The Hopf Algebra of Heisenberg-Weyl Diagrams

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Abstract. It has been found by A. Varvak (TODO improve) and others that the structure constants of the Heisenberg-Weyl algebra (HW) could be computed through rook placement numbers. We here give a diagrammatic realization of this formula in terms of graphs which constitute an algebra. It is also possible to keep track of the number cancellation in Wick's formula and this provides at once a q-analogue $\tilde{B}_q$ of the previous algebra. It tuns out that the latter can be very naturally endowed with the structure of a Hopf algebra. Returning to HW, it is well known that it has a 3-dimensional $\mathcal{L}_{HW}$ Lie subalgebra which generates (as an algebra) HW. We construct a Hopf arrow between the enveloping algebra $\mathcal{U}(\mathcal{L}_{HW})$ and $\tilde{B}_q$. It is there, in our sense, that resides the obstruction of HW to have finite dimensional non-zero representation.
28-12-2008 17:09

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1. Introduction

Since the advent of Hopf algebras in Combinatorial Physics, when facing an algebra, one always asks oneself whether its structure could be (more or less naturally) be enriched to a Hopf algebra one. For one of the simplest in Quantum Physics, the Heisenberg-Weyl (HW) algebra, the answer is "no" since the counit provides one with a one-dimensional representation HW and it is well-known (and easily checked) that this algebra has no non-zero finite dimensional representation (the ground field is supposed to that of the complex numbers or another field of zero characteristic).
The paper is devoted ...

$$\tilde{B}_q$$

$$\varphi_3 \qquad \qquad \varphi_4$$

$$\tilde{B} \qquad \varphi_5 \qquad \mathcal{U}(\mathcal{L}_{HW})$$

$$\varphi_2 \qquad \qquad \varphi_1$$

$$HW$$

2. Construction of the diagram algebras $\tilde{B}$ and $\tilde{B}_q$.

2.1. Heisenberg-Weyl graphs and their matching

As usual in graph theory, a graph Γ is given by the data of two sets $\Gamma^{(1)}$, the edges, $\Gamma^{(0)}$, the vertices and two functions (head and tail) $h, t \; : \; \Gamma^{(1)} \to \Gamma^{(0)}$. We are intesrested here by graphs with some edges having a "'free tail" or a "free head" but not both, so we allow the functions h, t to be partially defined with the condition that

$$dom(t) \cup dom(h) = \Gamma^{(1)} \tag{1}$$

(every edge has at least a tail or a head maybe both). The edges a with no origin i. e. such that $a \notin dom(t)$ (resp. with no endpoint i. e. such that $a \notin dom(h)$) will be called "ingoing edges" and denoted Γ^- (resp. "outgoing edges", denoted Γ^+).
A cycle is a sequence of edges $a_1 a_2 \cdots a_n$ such that, forall $k < n$, $h(a_k) = t(a_{k+1})$ and $h(a_n) = t(a_1)$. We will call HW-graph a finite graph with (possibly) ingoing and outgoing edges and no cycle.

`Figure TODO : Heisenberg-Weyl Graphs`
One can then see that a HW-graph has a natural partition in three parts :

- the outgoing edges $dom(t) - dom(h)$ and its associated subgraph Γ^+ (with restriction of the tail mapping and void head mapping)
- the ingoing edges $dom(h) - dom(t)$ and its associated subgraph Γ^- (with restriction of the head mapping and void tail mapping)
- the inner edges $dom(t) \cap dom(h)$ and its associated subgraph $\Gamma^{(in)}$ (with restriction of the tail and head mappings everywhere defined)

Now, in order to define algebra structures on these graphs, one has to define what is a matching between two HW-graphs. In general set theory a *matching* or one-to-one correspondence is a set of pairs

$$m \subset pr_1(m) \times pr_2(m) \tag{2}$$

such that two different pairs have different components i. e. for $c_1, c_2 \in m$ with $c_i = (x_i, y_i)$ one has

$$c_1 \neq c_2 \Longrightarrow (x_1 \neq x_2) \text{ and } (y_1 \neq y_2) \ . \tag{3}$$

If U, V are finite sets $|U| = p,\ |V| = q$ the number of matchings $m \subset U \times V$ with $|m| = i$ is

$$\binom{p}{i}\binom{q}{i} i! \ . \tag{4}$$

This formula will do the job when we construct the arrow φ_2.
For two HW-graphs Γ_2, Γ_1 with disjoint sets of vertices (i. e. $\Gamma_2^{(0)} \cap \Gamma_1^{(0)} = \emptyset$), we will call *matching* of Γ_1 over Γ_2 any matching between Γ_2^+ and Γ_1^-, their set will be denoted $match(\Gamma_2, \Gamma_1)$. Any matching of $match(\Gamma_2, \Gamma_1)$ allows the "pluging" of Γ_2 into Γ_1 as follows

`Figure TODO : a matching between two Heisenberg-Weyl Graphs`

The result will be denoted

$$\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}$$

which can be formally defined by

$$\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}^{(1)} = \Gamma_1^{(1)} \cup \Gamma_2^{(1)} \cup m - (pr_1(m) \cup pr_2(m)) \ . \tag{5}$$

Tail and head functions defined accordingly: remark that condition (1) implies that,

$$\Gamma_2^{(0)} \cap \Gamma_1^{(0)} = \emptyset \Longrightarrow \Gamma_2^{(1)} \cap \Gamma_1^{(1)} = \emptyset \tag{6}$$

so, there is no ambiguity to define them on $\Gamma_1^{(1)} \cup \Gamma_2^{(1)} - (pr_1(m) \cup pr_2(m))$ and we define tail and head functions on m (a subset, remind it, of $\Gamma_2^+ \times \Gamma_1^-$) by

$$t(a_2, a_1) := t(a_2) \ ; \ h(a_2, a_1) := h(a_1) \text{ where } (a_2, a_1) \in m \ . \tag{7}$$

We have a sort of associativity of the matchings which will be useful later. Let $\Gamma_i,\ i = 1, 2, 3$ be three disjoint HW-graphs. Let

$$m' \in match(\Gamma_2, \Gamma_1) \text{ and } m'' \in match(\Gamma_3, \begin{pmatrix}\Gamma_1\\ m'\\ \Gamma_2\end{pmatrix})$$

then m'' can be partitioned in $m''_{3,2} = m'' \cap (\Gamma_3^+ \times \Gamma_2^-)$ and $m''_{3,1} = m'' \cap (\Gamma_3^+ \times \Gamma_1^-)$ and one has

$$\begin{pmatrix} \begin{pmatrix} \Gamma_1 \\ m' \\ \Gamma_2 \end{pmatrix} \\ m'' \\ \Gamma_3 \end{pmatrix} = \begin{pmatrix} \Gamma_1 \\ m' \cup m''_{31} \\ \begin{pmatrix} \Gamma_2 \\ m''_{32} \\ \Gamma_3 \end{pmatrix} \end{pmatrix} \tag{8}$$

2.2. Multiplication of graphs and structure of the algebra $\tilde{B}_q$.

Following the general conventions in graph theory, an isomorphism of graphs between $\Gamma_1 = (\Gamma_1^{(1)}, \Gamma_1^{(0)}, t_1, h_1)$ and $\Gamma_2 = (\Gamma_2^{(1)}, \Gamma_2^{(0)}, t_2, h_2)$ is the data of two one-to-one mappings $\alpha^{(1)} : \Gamma_1^{(1)} \to \Gamma_2^{(1)}$ and $\alpha^{(0)} : \Gamma_1^{(0)} \to \Gamma_2^{(0)}$ such that the following commutative diagrams of partially defined functions hold

$$\begin{array}{ccc} \Gamma_1^{(1)} & \xrightarrow{t_1} & \Gamma_1^{(0)} \\ \downarrow{\scriptstyle \alpha^{(1)}} & & \downarrow{\scriptstyle \alpha^{(0)}} \\ \Gamma_2^{(1)} & \xrightarrow{t_2} & \Gamma_2^{(0)} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Gamma_1^{(1)} & \xrightarrow{h_1} & \Gamma_1^{(0)} \\ \downarrow{\scriptstyle \alpha^{(1)}} & & \downarrow{\scriptstyle \alpha^{(0)}} \\ \Gamma_2^{(1)} & \xrightarrow{h_2} & \Gamma_2^{(0)} \end{array} . \tag{9}$$

It is straightforward to check that the class of HW-graphs is closed under isomorphisms. We will call $\mathcal{B}$, the set (it can be shown to be denumerable) of classes of HW-graphs "up to isomorphisms".
The algebra $\tilde{B}$ (resp. $\tilde{B}_q$) is the vector space $\mathbb{C}\mathcal{B}$ (resp. the module $\mathbb{C}[q]\mathcal{B}$). The multiplication table in $\tilde{B}_q$ is given by

$$\tilde{\Gamma}_1 * \tilde{\Gamma}_2 = \sum_{m \in match(\Gamma_2, \Gamma_1)} q^{|m|} \widetilde{\begin{pmatrix} \Gamma_1 \\ m \\ \Gamma_2 \end{pmatrix}} \tag{10}$$

where $\Gamma_1,\ \Gamma_2$ are representatives of the classes $\tilde{\Gamma}_1,\ \tilde{\Gamma}_2$ with disjoint set of vertices (i. e. $\Gamma_1^{(0)} \cap \Gamma_2^{(0)} = \emptyset$).
It can be checked easily that the result is independent of the choice of disjoint representatives.
It is straightforward that, for all Γ, $\tilde{\Gamma} * \tilde{\emptyset} = \tilde{\emptyset} * \tilde{\Gamma} = \tilde{\Gamma}$. Hence $\tilde{\emptyset}$ (which, in fact is $\{\emptyset\}$) is neutral for the law and will be denoted here $1_{\tilde{B}_q}$. Let us prove now that the law is associative. To this end, take three disjoint HW-graphs $\Gamma_i,\ i = 1, 2, 3$ (we need two and a third which is disjoint of the results of the matchings of the two first). Then one has, using the denotations of formula (8)

$$(\tilde{\Gamma}_1 * \tilde{\Gamma}_2) * \tilde{\Gamma}_3 = \sum_{m' \in match(\Gamma_2, \Gamma_1)} \ \sum_{m'' \in match\left(\Gamma_3, \begin{pmatrix} \Gamma_1 \\ m' \\ \Gamma_2 \end{pmatrix}\right)} q^{|m'|} q^{|m''|} \widetilde{\begin{pmatrix} \begin{pmatrix} \Gamma_1 \\ m' \\ \Gamma_2 \end{pmatrix} \\ m'' \\ \Gamma_3 \end{pmatrix}} =$$

$$\sum_{m'\in match(\Gamma_2,\Gamma_1)} \sum_{m''\in match(\Gamma_3,\begin{pmatrix}\Gamma_1\\ m'\\ \Gamma_2\end{pmatrix})} q^{|m'|}q^{|m''|}\widetilde{\begin{pmatrix}\Gamma_1\\ m'\cup m''_{31}\\ \begin{pmatrix}\Gamma_2\\ m''_{32}\\ \Gamma_3\end{pmatrix}\end{pmatrix}} =$$

$$\sum_{m'\in match(\Gamma_2,\Gamma_1)} \sum_{m''\in match(\Gamma_3,\begin{pmatrix}\Gamma_1\\ m'\\ \Gamma_2\end{pmatrix})} q^{|m'\cup m''_{31}|}q^{|m''_{32}|}\widetilde{\begin{pmatrix}\Gamma_1\\ m'\cup m''_{31}\\ \begin{pmatrix}\Gamma_2\\ m''_{32}\\ \Gamma_3\end{pmatrix}\end{pmatrix}} \quad (11)$$

the result comes from the fact that the correspondence $(m', m'') \mapsto (m' \cup m''_{31}, m''_{32})$ between pairs of matchings in $match(\Gamma_2, \Gamma_1) \times match(\Gamma_3, \begin{pmatrix}\Gamma_1\\ m'\\ \Gamma_2\end{pmatrix})$ and pairs in $match(\begin{pmatrix}\Gamma_2\\ m''_{32}\\ \Gamma_3\end{pmatrix}, \Gamma_1) \times match(\Gamma_3, \Gamma_2)$ is one-to-one. Then the sum amounts to

$$\sum_{m''\in match(\Gamma_3,\Gamma_2)} \sum_{m'\in match(\begin{pmatrix}\Gamma_2\\ m''\\ \Gamma_3\end{pmatrix},\Gamma_1)} q^{|m''|}q^{|m'|}\widetilde{\begin{pmatrix}\Gamma_1\\ m'\\ \begin{pmatrix}\Gamma_2\\ m''\\ \Gamma_3\end{pmatrix}\end{pmatrix}} = \tilde{\Gamma}_1 * (\tilde{\Gamma}_2 * \tilde{\Gamma}_3)\ . \quad (12)$$

2.3. Specialization to $q = 1$: the algebra $\tilde{B}$.

Having constructed $\tilde{B}_q$, one can specialize q to one (i. e. send each sum $\sum_{i\in I} p_i(q)\tilde{\Gamma}_i$ to $\sum_{i\in I} p_i(1)\tilde{\Gamma}_i$) this ($\mathbb{C}$-linear) correspondence (which will be called φ_3 below) ranges in $\tilde{B}$ and endows the latter with the structure of $\mathbb{C}$-AAU.

3. Ado's theorem, the Lie algebra $\mathcal{L}_{HW}$ and its enveloping algebra. TODO : This paragraph will be put as a remark. Replace it by physical motivations for central elements.

It is well known that, in characteristic 0, HW admits no finite dimensional proper representation (its admits neither no representation in a Banach unital algebra). But one observes that the subspace $L = \mathbb{C}a \oplus \mathbb{C}a^+ \oplus \mathbb{C}1_{HW}$ is closed for the Lie bracket and hence a Lie subalgebra of $(HW, [,])$. Now, by Ado theorem such a finite-dimensional Lie algebra admits finite dimensional faithful representations. As an example, one can give the following one (noted ρ)

$$\rho(a) = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\ ;\ \rho(a^+) = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\ ;\ \rho(1_{HW}) = \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\ . \quad (13)$$

One sees that the image of 1_{HW} is not a unity and, in order not to get confused in our forthcoming development, we will take a copy of the Lie algebra $L = \mathbb{C}a \oplus \mathbb{C}a^+ \oplus \mathbb{C}1_{HW}$ as $\mathcal{L}_{HW} = \mathbb{C}\{b, b^+, u\}$ with the multiplication table

$[\downarrow,\rightarrow]$	b^+	b	u
b^+	0	$-u$	0
b	$-u$	0	0
u	0	0	0

Now, we have a morphism of Lie algebras $\phi_1 : \mathcal{L}_{HW} \to HW$ given by

$$\phi_1(b) = a \; ; \phi_1(b^+) = a^+ \; ; \phi_1(u) = 1_{HW} \tag{14}$$

and, by the universal propoerty of the envelopping algebras this can be extended uniquely as a morphism of AAU $\varphi_1 : \mathcal{U}(\mathcal{L}_{HW}) \to HW$. We can make it explicit, using the basis of Poincaré-Birkhoff-Witt of $\mathcal{U}(\mathcal{L}_{HW})$ which is given (using the order $b^+ \prec b \prec u$ as in the multiplication table above) by the products $\{(b^+)^p\, b^q\, u^r\}_{p,q,r\in\mathbb{N}}$. We then have $\varphi((b^+)^p\, b^q\, u^r) = (a^+)^p\, a^q$.

4. The arrow $\varphi_4 : \; \tilde{B}_q \to \mathcal{U}(\mathcal{L}_{HW})$.

The algebra $\tilde{B}_q$ is a $\mathbb{C}[q]$-AAU, but onee can restrict the scalars to $\mathbb{C}$. In this case, it is a $\mathbb{C}$-AAU with basis $q^n\tilde{\Gamma}$ (n in $\mathbb{N}$ and Γ choosen in a set of distinguished representatives). One sets

$$\varphi_4(q^n\tilde{\Gamma}) := (b^+)^{|\Gamma^+|} b^{|\Gamma^-|} u^n \; . \tag{15}$$

In order to prove that φ_4 is a morphism of $\mathbb{C}$-AAU, one has to prove that

$$\varphi_4(q^s\tilde{\Gamma_1} * q^t\tilde{\Gamma_2}) = \varphi_4(q^s\tilde{\Gamma_1})\varphi_4(q^t\tilde{\Gamma_2})$$

which can be rephrased as

$$\varphi_4(\tilde{\Gamma_1} * \tilde{\Gamma_2})u^{s+t} = \varphi_4(\tilde{\Gamma_1})\varphi_4(\tilde{\Gamma_2})u^{s+t}$$

so, one only has to prove the equality for $s = t = 0$.
Suppose that two disjoint HW-graphs $\Gamma_i, i = 1, 2$ are given. For convenience, set

$$match(\Gamma_2, \Gamma_1; i) = \{m \in match(\Gamma_2, \Gamma_1) |\; |m| = i\}$$

it is not difficult to get the following enumerations

$$|match(\Gamma_2, \Gamma_1; i)| = \begin{cases} \binom{|\Gamma_2^+|}{i}\binom{|\Gamma_1^-|}{i} i! & \text{for} \quad i \le min(|\Gamma_2^+|, |\Gamma_1^-|) \\ 0 & \text{for} \quad i > min(|\Gamma_2^+|, |\Gamma_1^-|) \end{cases} \tag{16}$$

then, one has

$$\varphi_4(\tilde{\Gamma_1} * \tilde{\Gamma_2}) = \varphi_4\Big(\sum_{m\; match(\Gamma_2,\Gamma_1)} q^{|m|} \widetilde{\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}}\Big) =$$

$$\sum_{i=0}^{min(|\Gamma_2^+|,|\Gamma_1^-|)} u^i \sum_{m\; match(\Gamma_2,\Gamma_1;i)} \varphi_4\Big(\widetilde{\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}}\Big) =$$

$$\sum_{i=0}^{min(|\Gamma_2^+|,|\Gamma_1^-|)} u^i \sum_{m\ \ match(\Gamma_2,\Gamma_1;i)} (b^+)^{\left|\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}^+\right|}\ b^{\left|\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}^-\right|} =$$

$$\sum_{i=0}^{min(|\Gamma_2^+|,|\Gamma_1^-|)} u^i \sum_{m\ \ match(\Gamma_2,\Gamma_1;i)} (b^+)^{|\Gamma_1^+|+|\Gamma_2^+|-i}\ b^{|\Gamma_1^-|+|\Gamma_2^-|-i}\ . \tag{17}$$

Now it is a easy exercise to get, in $\mathcal{U}(\mathcal{L}_{HW})$, the formula

$$(b^+)^{p_1}\, b^{q_1}\ (b^+)^{p_2}\, b^{q_2} = \sum_{i=0}^{min(q_1,p_2)} \begin{pmatrix} q_1\\ i\end{pmatrix} \begin{pmatrix} p_2\\ i\end{pmatrix} i!\, (b^+)^{p_1+p_2-i}\, b^{q_1+q_2-i}\, u^i \tag{18}$$

which implies that the sum in eq (17) is equal to

$$(b^+)^{|\Gamma_1^+|}\, b^{|\Gamma_1^-|}\ (b^+)^{|\Gamma_2^+|}\, b^{|\Gamma_2^-|} = \varphi_4(\tilde{\Gamma_1})\varphi_4(\tilde{\Gamma_2}) \tag{19}$$

5. Construction of the section $s_3:\ \tilde{B} \to \tilde{B}_q$ and of $\varphi_5:\ \tilde{B} \to \mathcal{U}(\mathcal{L}_{HW})$

For a HW-diagram Γ, we note Γ^{in} the set of inner edges i. e. $\Gamma^{in} = dom(t) \cap dom(h)$. Now, we construct an arrow $s_3:\ \tilde{B} \to \tilde{B}_q$ by

$$s_3(\Gamma) = q^{|\Gamma^{in}|}\Gamma \tag{20}$$

and one checks at once that

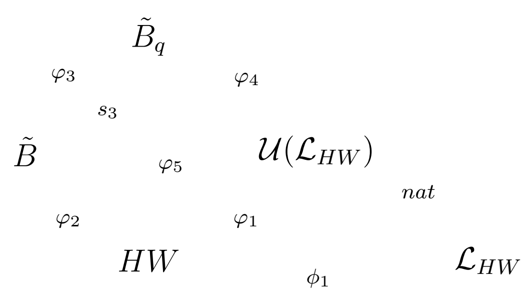
$$s_3(\Gamma_1 *_1 \Gamma_2) = \sum_{m\in match(\Gamma_2,\Gamma_1)} s_3(\widetilde{\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}}) = \sum_{m\in match(\Gamma_2,\Gamma_1)} q^{|\Gamma_1^{in}|+|\Gamma_2^{in}|+|m|} \widetilde{\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}} =$$

$$\sum_{m\in match(\Gamma_2,\Gamma_1)} q^{|m|} \widetilde{\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}} = s_3(\Gamma_1) * s_3(\Gamma_2)\ . \tag{21}$$

From there, φ_5 is defined as $\varphi_5 = \varphi_4 \circ s_3$. It should be noted that φ_5 is onto.

6. Diagram of algebras

We have, so far, constructed the following diagram of algebras



7. Coproducts and Hopf Algebra structures.

7.1. Comultiplications and section coefficients.

In a pioneering paper [31], G-C Rota introduced the concept of "section coefficients" and gave many combinatorial examples (set partitions, classes of intervals, placements ...).
Let us take, for simplicity, the example of binomial coefficients but presented here as "section coefficients". One has, for $n, p, q \in \mathbb{N}$

$$\binom{n}{p,q} = \begin{cases} \dfrac{n!}{p!\,q!} \;=\; \dbinom{n}{p} & \text{if } p+q=n \\ \quad 0 & \text{otherwise} \end{cases} \tag{22}$$

in this spirit, these coefficients count the number of ways of splitting a n-set into two disjoint subsets with, respectively p and q elements. Now consider the number of ways of splitting a n-set in three parts with respectively p, q, r elements (of course it will be 0 if $p+q+r \neq n$). This can be performed by first splitting the set in two parts and re-splitting the second part in two again or by first splitting the set in two parts and re-splitting the first part in two again. These two procedures provide the "trichotomy identity"

$$\sum_{m\in\mathbb{N}} \binom{n}{p,m}\binom{m}{q,r} = \sum_{m\in\mathbb{N}} \binom{n}{m,r}\binom{m}{p,q} \tag{23}$$

which, in this case is obvious as the two sums have only one term.
Now, one has also an interplay with the sum, called Van der Monde identity [31]

$$\binom{n_1+n_2}{p\,,\,q} = \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q}} \binom{n_1}{p_1,q_1}\binom{n_2}{p_2,q_2}. \tag{24}$$

In general, section coefficients $(i|j,k)$ count the number of ways of splitting an object i into two sub-objects j, k. they may or may not fulfill identities (23) and/or (27) and have a deep relationship with modern algebra.
Let us give another example which generalizes the preceding. For a word $w = a_1a_2\cdots a_n$ and $I \subset \{i_1, i_2, \cdots i_k\}$ $(i_1 < i_2 < \cdots < i_k)$ one defines the subword $w[I]$ as $w[I] = a_{i_1}a_{i_2}\cdots a_{i_k}$. For three words u, v, w, one can define the section coefficient

$$(w \mid u\,,v) = \#\Big\{(I,J) \mid I+J = [1..|w|],\ w[I]=u,\ w[J]=v\Big\} \tag{25}$$

for example, one has

$$(abab \mid aa\,,bb) = 1\ ;\ (abab \mid ba\,,ba) = 0\ ;\ (abab \mid ab\,,ba) = 1\ ;\ (abab \mid ab\,,ab) = 2.$$

These coefficients can be considered as "word functions", they are a generalization of the binomial section coefficients as

$$(a^n \mid a^p\,,a^q) = \binom{n}{p,q}\ .$$

For these coefficients, one has the trichotomy identity

$$\sum_{m\in A^*} (w \mid s\,, m)(m \mid t\,, u) = \sum_{m\in\mathbb{N}} (w \mid m\,, u)(m \mid s\,, t) \tag{26}$$

and the Van der Monde one reads

$$(w_1 w_2 \mid u\,, v) = \sum_{\substack{u=u_1u_2\\ v=v_1v_2}} (w_1 \mid u_1\,, v_1)(w_2 \mid u_2\,, v_2)\ . \tag{27}$$

The connection with modern algebra is done through the notion of a comultiplication. Taking C the vector space freely generated by the objects under consideration (the basis will be denoted $(e_i)_{i\in I}$), one can consider the section coefficients as structure constants for a linear mapping (called comultiplication)

$$\Delta\ :\ C \to C\otimes C \tag{28}$$

by

$$\Delta(\sum_{i\in I}\alpha_i e_i) = \sum_{i\in I}\alpha_i\Delta(e_i) = \sum_{i\in I}\sum_{j,k\in I}\alpha_i(e_i \mid e_j\,, e_k)e_j\otimes e_k\ . \tag{29}$$

(for the notion of structure constants of a comultiplication see [20]). Calling I the identity mapping $C\to C$, the trichotomy identity reads

$$(\Delta\otimes I)\circ\Delta = (I\otimes\Delta)\circ\Delta \tag{30}$$

whereas the Van der Monde Identity (providing C be given an algebra law $\mu\ :\ C\otimes C\to C$) reads

$$\Delta\circ\mu = \mu^{\otimes\ 2}\circ\Delta \tag{31}$$

where $\mu^{\otimes\ 2}$ is the law on $C\otimes C$ defined by

$$\mu^{\otimes\ 2}(x_1\otimes y_1, x_2\otimes y_2) = \mu(x_1\otimes y_1)\otimes\mu(x_2\otimes y_2)\ . \tag{32}$$

7.2. Coproducts and Hopf Algebra structures.

As it is classical, $\mathcal{U}(\mathcal{L}_{HW})$ is endowed with a structure of Hopf Algebra by the comultiplication $\Delta_{\mathcal{U}}$ given by the fact that the generators b, b^+ and u are primitive i. e.

$$\Delta_{\mathcal{U}}(x) = x\otimes 1_{\mathcal{U}} + 1_{\mathcal{U}}\otimes x \text{ for } x = b, b^+, u \tag{33}$$

one then gets at once the comultiplication of an element of the PBW basis

$$\Delta_{\mathcal{U}}((b^+)^p\, b^q\, u^r) = \sum_{(\alpha,\beta,\gamma)\le(p,q,r)} \binom{p}{\alpha}\binom{q}{\beta}\binom{r}{\gamma}(b^+)^\alpha\, b^\beta\, u^\gamma\otimes(b^+)^{p-\alpha}\, b^{q-\beta}\, u^{r-\gamma} \tag{34}$$

the counity is given by $\epsilon_{\mathcal{U}}((b^+)^p\, b^q\, u^r) = 0$ unless $p = q = r = 0$ in which case it equals 1. The antipode is given classically by $S(\Delta_{\mathcal{U}}((b^+)^p\, b^q\, u^r)) = (-1)^{p+q+r}u^r\, b^q\, ((b^+)^p$. This is the standard structure of any enveloping algebra as a Hopf algebra. The question solved in this paper is a construction on $\tilde{B}_q$ of a structure of Hopf algebra such that φ_4 be a Hopf arrow.

7.3. Partitions of a HW-graph and the comultplication.

Let Γ be a HW-graph and $R \subset \Gamma^{(1)}$, one defines $\Gamma[R]$ as the subgraph with lines R, tail and head functions restricted to R (call them t_R, h_R) and vertices $\Gamma[R]^{(0)}$ as $t_R(R) \cup h_R(R)$. It is easy to check that $\Gamma[R]$ is a HW-graph. It is also not difficult to check that, if $R_1 \subset R_2 \subset \Gamma^{(1)}$, on has the transitivity of restrictions i. e.

$$\Gamma[R_2][R_1] = \Gamma[R_1] \ . \tag{35}$$

We define a comultiplication on $\tilde{B}$ by

$$\Delta(\tilde{\Gamma}) = \sum_{R+Y=\Gamma^{(1)}} \widetilde{\Gamma[R]} \otimes \widetilde{\Gamma[Y]} \tag{36}$$

of course, one has to check that the result does not depend of the choice of the representative Γ which is straightforward. The proof of coassociativity rests on (35), i. e.

$$\begin{aligned}(\Delta \otimes I)\Delta(\tilde{\Gamma}) = (\Delta \otimes I)\Big(\sum_{R+Y=\Gamma^{(1)}} \widetilde{\Gamma[R]} \otimes \widetilde{\Gamma[Y]}\Big) = \\ \sum_{R+Y=\Gamma^{(1)}} \ \sum_{R_1+R_2=\Gamma[R]^{(1)}} \widetilde{\Gamma[R][R_1]} \otimes \widetilde{\Gamma[R][R_2]} \otimes \widetilde{\Gamma[Y]}\end{aligned} \tag{37}$$

but, by construction $\Gamma[R]^{(1)} = R$ and then, using (35), one has

$$\begin{aligned}(\Delta \otimes I)\Delta(\tilde{\Gamma}) = \sum_{R+Y=\Gamma^{(1)}} \ \sum_{R_1+R_2=R} \widetilde{\Gamma[R_1]} \otimes \widetilde{\Gamma[R_2]} \otimes \widetilde{\Gamma[Y]} = \\ \sum_{R_1+R_2+Y=R} \widetilde{\Gamma[R_1]} \otimes \widetilde{\Gamma[R_2]} \otimes \widetilde{\Gamma[Y]}\end{aligned} \tag{38}$$

which is the same as $(I \otimes \Delta)\Delta(\tilde{\Gamma})$.
One has now to prove that $\Delta\tilde{B} \to \tilde{B} \otimes \tilde{B}$ is a morphism of algebras. Let $\Gamma_i, i = 1, 2$ be two disjoint HW-graphs, one has

$$\begin{aligned}\Delta(\tilde{\Gamma_1}) *^{\otimes^2} \Delta(\tilde{\Gamma_2}) = \Big(\sum_{R_1+Y_1=\Gamma_1^{(1)}} \widetilde{\Gamma_1[R_1]} \otimes \widetilde{\Gamma_1[Y_1]}\Big) *^{\otimes^2} \Big(\sum_{R_1+Y_1=\Gamma_1^{(1)}} \widetilde{\Gamma_1[R_1]} \otimes \widetilde{\Gamma_1[Y_1]}\Big) = \\ \sum_{\substack{R_1+Y_1=\Gamma_1^{(1)} \\ R_2+Y_2=\Gamma_2^{(1)}}} (\widetilde{\Gamma_1[R_1]} * \widetilde{\Gamma_2[R_2]}) \otimes (\widetilde{\Gamma_1[Y_1]} * \widetilde{\Gamma_2[Y_2]}) = \\ \sum_{\substack{R_1+Y_1=\Gamma_1^{(1)} \\ R_2+Y_2=\Gamma_2^{(1)}}} \ \sum_{\substack{m_R \in match(\Gamma_2[R_2], \Gamma_1[R_1]) \\ m_Y \in match(\Gamma_2[Y_2], \Gamma_1[Y_1])}} \begin{pmatrix} \widetilde{\Gamma_1[R_1]} \\ m_R \\ \Gamma_2[R_2] \end{pmatrix} \otimes \begin{pmatrix} \widetilde{\Gamma_1[Y_1]} \\ m_R \\ \Gamma_2[Y_2] \end{pmatrix} =\end{aligned}$$

$$\sum_{m\in match(\Gamma_2,\Gamma_1)} \sum_{R+Y=\begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}^{(1)}} \begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}[R] \otimes \begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}[Y] \tag{39}$$

as the correspondence

$$(m, R, Y) \mapsto (R_1, Y_1, R_2, Y_2, m_R, m_Y) \tag{40}$$

is one to one between the sets

$$\{(m, R, Y) \text{ such that } (m \in match(\Gamma_2, \Gamma_1)) \text{ and } (R + Y = \begin{pmatrix}\Gamma_1\\ m\\ \Gamma_2\end{pmatrix}^{(1)})\} \tag{41}$$

and

$$\begin{aligned}&\{(R_1, R_2, Y_1, Y_2, m_R, m_Y) \text{ such that } (R_1 + Y_1 = \Gamma_1^{(1)},\ R_2 + Y_2 = \Gamma_2^{(1)})\\ &(m_R \in match(\Gamma_2[R_2], \Gamma_1[R_1]),\ m_Y \in match(\Gamma_2[Y_2], \Gamma_1[Y_1]))\}\end{aligned} \tag{42}$$

8. Endnotes

Formula (11) : Strictly speaking, the (double) sum in the LHS of this formula is indexed by the set

$$I_1 := \bigcup_{m'\in match(\Gamma_2,\Gamma_1)} \{m'\} \times match(\Gamma_3, \left(\begin{smallmatrix}\Gamma_1\\ m'\\ \Gamma_2\end{smallmatrix}\right)) \tag{43}$$

and the RHS of this formula is indexed by the set

$$I_2 := \bigcup_{m''\in match(\Gamma_3,\Gamma_2)} match(\begin{pmatrix}\Gamma_2\\ m''\\ \Gamma_3\end{pmatrix}, \Gamma_1) \times \{m''\} \tag{44}$$

the (one-to-one) correspondence $I_1 \to I_2$ is given by $(m', m'') \mapsto (m' \cup m''_{31}, m''_{32})$ and its inverse by $(m', m'') \mapsto (m'_{21}, m'' \cup m'_{31})$ with $m_{ij} = m \cap (\Gamma_i^+ \times \Gamma_j^-)$.

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