W}$ the relation of equation (1), i.e. $\mathcal{H} = \mathcal{W}/_{\{aa^\dagger = a^\dagger a + I\}}$. In $\mathcal{P}$, this relation takes the symbolic form

$$\ulcorner = \lrcorner + \because, \tag{13}$$

by which we mean that a given staircase path is equivalent to the sum of two staircase paths obtained by

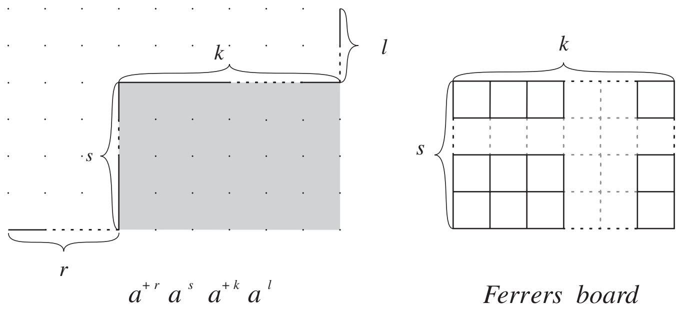

Figure 3. Path defined by the word $\mathtt{w} = a^{\dagger r} a^s a^{\dagger k} a^l$ and the associated Ferrers board.

(i) replacing an *upper-left-hand* corner (⌜) by a *lower-right-hand* corner (⌟), and
(ii) removing the row and column which intersect in the given cell (⁘).

Note that this reduction is exactly equivalent to the decomposition of the associated Ferrers board induced by the rook problem. In fact, any path can be uniquely decomposed into a finite sum of paths without steps (paths pertaining to monomials $a^{\dagger r} a^s$). The latter constitute the basis in $\mathcal{P}/_{\{⌜=⌟+⁘\}}$ corresponding to the normally ordered basis in $\mathcal{H}$.

In this way, we obtain the combinatorial model of the Heisenberg Weyl algebra as the algebra of paths with relation (13), i.e.

$$\mathcal{H} = \mathcal{W}/_{\{aa^\dagger = a^\dagger a + I\}} \cong \mathcal{P}/_{\{⌜=⌟+⁘}. \tag{14}$$

With this equivalence the algebraic structure of $\mathcal{H}$ and its calculus are recast in combinatorial terms, thus providing different perspectives and making accessible intuitive combinatorial arguments.

3.4. Example: structure constants

To illustrate the above equivalence, we show how to calculate the structure constants equation (3) of the Heisenberg–Weyl algebra by simple combinatorial enumeration. In the form suited for our purposes here we multiply two elements of the basis of equation (6), say $b_{(r,s)}$ and $b_{(k,l)}$, defining the structure constants $\gamma^{(p,q)}_{(r,s)(k,l)}$ in the form

$$b_{(r,s)}\, b_{(k,l)} = \sum_{p,q} \gamma^{(p,q)}_{(r,s)(k,l)}\, b_{(p,q)}. \tag{15}$$

Expansion in this basis of the product $b_{(r,s)}\, b_{(k,l)}$ essentially comes from the normal ordering of the word

$$\mathtt{w} = a^{\dagger r} a^s a^{\dagger k} a^l. \tag{16}$$

Following the scheme of section 3.1, we draw the associated path and subsequently read off the Ferrers board $\mathcal{B}_{\mathtt{w}}$ which has a simple rectangular form; see figure 3. Enumeration of non-capturing rook arrangements on the board is a direct result of the following observations:

(1) the maximum number of rooks on the board is $\min\{s, k\}$, and
(2) there are as many possible arrangements of i rooks as unordered choices of i columns and rows in the board $\mathcal{B}_{\mathrm{w}}$.

Hence, the rook numbers are

$$r_{\mathcal{B}_{\mathtt{w}}}(i) = i!\binom{s}{i}\binom{k}{i}, \tag{17}$$

for $i = 0 \ldots \min\{s, k\}$ and zero otherwise. Consequently, equation (11) gives the normally ordered form of the word $\mathtt{w}$:

$$a^{\dagger r} a^{s} a^{\dagger k} a^{l} = \sum_{i=0}^{\min\{k,s\}} i!\binom{s}{i}\binom{k}{i} a^{\dagger\, r+k-i} a^{s+l-i}. \tag{18}$$

Finally, from equation (15) one may read off the structure constants of the algebra $\mathcal{H}$ in the normally ordered basis. For fixed (r, s) and (k, l), the only non-vanishing γ's are

$$\gamma_{(r,s)(k,l)}^{(r+k-i,s+l-i)} = i!\binom{s}{i}\binom{k}{i} \qquad \text{for} \quad i = 0, \ldots, \min\{k, s\}. \tag{19}$$

Note that the right-hand side of equation (19) is, in fact, independent of r and l since the outer elements are not included in the commutation.

4. Summary

We considered the Heisenberg–Weyl algebra $\mathcal{H}$ starting from the (two generator) free algebra of words $\mathcal{W}$ and imposing the relation of equation (1), i.e.

$$\mathcal{W} \xrightarrow{[a,a^{\dagger}]=I} \mathcal{H}$$

We showed that words can be uniquely encoded as staircase paths on a plane, thus providing a realization of $\mathcal{W}$ as a combinatorial algebra of paths $\mathcal{P}$. This allowed us to construct a model of $\mathcal{H}$ in terms of paths with the decomposition rule of equation (13) in $\mathcal{P}$ which reflects the defining relation of equation (1) in $\mathcal{W}$. The following diagram illustrates the whole scheme:

$$\begin{tikzcd}
\mathcal{W} \arrow[rr, leftrightarrow, "\sim"] \arrow[d] & & \mathcal{P} \arrow[d] \\
\mathcal{W}/_{aa^{\dagger}=a^{\dagger}a+I} \arrow[rr, leftrightarrow, "\sim"] & & \mathcal{P}/_{\{\ulcorner=\lrcorner+\cdot\vdots\cdot\}}
\end{tikzcd}$$

We further looked at the normally ordered basis in $\mathcal{H}$ and the corresponding basis in $\mathcal{P}/_{\{\ulcorner=\lrcorner+\cdot\vdots\cdot\}}$. We pointed out that the decomposition rule of equation (13), reducing the number of steps in a path, is closely related to the familiar rook problem on the associated Ferrers board. This permits a graphical illustration of the calculus in this basis and, more generally, of the normal ordering problem.

In this paper, we have advocated a combinatorial approach to the Heisenberg–Weyl algebra by showing that it can be conceived as having a purely combinatorial origin. This gives a new perspective on the whole algebraic framework which is easily amenable to the sophisticated methods of discrete mathematics. Finally, we should mention other models of the algebra based on various discrete structures such as set partitions, graphs or urn models [22–24] as well as results deriving from combinatorial methodology; see [17] and references therein.

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