Algebras of Diagrams and the Normal Ordering Problem

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Content of talk

- **Introduction : The Hadamard product of two EGFs (Exponential Generating Fonctions) and its diagrammatic expansion.**

- **First part : A single exponential**
  - **One-parameter groups and the Normal Ordering Problem**
  - **Substitutions and explicit computation**
  - **The correspondence :**

    one-parameter group <--> matrix of normal forms

- **Second part : Two exponentials**
  - **Link with packed matrices**
  - **Hopf algebra structures and deformations**
  - **Discussion of the second part**

- **Conclusion & remarks**

A simple formula giving the Hadamard product of two EGFs

In a their paper, ***Quantum field theory of partitions,*** Bender, Brody and Meister introduce a special Field Theory described by a product formula in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of graphs.
These graphs label monomials and are obtained in the case of special interest when the functions have 1 as constant term.

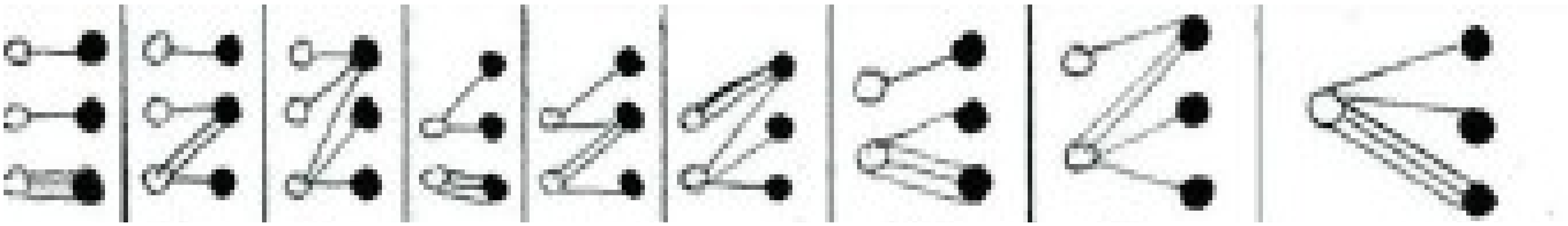
Bender, C.M, Brody, D.C. and Meister,
Quantum field theory of partitions, J. Math. Phys. Vol 40 (1999)

• If we write these functions as exponentials, we are led to witness a surprising interplay between the following aspects: **algebra** (of normal forms or of the exponential formula), **geometry** (of one-parameter groups of transformations and their conjugates) and **analysis** (parametric Stieltjes moment problem and convolution of kernels).

$$U_\lambda(f) = x^{-\frac{3}{2}} f(T(\lambda, x)).(T(\lambda, x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

• Writing *F* and *G* as free exponentials we shall see that the expansion can be indexed by specific diagrams (which are bicoloured graphs).



Some 5-line diagrams

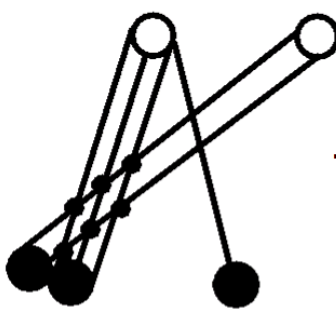
• These diagrams are in fact labelling monomials. We are then in position of imposing two types of rules:

- On the diagrams (Selection rules) : on the outgoing, ingoing degrees, total or partial weights.
- On the set of diagrams (Composition and Decomposition rules) : product and coproduct of diagram(s)

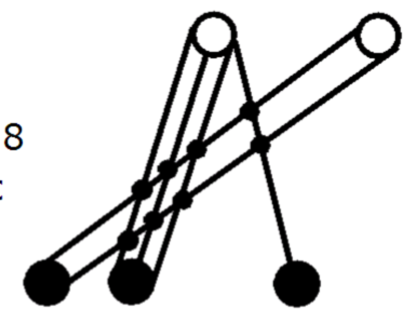
• This leads to structures of Hopf algebras for spaces freely generated by the two sorts of diagrams (labelled and unlabelled).

- Labelled diagrams generate the space of Matrix Quasisymmetric Functions, we thus obtain a new Hopf algebra structure on this space.
- Natural deformations (counting graph parameters as crossings and superpositions) can be introduced in the product law to give a three parameter (two formal - or continuous - and one boolean) true Hopf deformation of this algebra of diagrams.

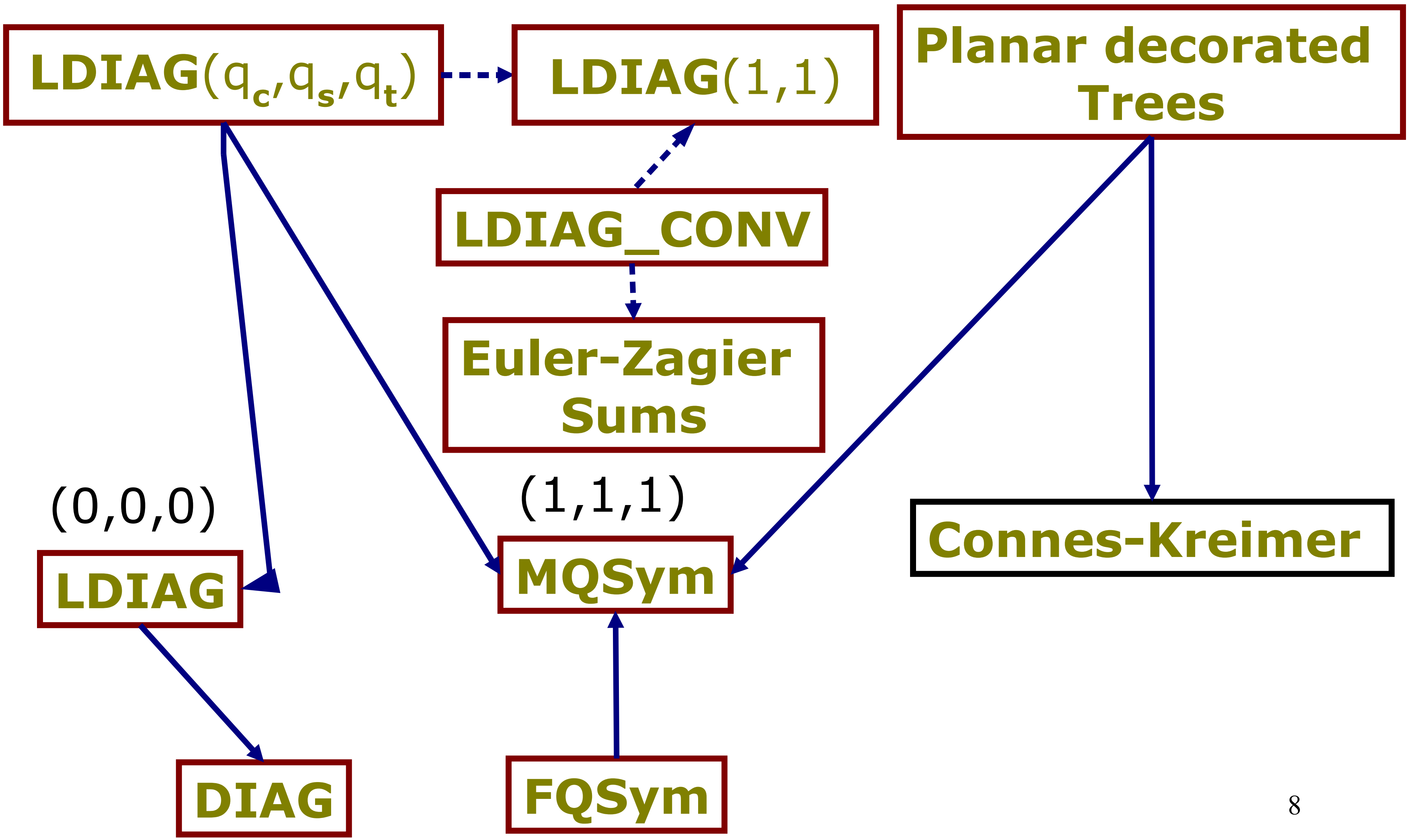
$q_c^2 q_s^6$



$+q_c^8$



Images and Specializations



Product formula

The Hadamard product of two sequences

$$(a_n)_{n\geq 0} \quad (b_n)_{n\geq 0}$$

is given by the pointwise product

$$(a_n b_n)_{n\geq 0}$$

We can at once transfer this law on EGFs by

$$F = \sum_{n\geq 0} a_n \frac{y^n}{n!};\; G = \sum_{m\geq 0} b_m \frac{y^m}{m!};\; \mathcal{H}(F,G) := \sum_{n\geq 0} a_n b_n \frac{y^n}{n!}$$

but, here, as $$\frac{(y\frac{d}{dx})^n}{n!} \frac{x^m}{m!}\Big|_{x=0} = \delta_{mn} \frac{y^n}{n!}$$

we get $$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0}$$

When the constant terms are 1, i. e. F(0)=G(0)=1, we can write with free alphabets

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

and

$$F(y) = \sum_{n\geq 0} \frac{y^n}{n!} P_n(L_1, L_2, \cdots, L_n, \cdots)$$

```
> f1:=exp(L1*z+L2*z^2/2);
```

$$fl := \mathbf{e}^{(Ll\,z + 1/2\,L2\,z^2)}$$

```
> taylor(f1,z=0,5);
```

$$1 + Ll\,z + \left(\frac{L2}{2} + \frac{Ll^2}{2}\right) z^2 + \left(\frac{1}{2}\,Ll\,L2 + \frac{1}{6}\,Ll^3\right) z^3 + \left(\frac{1}{8}\,L2^2 + \frac{1}{4}\,L2\,Ll^2 + \frac{1}{24}\,Ll^4\right) z^4 + \mathrm{O}(z^5)$$

```
> f2:=exp(L1*z+1/2*L2*z^2+1/6*L3*z^3+1/24*L4*z^4);
```

$$f2 := \mathrm{e}^{\left(L1\,z+\frac{L2\,z^2}{2}+\frac{L3\,z^3}{6}+\frac{L4\,z^4}{24}\right)}$$

```
> t1:=taylor(f2,z=0,5);
```

$$t1 := 1 + L1\,z + \left(\frac{L2}{2}+\frac{L1^2}{2}\right)z^2 + \left(\frac{1}{6}L3+\frac{1}{2}L1\,L2+\frac{1}{6}L1^3\right)z^3 + \left(\frac{L4}{24}+\frac{L1\,L3}{6}+\frac{L2^2}{8}+\frac{L2\,L1^2}{4}+\frac{L1^4}{24}\right)z^4 + \mathrm{O}(z^5)$$

```
> seq([coeff(t1,z,n)*n!],n=1..4);
```

$$[L1],\ [L2+L1^2],\ [L3+3\,L1\,L2+L1^3],\ [L4+4\,L1\,L3+3\,L2^2+6\,L2\,L1^2+L1^4]$$

In general, we adopt the notation

$$\alpha = 1^{a_1} 2^{a_2} \cdots r^{a_r}$$

for the *type* of a (set) partition which means that there are a_1 singletons a_2 pairs a_3 3-blocks a_4 4-blocks and so on.

The number of set partitions of type α as above is well known (see Comtet for example)

$$numpart(\alpha) = \frac{|\alpha|!}{(1!)^{a_1}(2!)^{a_2}\cdots(r!)^{a_r}(a_1)!(a_2)!\cdots(a_r)!}$$

Thus, using what has been said in the beginning, with

$$F(y) = exp(\sum_{n\geq 1} L_n \frac{y^n}{n!}) \quad G(x) = exp(\sum_{n\geq 1} V_m \frac{x^m}{m!})$$

one has

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

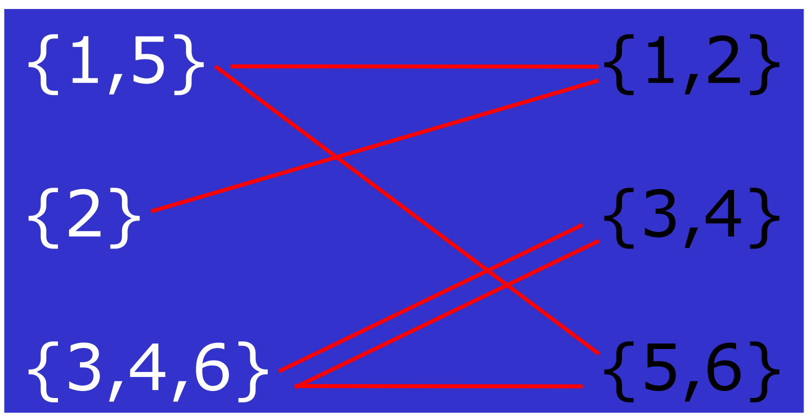
$$\sum_{n\geq 0} \frac{y^n}{n!} \sum_{|\alpha|=|\beta|=n} numpart(\alpha)numpart(\beta)\mathbb{L}^\alpha \mathbb{V}^\beta$$

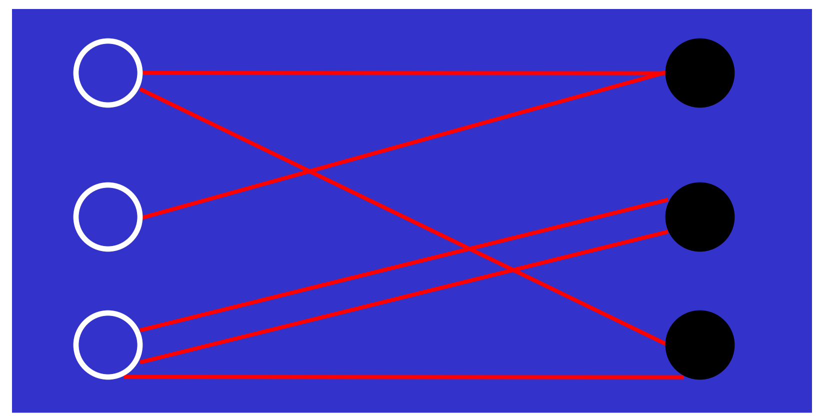
Now, one can count in another way the expression *numpart(α)numpart(β),* remarking that this is the number of pair of set partitions (P1,P2) with type(P1)=α, type(P2)=β. But every couple of partitions (P1,P2) has an intersection matrix ...

	{1,5}	{2}	{3,4,6}
{1,2}	1	1	0
{3,4}	0	0	2
{5,6}	1	0	1

Packed matrix see NCSF 6 (GHED, Hivert, and Thibon)

Feynman-type diagram (Bender & al.)





Now the product formula for EGFs reads

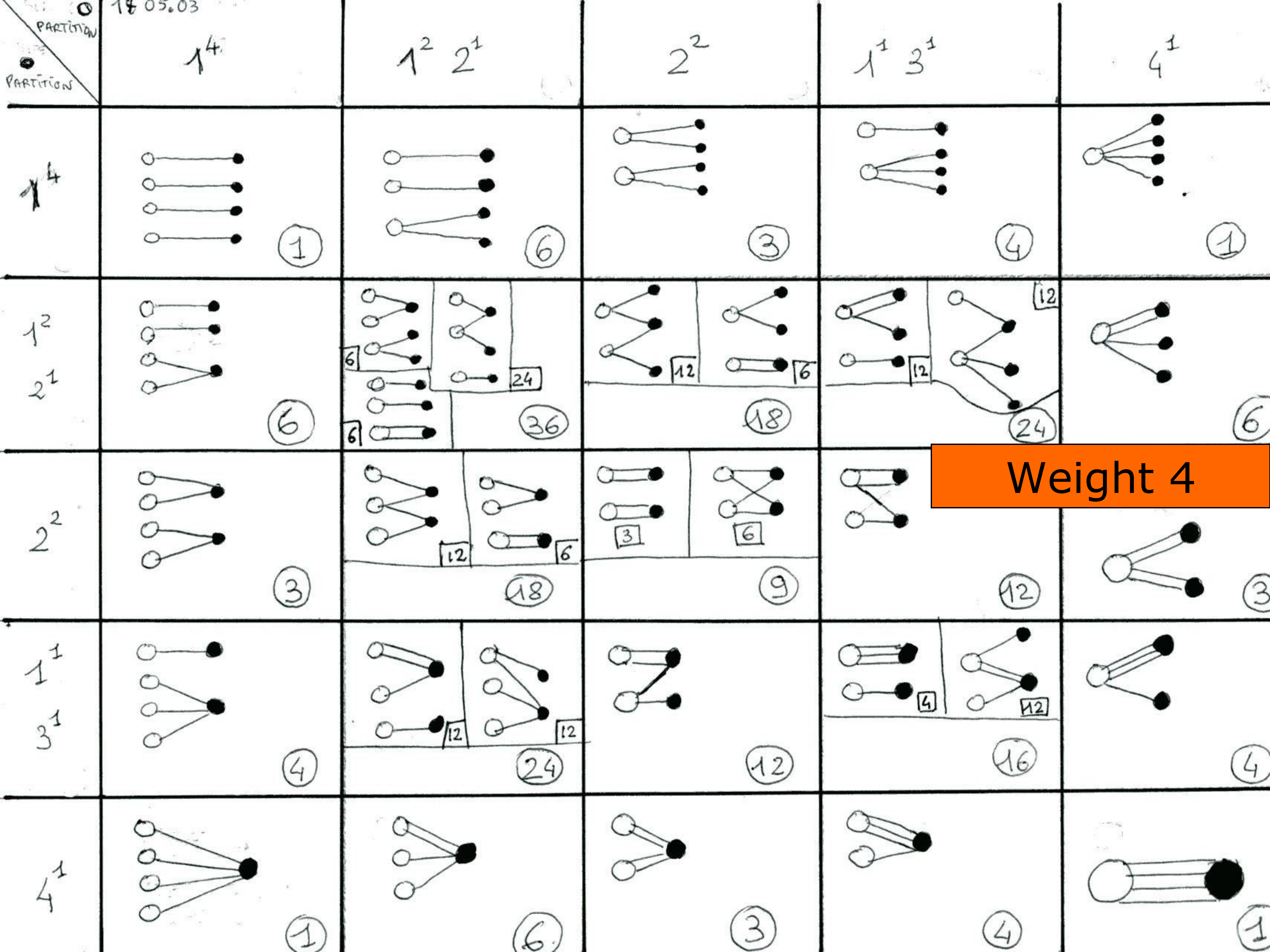
$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{d \text{ diagram}} mult(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}\frac{y^{|d|}}{|d|!}$$

and

$$\sum_{d} mult(d) = B(n)^2$$

The main interest of this new form is that we can impose rules on the counted graphs.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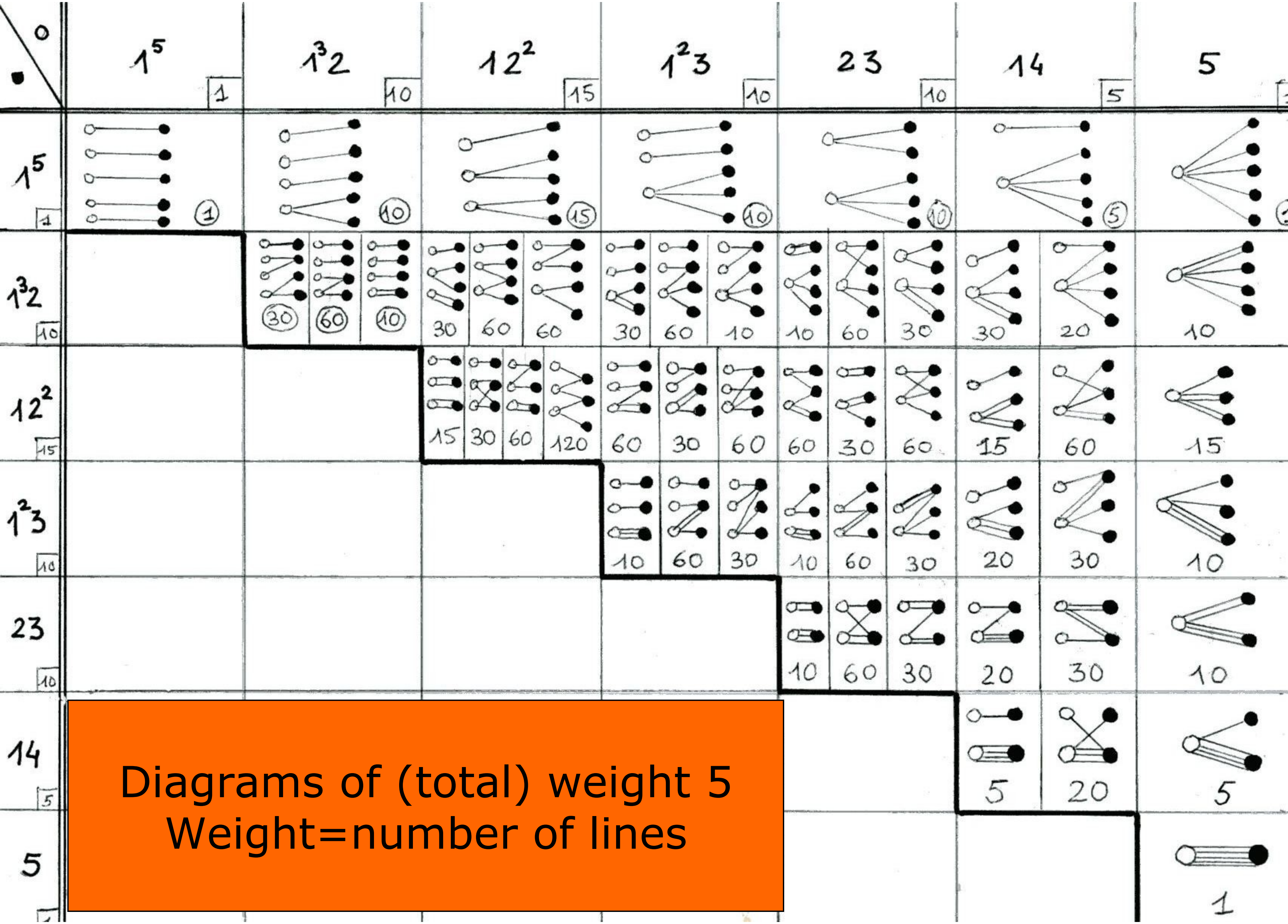

Diagrams of (total) weight 5
Weight=number of lines

A single exponential

We want to specialize our exponentials in known algebras of operators, for example the Heisenberg-Weyl algebra HW.

- HW (of GK dimension 2) is defined by generators and relations (in the category AAU) as

$$< a^+, a \; ; [a \, , a^+]=1 >_{C\text{-}AAU}$$

- It is known to have no representation in a Banach algebra, hence no representation by bounded operators in any Banach space.

There are many (faithful) representations by (unbounded) operators. One of them is the Bargmann-Fock representation

$$a \to d/dx \;;\; a^{+} \to x$$

Where, when seen as acting on polynomials, a has degree -1 and a^{+} has degree 1.

A typical element in the Weyl algebra is of the form

$$\Omega = \sum_{k,l \geq 0} c(k,l)(a^{+})^{k} a^{l}$$

(normal form).

But HW is graded by the excess defined on a string $w(a^{+},a)$ by

$$\text{excess}(w) = |w|_{a+} - |w|_{a}$$

Ω is then homogeneous of degree e (excess) iff one has

$$\Omega = \sum_{\substack{k,l \geq 0 \\ k-l=e}} c(k,l)(a^{+})^{k} a^{l}$$

Due to the symmetry of the Weyl algebra, we can suppose, with no loss of generality that $e \geq 0$. For homogeneous operators one has generalized Stirling numbers defined by

$$\Omega^{n} = (a^{+})^{ne} \sum_{k \geq 0} S_{\Omega}(n,k)(a^{+})^{k} a^{k}$$

Example: $\Omega_1 = a^{+2}a\, a^{+4}a + a^{+3}a\, a^{+2}$ (e=4)

$\Omega_2 = a^{+2}a\, a^{+} + a^{+}a\, a^{+2}$ (e=2)

If there is only one « a » in each monomial as in Ω_2, one can use the integration techniques of the Frascati(*) school (even for inhomogeneous) operators of the type $\Omega = q(a^{+})a + v(a^{+})$

(*) ***G. Dattoli, P.L. Ottaviani, A. Torre and L. Vàsquez, Evolution operator equations: integration with algebraic and finite difference methods, La Rivista del Nuovo Cimento 20 1 (1997).***

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3)$$

For $w = a^+aa^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+aaa^+a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

It can be proved that the matrices of coefficients for expressions with only a single « a » are matrices of special type : that of substitutions with prefunction factor.

2. The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of sequence transformations

Let $\mathbf{C}^{\mathbf{N}}$ be the vector space of all complex sequences, endowed with the Frechet product topology [23]. It is easy to check that the algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of all continuous operators $\mathbf{C}^{\mathbf{N}} \to \mathbf{C}^{\mathbf{N}}$ is the space of *row-finite* matrices with complex coefficients. Such a matrix M is indexed by $\mathbf{N} \times \mathbf{N}$ and has the property that, for every fixed row index n, the sequence $(M(n,k))_{k\geq 0}$ has finite support. For a sequence $A = (a_n)_{n\geq 0}$, the transformed sequence $B = MA$ is given by $B = (b_n)_{n\geq 0}$ with

$$b_n = \sum_{k\geq 0} M(n,k) a_k \tag{6}$$

Remark that the combinatorial coefficients S_w defined above are indeed row-finite matrices.

2.1. *Substitutions with prefunctions*

Let $(d_n)_{n\geq 0}$ bet a fixed set of denominators. We consider, for a generating function f, the transformation

$$\Phi_{g,\phi}[f](x) = g(x)f(\phi(x)). \tag{9}$$

Where $\varphi(x)=\lambda x+$higher terms and $g(x)=1+$higher terms. The fact that, in the case of a single "a", the matrices of generalized Stirling numbers are matrices of substitutions with prefunctions is due to the fact that the one-parameter groups associated with the operators of type $\Omega=q(x)d/dx+v(x)$ are conjugate to vector fields on the line.

Conjugacy trick :

Let $u_2=\exp(\int(v/q))$ and $u_1=q/u_2$ then

$u_1u_2=q$; $u_1u'_2=v$ and the operator $q(a^+)a+v(a^+)$

reads, via the Bargmann-Fock correspondence

$$(u_2u_1)d/dx+ u_1u'_2=u_1(u'_2 + u_2d/dx)= u_1d/dx\ u_2 =$$

$$1/u_2\,(u_1\, u_2\, d/dx\,)\, u_2$$

Which is conjugate to a vector field and integrates as a substitution with prefunction factor.

Example: The expression $\Omega = a^{+2} a\, a^{+} + a^{+} a\, a^{+2}$ above corresponds to the operator
(the line below ω is in form q(x)d/dx+v(x))

$$\omega = x^2 \frac{d}{dx} x + x \frac{d}{dx} x^2 =$$

$$2x^3 \frac{d}{dx} + 3x^2 = x^{-3/2}(2x^3 \frac{d}{dx}) x^{3/2} = x^{-3/2}(\phi) x^{3/2}$$

Now, ϕ is a vector field and its one-parameter group acts by a one parameter group of substitutions. We can compute the action by another conjugacy trick which amounts to straightening ϕ to a constant field.

Thus set

$$\exp(\lambda\, \phi)[f(x)]=f(u^{-1}(u(x)+\lambda)) \text{ for some } u \ldots$$

By differentiation w.r.t. λ at ($\lambda=0$) one gets

$$u'=1/(2x^3)\ ;\ u=-1/(4x^2)\ ;\ u^{-1}(y)=(-4y)^{-1/2}$$

Maple 9 - [fc_sem_algo.mws - [Server 1]]

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```
> expand(x^(-3/2)*2*x^3*diff(f(x)*x^(3/2),x));
```

$$2\,x^3\left(\frac{d}{dx}\,f(x)\right)+3\,x^2\,f(x)$$

The one-parameter group given by f(v(u(x)+λ); v being the (compositional) inverse of u,

reads

```
> T1:=(lambda,x)->x*(1-4*lambda*x^2)^(-1/2);
```

$$T1 := (\lambda, x) \rightarrow \frac{x}{\sqrt{1-4\,\lambda\,x^2}}$$

Checking the tangent vector at the origin

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2\,x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))^2-T1(lambda1+lambda2,x)^2);
```

$$0$$

In view of the conjugacy established previously we have that exp(λ ω)[f(x)] acts as

$$U_\lambda(f) = x^{-\frac{3}{2}} f(T(\lambda, x)).(T(\lambda, x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

which explains the prefactor. Again we can check by computation that the composition of (U_λ)s amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of Eq1 (slide 9) we must have

$$\exp(\lambda\,\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k \left(\frac{d}{dx}\right)^k$$

Hence, introducing the eigenfunctions of the derivative (a method which is equivalent to the computation with coherent states) one can recover the mixed generating series of $S_\omega(n,k)$ from the knowledge of the

one-parameter group of transformations.

$$\exp(\lambda\,\omega)\left[e^{yx}\right] = \Big(\sum_{n\geq 0}\frac{\lambda^n}{n!}x^{ne}\sum_{k=0}^{ne}S_\omega(n,k)x^k y^k\Big)e^{yx}$$

Thus, one can state

Proposition (*): With the definitions introduced, the following conditions are equivalent (where f → U_λ[f] is the one-parameter group exp(λω)).

$$1.\ \sum_{n,k\geq 0} S_\omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)}$$

$$2.\ U_\lambda[f](x) = g(\lambda x^e)f(x(1+\phi(\lambda x^e)))$$

Remark : Condition 1 is known as saying that S(n,k) is of « Sheffer » type.

G. Duchamp, A.I. Solomon, K.A. Penson, A. Horzela and P. Blasiak, One-parameter groups and combinatorial physics, World Scientific Publishing. arXiv: quant-ph/04011262

Example : With $\Omega = a^{+2}a\, a^{+} + a^{+}a\, a^{+2}$ (previous slide), we had e=2 and

$$U_\lambda[f](x) = \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt[2]{\frac{x^2}{1-4\lambda x^2}}\right)$$

Then, applying the preceding correspondence one gets

$$\sum_{n,k\geq 0} S_\omega(n,k)\frac{x^n}{n!}y^k = \sqrt[4]{\frac{1}{(1-4x)^3}}\, e^{y\left(\sqrt{\frac{1}{1-4x}}-1\right)} =$$

$$\sqrt[4]{\frac{1}{(1-4x)^3}}\, e^{y\left(\sum_{n\geq 1} c_n x^n\right)}$$

Where $c_n = \binom{2n}{n}$ are the central binomial coefficients.

```
> E1:=(1/((1-4*x)^3))^(1/4)*exp(y*(1/(1-4*x)^(1/2)-1));
```

$$E1 := \left(\frac{1}{(1-4x)^3}\right)^{(1/4)} \mathrm{e}^{\left(y\left(\frac{1}{\sqrt{1-4x}}-1\right)\right)}$$

```
> T1:=taylor(E1,x=0,6);
```

$$T1 := 1 + (2y+3)x + \left(12y + 2y^2 + \frac{21}{2}\right)x^2 + \left(59y + 18y^2 + \frac{4}{3}y^3 + \frac{77}{2}\right)x^3 + \left(270y + 115y^2 + 16y^3 + \frac{2}{3}y^4 + \frac{1155}{8}\right)x^4 + \left(\frac{4389}{8} + \frac{4767}{4}y + 637y^2 + 126y^3 + 10y^4 + \frac{4}{15}y^5\right)x^5 + \mathrm{O}(x^6)$$

```
> seq([sort(coeff(T1,x,n)*n!)],n=1..5);
```

$$[2y+3], [4y^2+24y+21], [8y^3+108y^2+354y+231], [16y^4+384y^3+2760y^2+6480y+3465], [32y^5+1200y^4+15120y^3+76440y^2+143010y+65835]$$

```
> M1:=matrix(5,5,(n,k)->coeff(coeff(T1,x,n)*n!,y,k));
```

$$M1 := \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 24 & 4 & 0 & 0 & 0 \\ 354 & 108 & 8 & 0 & 0 \\ 6480 & 2760 & 384 & 16 & 0 \\ 143010 & 76440 & 15120 & 1200 & 32 \end{bmatrix}$$

Proposition (*): With the definitions introduced, the following conditions are equivalent (where f → U_λ[f] is the one-parameter group exp(λω)).

$$1.\ \sum_{n,k\geq 0} S_\omega(n,\ k)\frac{x^n}{n!}\,y^k = g(x)e^{y\phi(x)}$$

$$2.\ U_\lambda[f](x) = g(\lambda x^e)f(x(1+\phi(\lambda x^e)))$$

Remark : Condition 1 is known as saying that S(n,k) is of « Sheffer » type.

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Remarks on the proof of the proposition :

2) → 1) Can be proved by direct computation.
1) → 2) Firstly the operator $\exp(\lambda\omega)$ is continuous for the Treves topology on the EGF. Secondly, the equality in (2) is linear and continous in f (both sides). Thirdly the set of $\exp(yx)$ for y complex is total in the spaces of EGF endowed with this topology and the equality is satisfied on this set.

A bit more on the correspondence
Subs. w. pref. <--> Vector fields

Proposition : Let

$$USWP=\{M\in U(\mathbf{N},\mathbf{C}) | \hat{}f(z)= g(z)f(\varphi(z))\}$$

with $g(z)=1+\ldots$higher terms ; $\varphi(z)=z+\ldots$higher terms and τ_n be the usual truncation

$$\tau_n : U(\mathbf{N},\mathbf{C}) \to U([0..n]\mathrm{x}[0..n],\mathbf{C})$$

Then

a) The images $\mathbf{AS}_n=\tau_n(U(\mathbf{N},\mathbf{C}))$ are algebraic groups

b) USWP is the projective limit of the $\mathbf{AS}_n$

c) Therefore, for every $z\in\mathbf{C}$, $M\in USWP \Rightarrow M^z\in USWP$

d) The Lie algebra of USWP is the set of matrices associated with the differential operators $q(z)D+v(z)$; $q(z)=\beta z^2+\ldots$higher t. ; $v(z)=\eta z+\ldots$higher t.

Substitutions and the « connected graph theorem (*)»

A great, powerful and celebrated result:
(For certain classes of graphs)

If $C(x)$ is the EGF of CONNECTED graphs, then
exp(C(x)) is the EGF of ALL graphs.
(Uhlenbeck, Mayer, Touchard,…)

This implies that the matrix

M(n,k)=number of graphs with n vertices and having k connected components

is the matrix of a substitution (like $S_{\Omega}(n,k)$ previously but without prefactor).

(*) i.e. the « exponential formula » of combinatorialists.

One proves, using a Zariski-like argument, that, if *M* is such a matrix (with identity diagonal) then, all its powers (positive, negative and fractional) are substitution matrices and form a one-parameter group of substitutions, thus coming from a vector field on the line which could (in theory) be computed.

We are in search of a nice combinatorial principle.

For example, to begin with, the Stirling substitution $z \rightarrow e^z - 1$. We know that there is a unique one-parameter group of substitutions $s_\lambda(z)$ such that, for λ integer, one has the value ($s_2(z) \leftrightarrow$ partition of partitions)

$$s_2(z) = e^{(e^z-1)} - 1;\ \ s_3(z) = e^{(e^{(e^z-1)}-1)} - 1;\ \ s_{-1}(z) = log(1+z)$$

But we have no nice description of this group nor of the vector field generating it.

Hopf algebra structures on the diagrams

Hopf algebra structures on the diagrams

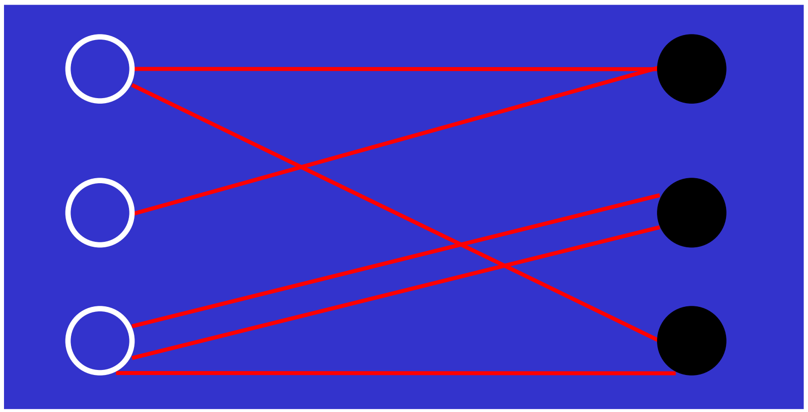
From our product formula expansion

$$\mathcal{H}(F,G) = F(y\frac{d}{dx})G(x)|_{x=0} =$$

$$\sum_{d \text{ diagram}} mult(d)\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}\frac{y^{|d|}}{|d|!}$$

one gets the diagrams as multiplicities for monomials in the (L_n) and (V_m).

For example, the diagram below corresponds to the monomial $(L_1\ L_2\ L_3)\ (V_2)^3$



	V_2	V_2	V_2
L_2	1	0	1
L_1	1	0	0
L_3	0	2	1

We get here a correspondence

diagram → monomial in (L_n) and (V_m).

Set

$$m(d,\mathbf{L},\mathbf{V},z)=\mathbf{L}^{\alpha(d)}\ \mathbf{V}^{\beta(d)}\ z^{|d|}$$

Question Can we define a (Hopf algebra) structure on the space spanned by the diagrams which represents the operations on the monomials (multiplication and doubling of variables) ?

Answer : Yes

First step: Define the space
Second step: Define a product
Third step: Define a coproduct

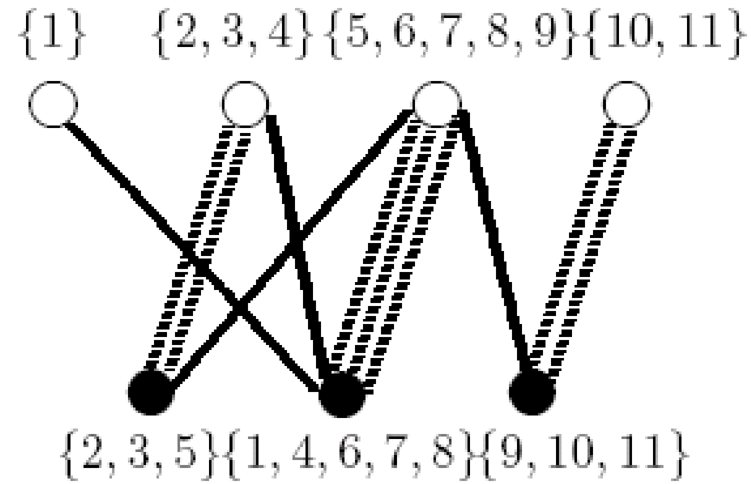


Fig 1. — *Diagram from P_1, P_2 (set partitions of $[1 \cdots 11]$).*

$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ and $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ (respectively black spots for P_1 and white spots for P_2).

The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. the diagram could be represented by the matrix $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ as well.

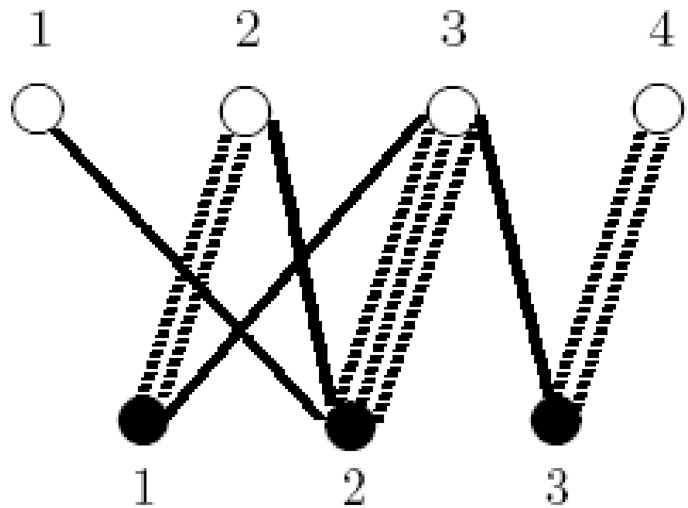


Fig 2. — *Labelled diagram of format* 3×4 *corresponding to the one of Fig 1.*

First step: Define the spaces

$Diag = \oplus_{d \in \text{diagrams}} \mathbf{C}\, d \quad LDiag = \oplus_{d \in \text{labelled diagrams}} \mathbf{C}\, d$

at this stage, we have an arrow *LDiag* → *Diag* (finite support functionals on the set of diagrams).

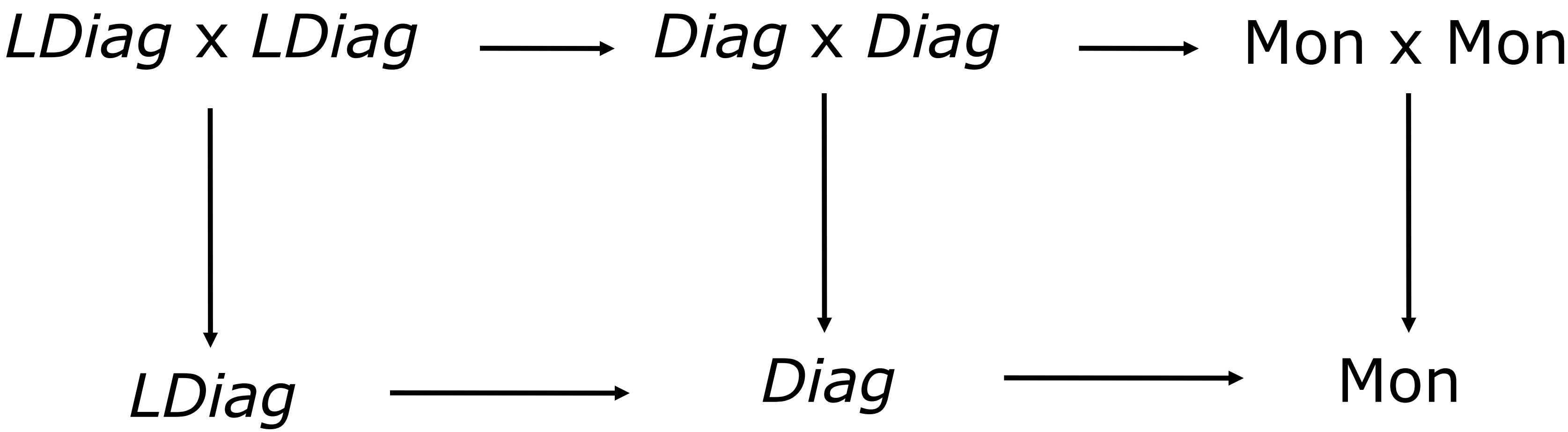
Second step: The product on *Ldiag* is just the concatenation of diagrams (we draw diagrams with their black spots downwards)

$$d_1 \star d_2 = d_1 d_2$$

So that $m(d_1 * d_2, \mathbf{L}, \mathbf{V}, z) = m(d_1, \mathbf{L}, \mathbf{V}, z) m(d_2, \mathbf{L}, \mathbf{V}, z)$

Remark: Concatenation of diagrams amounts to do the blockdiagonal product of the corresponding matrices.

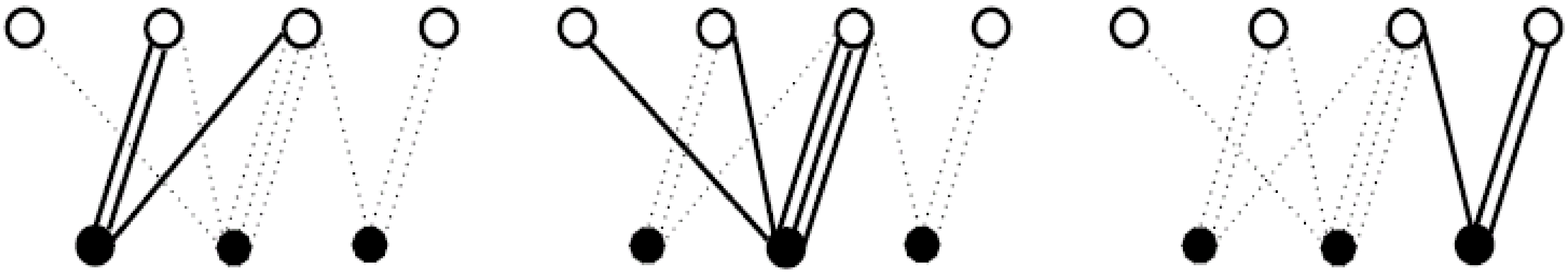
This product is associative with unit (the empty diagram). It is compatible with the arrow *LDiag* → *Diag* and so defines the product on *Diag* which, in turn is compatible with the product of monomials.



Third step: For the coproduct on *Ldiag*, we have several possibilities :

a) Split wrt to the white spots (two ways)
b) Split wrt the black spots (two ways)
c) Split wrt the edges

Comments : (c) does not give a nice identity with the monomials (when applying d → m(d,**?**,**?**,?)) nor do (b) and (c) by intervals.
(b) and (c) are essentially the same (because of the WS → BS symmetry)
In fact (b) and (c) by subsets give a good representation and, moreover, they are appropriate for several physical models.
Let us choose (b) by subsets, for instance…



$$d \otimes 1 + d_1 \otimes (d_2 \cup d_3) + d_2 \otimes (d_1 \cup d_3) + d_3 \otimes (d_1 \cup d_2) + \text{flips of those}$$

This coproduct is compatible with the usual coproduct on the monomials.

If $\Delta_{bs}(d)=\sum d_{(1)} \otimes d_{(2)}$

then

$$\sum m(d_{(1)},1,V',z)\, m(d_{(2)},1,V'',z) = m(d,1,V'+V'',z)$$

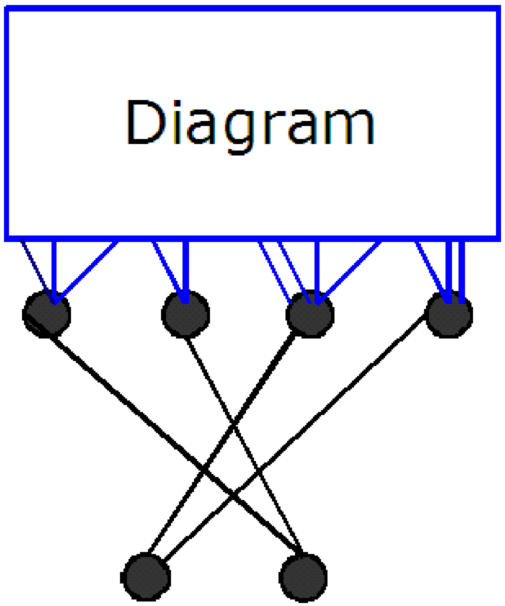
It can be shown that, with this structure (product with unit, coproduct and the counit $d \to \delta_{d,\emptyset}$), *Ldiag* is a Hopf algebra and that the arrow *Ldiag*→*Diag* endows *Diag* with a structure of Hopf algebra.

Remark: The labelled diagram are in one-to-one correspondence with the packed matrices as explained above. The product defined on diagrams is the product of the functions $(\phi S_P)_{p \text{ packed}}$ of NCSF VI p 709 (*).

$$\Delta\left(\mathrm{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}\right)=1\otimes\mathrm{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}+\mathrm{MS}_{[1\ 3]}\otimes\mathrm{MS}\begin{bmatrix}0&2&1\\0&0&3\\1&0&2\end{bmatrix}+\mathrm{MS}\begin{bmatrix}1&0&3\\0&2&1\end{bmatrix}\otimes\mathrm{MS}\begin{bmatrix}0&3\\1&2\end{bmatrix}$$
$$+\mathrm{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\end{bmatrix}\otimes\mathrm{MS}_{[1\ 2]}+\mathrm{MS}\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}\otimes 1$$

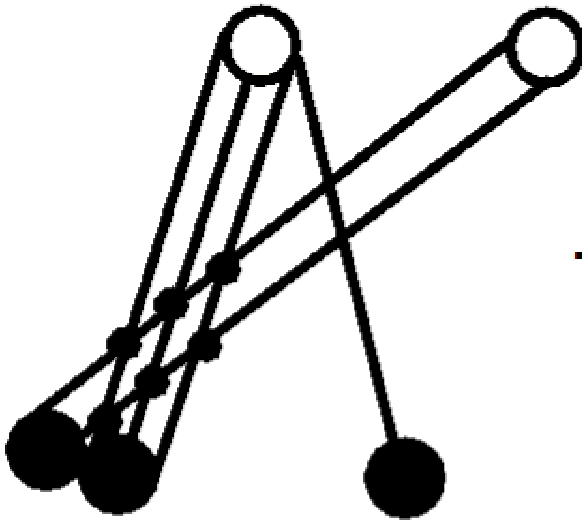
The question now is to interpolate between the two algebras in order to examine perturbations and deformations on direct and dual laws.

In order to connect these Hopf algebras to others of interest for physicists, we have to deform the product. The most popular technic is to use a monoidal action with many parameters (as braiding etc.).
Here, it is an analogue of the symmetric semigroup (the stacking-concatenation monoid) which acts on the black spots

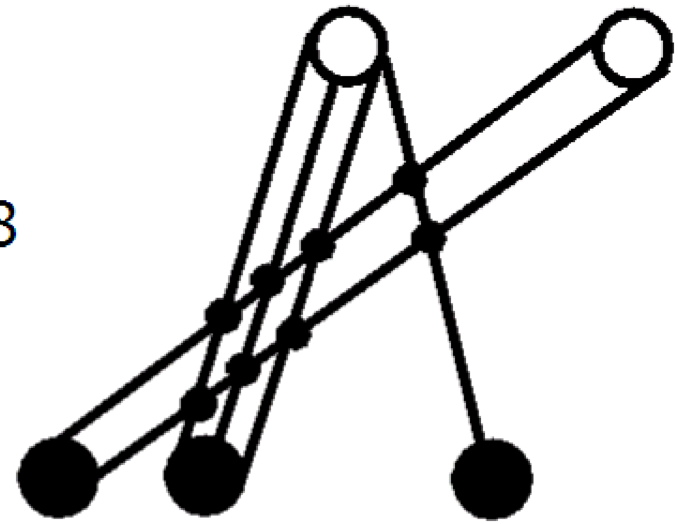


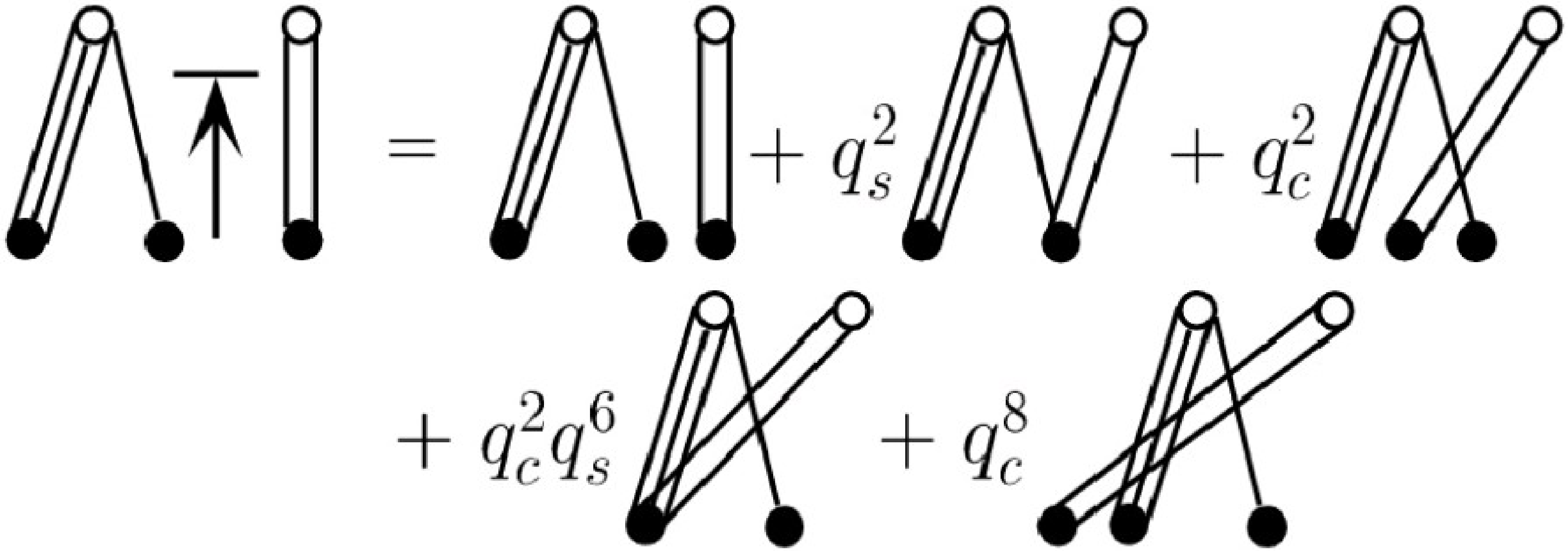
We tried the shuffle with superpositions. The weights being given by the intersection numbers.

$q_c^2 q_s^6$



$+ q_c^8$





What is striking is that this law is associative.

$$(au \uparrow bv) \uparrow cw = \big(a(u \uparrow bv) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v) + q^{|au||b|}b(au \uparrow v)\big) \uparrow cw$$

$$\Big[a((u \uparrow bv) \uparrow cw) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} ((u \uparrow bv) \uparrow w) + q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w)\Big]$$

$$\Big[q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|u||b|+(|u|+|v|)|c|}t^{|a||b|}t^{(|a|+|b|)|c|} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow \ w)$$

$$q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w)\Big]$$

$$\Big[q^{|au||b|}b((au \uparrow v) \uparrow cw) + q^{|au||b|+(|au|+|v|)|c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w) + q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w$$

$$au \uparrow (bv \uparrow cw) = au \uparrow \Big(b(v \uparrow cw) + q^{|v||c|}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w) + q^{|bv||c|}c(bv \uparrow w)\Big) =$$

$$\Big[a(u \uparrow b(v \uparrow cw)) + q^{|u||b|}t^{|a||b|} \begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v \uparrow cw) + q^{|au||b|}b(au \uparrow v \uparrow cw)\Big] +$$

$$\Big[q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|v||c|+|u|(|c|+|b|)}t^{|b||c|+|a|(|b|+|c|)} \begin{bmatrix} c \\ b \\ a \end{bmatrix} (u \uparrow v \uparrow w) +$$

$$q^{|v||c|+|au|(|b|+|c|)}t^{|b||c|} \begin{bmatrix} c \\ b \end{bmatrix} (au \uparrow v \uparrow w)\Big] +$$

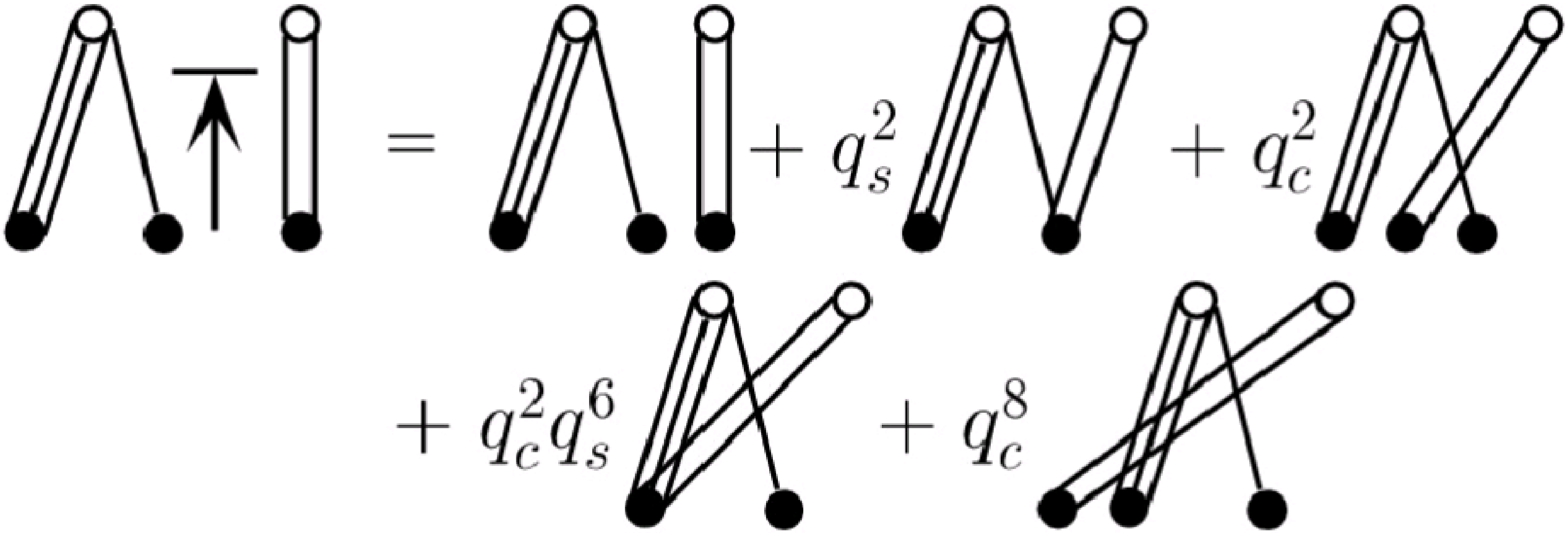
$$\Big[q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) + q^{(|u|+|bv|)|c|}t^{|a||c|} \begin{bmatrix} c \\ a \end{bmatrix} (u \uparrow bv \uparrow w) + q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w)\Big] \quad (3)$$

dans la deuxième expression, on regroupe les trois termes de tête des crochets et on trouve

$$a(u \uparrow b(v \uparrow cw)) + q^{|v||c|}t^{|b||c|}a(u \uparrow \begin{bmatrix} c \\ b \end{bmatrix} (v \uparrow w)) + q^{|bv||c|}a(u \uparrow c(bv \uparrow w)) = a(u \uparrow bv \uparrow cw) \quad (4)$$

dans la première expression, on regroupe les trois termes de queue des crochets et on trouve

$$q^{(|au|+|bv|)|c|}c(a(u \uparrow bv) \uparrow w) + q^{|u||b|+(|au|+|bv|)|c|}t^{|a||b|}c((\begin{bmatrix} b \\ a \end{bmatrix} (u \uparrow v)) \uparrow w) +$$
$$q^{|au||b|+(|au|+|bv|)|c|}c(b(au \uparrow v) \uparrow w) = q^{(|au|+|bv|)|c|}c(au \uparrow bv \uparrow w) \quad (5)$$



The labelled diagrams are in one to one correspondence with the packed matrices of MQSym and we can see easily that the product of the latter is obtained for

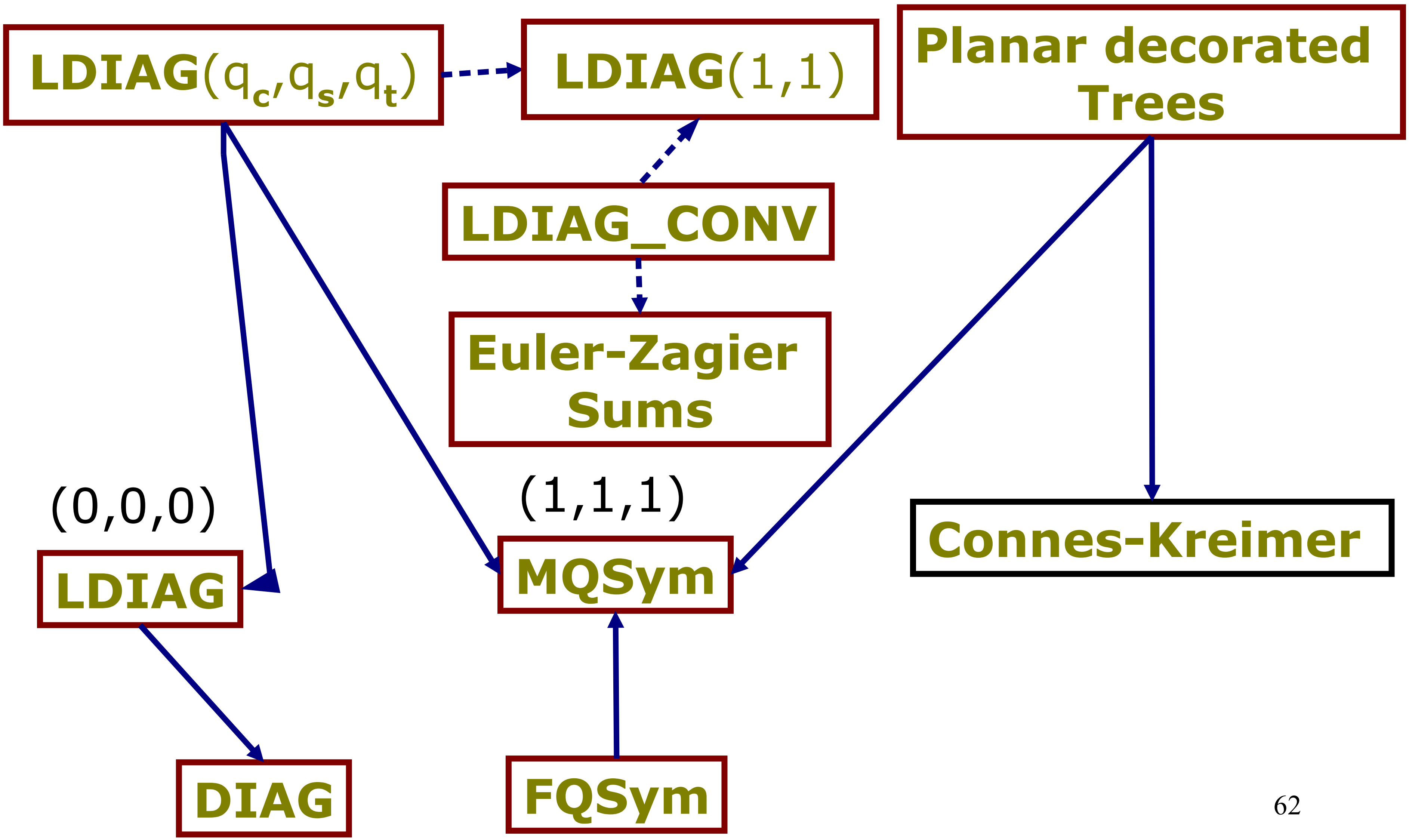
$$q_c = 1 = q_s$$

The algebra structure is that of a free algebra. The diagrams (under concatenation) form a free monoid the alphabet of which is the set of irreducible labelled diagrams irr(ldiag). Let us denote $\mathrm{ldiag}^{\leq n}$ the set of diagrams that are concatenation of less than n irreducibles and $\mathbf{LDIAG}^{\leq n}(q_c,q_s)$ the space linearly generated by them, it is not difficult to check that

$$\mathbf{LDIAG}^{\leq n} * \mathbf{LDIAG}^{\leq m} \subseteq \mathbf{LDIAG}^{\leq m+n}$$

and that the first term of the law (associated graded algebra) IS the concatenation. By a general theorem of algebra, $\mathbf{LDIAG}^{\leq n}(q_c,q_s)$ is a free algebra. We can then construct the third parameter q_t.

Images and Specializations



The Euler-Zagier sum

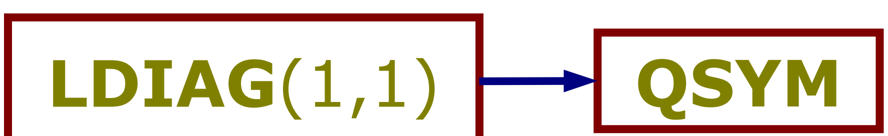
$$\zeta(s_1, \cdots, s_n) = \sum_{0<i_1<\cdots<i_n} \frac{1}{i_1^{s_1} \cdots i_n^{s_n}}$$

is the specialization to the alphabet {1/n} (n non-zero integer) of the monoial quasi-symmetric function $M_{[s1,s2,\ldots,sn]}$

One can check that the arrow

labelled diagram → list of the weights of the BS

provides a morphism of algebras.



One can check that the arrow
labelled diagram $\rightarrow$ list of the weights of the BS
provides a morphism of algebras.

Which, in turn can be adapted to Euler-Zagier sums

(A part of) The legacy of Schützenberger or how to compute efficiently in Sweedler's duals using Automata Theory

Sweedler's dual of a Hopf algebra

i) Multiplication

$$\mathcal{A}\otimes\mathcal{A} \xrightarrow{\ \mu\ } \mathcal{A}$$

ii) By dualization one gets

$$(\mathcal{A})^* \xrightarrow{\ {}^t\mu\ } (\mathcal{A}\otimes\mathcal{A})^*$$

but not a "stable calculus" as

$$(\mathcal{A})^* \otimes (\mathcal{A})^* \subseteq (\mathcal{A}\otimes\mathcal{A})^*$$

(strict in general). We ask for elements $x \in \mathcal{A}$ such that

$${}^t\mu(x) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$$

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These elements are easily characterized as the "representative linear forms" (see also the Group-Theoretical formulation in the last talk of Pierre Cartier)

Proposition : TFAE (the notations being as above)

i) ${}^t\mu(c)\in(\mathcal{A})^*\otimes(\mathcal{A})^*$

ii) There are functions f_i, g_i $i=1,2..n$ such that

$$c(xy)=\sum_{i=1}^{n} f_i(x)\, g_i(y)$$

forall x,y in $\mathcal{A}$.

iii) There is a morphism of algebras $\mu: \mathcal{A} \to k^{n\times n}$ (square matrices of size n x n), a line λ in $k^{1\times n}$ and a column ξ in $k^{n\times 1}$ such that, for all z in $\mathcal{A}$,

$$c(z)=\lambda\mu(z)\xi$$

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Theorem A: TFAE (the notations being as above)

i) ${}^t\mu(c) \in (\mathcal{A})^* \otimes (\mathcal{A})^*$

ii) There are functions f_i, g_i $i=1,2..n$ such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in A^* (the free monoid of alphabet A).

iii) There is a morphism of monoids μ: $A^* \dashrightarrow k^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If A is finite) c lies in the rational closure of A within the algebra k<<A>>.

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We can safely apply the first three conditions of **<u>Theorem A</u>** to *Ldiag.* The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more "basic" functions. The modification reads

iv) (A is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in A} p(a)\, a$$

within the algebra k<<A>>.

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iii) Schützenberger's theorem (known as the theorem of Kleene-Schützenberger) could be rephrased in saying that functions in a Sweedler's dual are behaviours of finite (state and alphabet) automata.

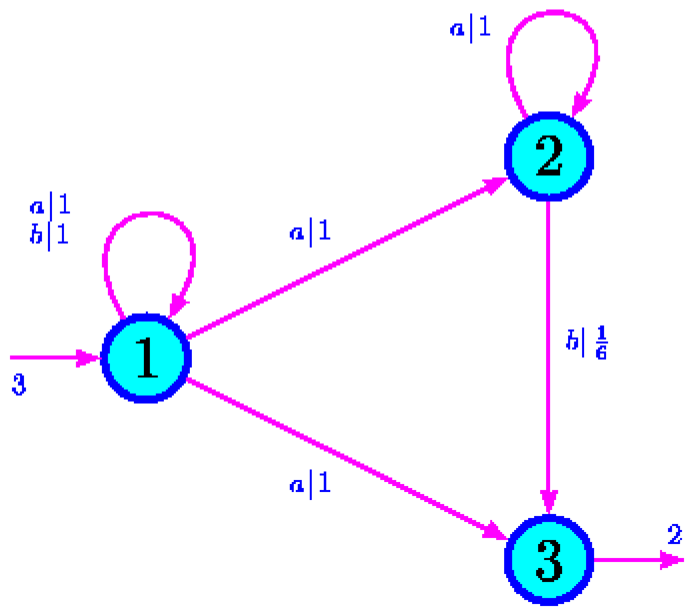
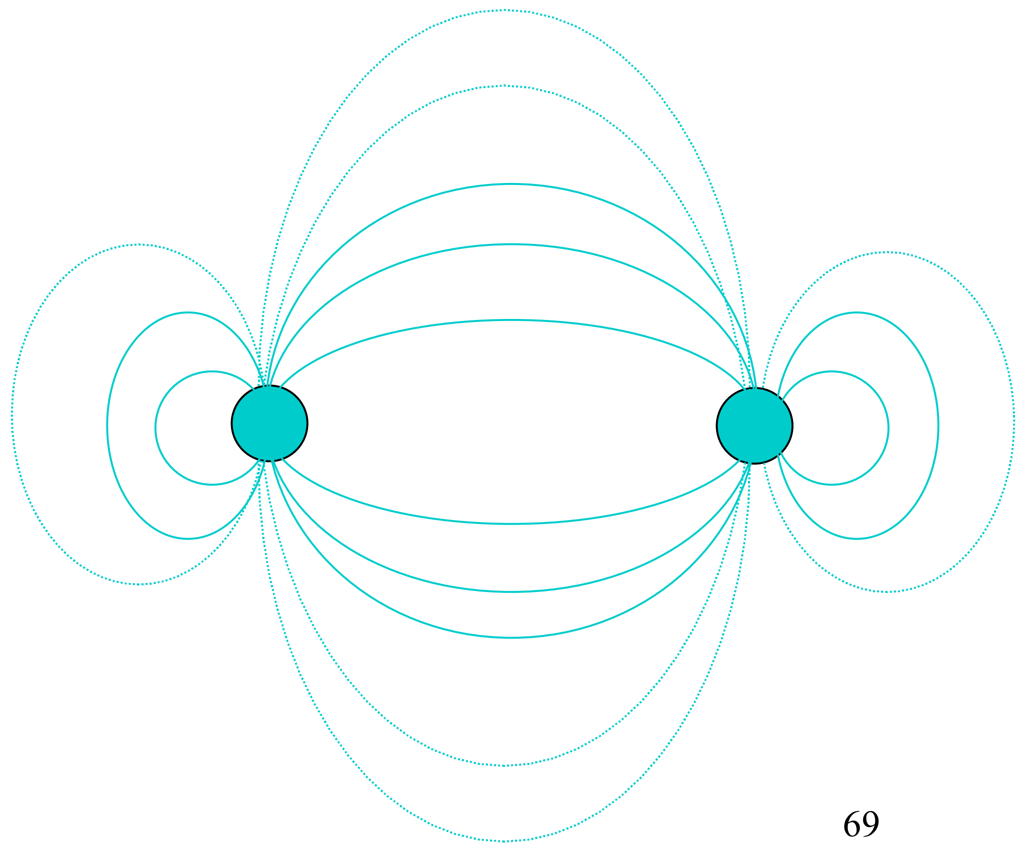


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6+a^*b).$$

In our case, we are obliged to allow infinitely many edges.



Concluding remarks

i) We have many informations on the structures of *Ldiag* and *Diag* and the deformed version.

ii) One can change the constant $L_k=1$ to a condition with level (i.e. $L_k=1$ for $k \leq N$ and $L_k= 0$ for $k>N$). We obtain then sub-Hopf algebras of the one constructed above.

iii) We possess deep explanations of the associativity of the deformation in terms of dual laws which also explains the link with the polyzeta functions.

iv) It seems that the parameter “t” (which is boolean) could be made continuous.

v) Many Hopf algebras of Combinatorial Physics and Combinatorial Hopf algebras being free as algebras, one can master their Sweedler’s duals by automata theory.

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End of the talk

Merci

Danke

Thank you

Dziękuję