Heisenberg–Weyl Diagrams: Combinatorial Algebra with Decomposition

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I. INTRODUCTION

Comprehension of abstract mathematical concepts always goes through concrete models. Oftentimes, convenient representations are attained in terms of combinatorial objects. Their advantage comes from simplicity based on intuitive notions of enumeration, composition and decomposition which allow for insightful interpretations and neat pictorial arguments. This makes combinatorial perspective particularly attractive to quantum physics in its active pursuit of proper outlook and better understanding of fundamental phenomena, *e.g.* see (Baez and Dolan, 2001; Louck, 2008; Spekkens, 2007; **?**) for a few recent developments in this direction. In the present paper we take up with an algebraic structure of Quantum Theory which shall be considered from a combinatorial point of view.

The present-day formalism and structure of Quantum Theory is founded on the theory of operators acting in a Hilbert space. According to a few basic postulates the physical concepts of a system, observables and transformations find their representation as operators which further processed give account for experimental results. An important role in this abstract description play the notions of addition, multiplication and tensor product which are responsible for peculiar quantum properties such as interference, non-compatibility of measurements or entanglement in composite systems (Hughes, 1989; Isham, 1995; Peres, 2002). From the algebraic point of view the appropriate structure capturing these features is a bi-algebra. It consists of a vector space with two operations multiplication and co-multiplicationdescribing how operators compose and decompose. In the following we shall be concerned with a combinatorial model providing an intuitive picture of this abstract structure.

The bare formalism by itself is, however, not enough for description of real quantum phenomena. One has yet to associate operators with physical quantities which always involve some algebraic structure describing physical concepts related to the system. In practice the most common correspondence rules are based on the Heisenberg–Weyl algebra. This mainly derives from the analogy with classical mechanics whose Poissonian structure is reflected in the commutator of position and momentum observables $[x, p] = i\hbar$ (Dirac, 1982), which immediately brings Lie algebra into play. Another important instance of its use is the creation–annihilation paradigm $[a, a^\dagger] = 1$ employed in the occupation number representation in quantum mechanics or the second quantization formalism of quantum field theory. Accordingly, we take the Heisenberg–Weyl algebra as the basis for our combinatorial approach.

In this paper we are interested in development of combinatorial perspective on the Heisenberg–Weyl algebra and present a comprehensive model of the algebra in terms of diagrams. We shall discuss natural notions of diagram composition and decomposition which provide a straightforward interpretation of abstract operations of multiplication and co-multiplication. Such constructed combinatorial algebra $\mathcal{G}$ can be seen as a lifting of the Heisenberg–Weyl algebra $\mathcal{H}$ to a richer structure of diagrams capturing all properties of the latter. Moreover, it will be shown to have a natural bi-algebra structure furnishing a concrete model the enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ as well. Schematically, these relationships can be pic-

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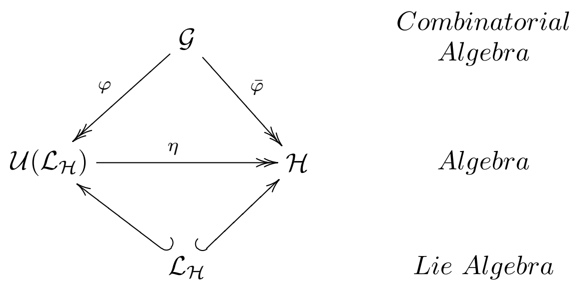

where all the arrows are (bi-)algebra morphisms. Whilst the lower part of the diagram is standard, the upper part and the construction of the combinatorial algebra $\mathcal{G}$ brings forth a genuine combinatorial underpinning of these abstract algebraic structures.
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(Flajolet and Sedgewick, 2008)(Bergeron *et al.*, 1998)(Hall, 2004)

II. HEISENBERG–WEYL ALGEBRA

Objective of this paper is development of a combinatorial model of the Heisenberg–Weyl algebra. In order to fully appreciate versatility of the following construction we start by briefly recalling some common algebraic structures and clearing up their relation to the Heisenberg–Weyl algebra.

A. Algebraic setting

Associative algebra with unit is one of the most basic structures used in theoretical description of physical phenomena. It consists of a *vector space* $\mathcal{A}$ over a field $\mathbb{K}$ which is equipped with a bilinear *multiplication* law $\mathcal{A}\times\mathcal{A}\ni(x,y)\longrightarrow x\,y\in\mathcal{A}$ which is associative and possesses a *unit* element I. Important notions in this framework are a *basis* of an algebra, which is a basis for its vector space structure, and the associated *structure constants*. For each basis $\{x_i\}$ the latter are defined as the coefficients $\gamma_{ij}^k\in\mathbb{K}$ in the expansion of the product $x_i\,x_j=\sum_k\gamma_{ij}^k\,x_k$. We note that structure constants uniquely determine the multiplication law in the algebra. A canonical example of the (noncommutative) associative algebra with unit is a matrix algebra, or more generally an algebra of linear operators acting in a vector space.

Description of composite systems is attained through the construction of a tensor product. Of particular importance for physical applications is how the algebra of transformations distribute among the components. A canonical example is the algebra of angular momentum and its representation in composite systems. In general, this issue is properly captured by the notion of a *bi-algebra* which consists of an associative algebra with unit $\mathcal{A}$ which is additionally equipped with a *co-product* and a *co-unit*. The co-product is defined as a co-associative linear mapping $\Delta:\mathcal{A}\longrightarrow\mathcal{A}\otimes\mathcal{A}$ prescribing the action of an algebra in a tensor product, whilst the co-unit $\varepsilon:\mathcal{A}\longrightarrow\mathbb{K}$ gives the representation in a trivial subsystem $\mathbb{K}$. Furthermore, the bi-algebra axioms require Δ and ε to be algebra morphisms, *i.e.* preserve multiplication in the algebra, which asserts the correct transfer of algebraic structure of $\mathcal{A}$ into tensor product.

It is instructive in this context to discuss the difference between Lie algebras and associative algebras which is often misconstrued. *Lie algebra* is a vector space $\mathcal{L}$ over a field $\mathbb{K}$ with a bilinear law $\mathcal{L}\times\mathcal{L}\ni(x,y)\to[x,y]\in\mathcal{L}$, called the Lie bracket, which is antisymmetric $[x,y]=-[y,x]$ and satisfies the Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$. As evident from the definition Lie algebras are not associative and lack the identity element. A standard remedy for these deficiencies consist in passing to its *enveloping algebra* $\mathcal{U}(\mathcal{L})$ which has a more familiar structure of an associative algebra with unit and at the same time captures all relevant properties of $\mathcal{L}$. The crucial step in this construction is the Poincaré-Birkhoff-Witt theorem providing an explicit construction of $\mathcal{U}(\mathcal{L})$ in terms of ordered monomials in the basis elements of $\mathcal{L}$. As such the enveloping algebras can be seen as faithful models of Lie algebras in terms of a structure with an associative law.

Below we shall illustrate these abstract algebraic constructions to explain the structure of the Heisenberg–Weyl algebra These abstract algebraic concepts gain on a concrete example.

B. Heisenberg–Weyl algebra revisited

In this paper we shall consider the *Heisenberg–Weyl algebra*, denoted by $\mathcal{H}$, which is an associative algebra with unit generated by two elements a and $a^\dagger$ subject to the relation

$$a\,a^\dagger=a^\dagger a+I\ . \tag{1}$$

This means that the algebra consists of elements $A\in\mathcal{H}$ which are linear combinations of finite products of the generators, *i.e.*

$$A=\sum_{\substack{r_k,\dots,r_1\\ s_k,\dots,s_1}}A_{\substack{r_k,\dots,r_1\\ s_k,\dots,s_1}}\ \ a^{\dagger\,r_k}\,a^{s_k}\,...\,a^{\dagger\,r_2}\,a^{s_2}\ a^{\dagger\,r_1}\,a^{s_1}, \tag{2}$$

where the sum ranges over a finite set of multi-indexes $r_k,...,r_1\in\mathbb{N}$ and $s_k,...,s_1\in\mathbb{N}$ (with the convention $a^0=a^{\dagger\,0}=I$). Throughout the paper we stick to the notation used in the occupation number representation in which a and $a^\dagger$ are interpreted the *annihilation* and *creation* operators. We note, however, that one should not attach too much weight to this choice as we consider algebraic properties only for which particular realizations are irrelevant and the crux of the study is the sole relation of Eq. (1). For example, one could equally well use the multiplication X and derivative operators $D = \partial_x$ acting in the space of polynomials or square integrable functions which also satisfy the relation $[D, X] = I$.

Observe that representation furnished by Eq. (2) is ambiguous since the rewrite rule of Eq. (1) renders different representations of the same element of the algebra, *e.g.* $aa^\dagger = a^\dagger a + I$. The remedy for this situation consists in fixing a preferred order of the generators. Conventionally, it is done by choosing the *normally ordered* form in which all annihilators stand to the right of creators. As a result, each element of the algebra $\mathcal{H}$ can be uniquely written in the normally ordered form as

$$A = \sum_{k,l} \alpha_{kl}\ a^{\dagger\, k}\, a^l\,. \tag{3}$$

In this way, we find out that the normally ordered monomials constitute a natural basis for the Heisenberg–Weyl algebra, *i.e.*

$$\text{Basis of } \mathcal{H}\ : \qquad \{\, a^{\dagger\, k} a^l \,\}_{k,l\in\mathbb{N}}\ ,$$

indexed by pairs of integers $k, l = 0, 1, 2, ...$, and Eq. (3) is the expansion of the element A in this basis. We should note that the normally ordered representation of the elements of the algebra suggests itself not only as the simplest one but is also of practical use and importance in applications in quantum optics (Glauber, 1963; Klauder and Skagerstam, 1985; Schleich, 2001) and quantum field theory (Bjorken and Drell, 1993; Mattuck, 1992). In the sequel we choose to work in this particular basis and for the complete algebraic description of $\mathcal{H}$ we still need structure constants of the algebra. They can be readily read off from the formula for expansion of the product of basis elements

$$a^{\dagger\, p} a^q\, a^{\dagger\, k} a^l = \sum_i \binom{q}{i}\binom{k}{i}\, i!\ a^{\dagger\, p+k-i} a^{q+l-i}\ . \tag{4}$$

We note that working a fixed basis is in general a nontrivial task. In our case the problem comes down reshuffling of a and $a^\dagger$ to the normally ordered form which oftentimes is be attained by insightful combinatorial methodology (Blasiak *et al.*, 2007; Wilcox, 1967).

C. Enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$

We recall that the *Heisenberg–Weyl Lie algebra*, denoted by $\mathcal{L}_\mathcal{H}$, is a 3-dimensional vector space with the basis $\{a^\dagger, a, e\}$ and the Lie bracket defined as $[a, a^\dagger] = e$, $[a^\dagger, e] = [a, e] = 0$. Passing to the associative algebra consist in imposing the linear order $a^\dagger \succ a \succ e$ and constructing the *enveloping algebra* $\mathcal{U}(\mathcal{L}_\mathcal{H})$ with the basis given by the family

$$\text{Basis of } \mathcal{U}(\mathcal{L}_\mathcal{H})\ : \qquad \{\, a^{\dagger\, k} a^l\, e^m \,\}_{k,l,m\in\mathbb{N}}\ ,$$

which is indexed by triples of integers $k, l, m = 0, 1, 2, ...$ Hence, elements $B \in \mathcal{U}(\mathcal{L}_\mathcal{H})$ are of the form

$$B = \sum_{k,l,m} \beta_{klm}\ a^{\dagger\, k} a^l\, e^m\,. \tag{5}$$

According to the the Poincaré-Birkhoff-Witt theorem, the associative multiplication law in the enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ is defined by concatenation subject to the rewrite rules

$$a\, a^\dagger = a^\dagger a + e\,, \tag{6}$$

$$e\, a^\dagger = a^\dagger e\,, \qquad e\, a\ = a\, e\,. \tag{7}$$

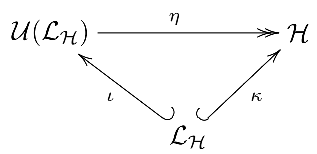
One checks that the formula for multiplication of basis elements in $\mathcal{U}(\mathcal{L}_\mathcal{H})$ is a slight generalization of Eq. (4) and reads

$$a^{\dagger\, p} a^q\, e^r\, a^{\dagger\, k} a^l\, e^m = \\ = \sum_i \binom{q}{i}\binom{k}{i}\, i!\ a^{\dagger\, p+k-i}\, a^{q+l-i}\, e^{r+l+i}\ . \tag{8}$$

Note that the algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ differs from $\mathcal{H}$ by the additional central element e which should not be confused with the unity I. This distinction plays an important role in some applications as explained below. In situations when this difference is insubstantial one may set $e \to I$ recovering the Heisenberg–Weyl algebra $\mathcal{H}$, *i.e.* we have the surjective morphism $\eta : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathcal{H}$ given by

$$\eta\left(a^{\dagger\, i} a^j\, e^k\right) = a^{\dagger\, i} a^j\ . \tag{9}$$

This completes the algebraic picture which can be subsumed in the following diagram



We emphasize that the inclusions $\iota : \mathcal{L}_\mathcal{H} \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$ and $\kappa = \eta \circ \iota : \mathcal{L}_\mathcal{H} \longrightarrow \mathcal{H}$ are Lie algebra morphisms, whilst the suriection $\eta : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathcal{H}$ is a morphism of associative algebras with unit. Note different structures carried over by these morphisms.

Finally, we observe that the enveloping algebra $\mathcal{U}(\mathcal{L}_{HW})$ is equipped with a bi-algebra structure. It is constructed in a standard way by determining the co-product $\Delta : \mathcal{U}(\mathcal{L}_{HW}) \longrightarrow \mathcal{U}(\mathcal{L}_{HW}) \otimes \mathcal{U}(\mathcal{L}_{HW})$ on the generators $x = a^\dagger, a, e$ setting $\Delta(x) = x \otimes I + I \otimes x$, which further extends to

$$\Delta\left(a^{\dagger\, p} a^q\, e^r\right) = \\ \sum_{i,j,k} \binom{p}{i}\binom{q}{j}\binom{r}{k}\ a^{\dagger\, i} a^j\, e^k \otimes a^{\dagger\, p-i} a^{q-j}\, e^{r-k}\,. \tag{10}$$

The co-unit $\varepsilon : \mathcal{U}(\mathcal{L}_{HW}) \longrightarrow \mathbb{K}$ is given by

$$\varepsilon\left(a^{\dagger\, p} a^q\, e^r\right)\ =\ \begin{cases} 1 & \text{if} \quad p, q, r = 0\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{11}$$

A word of warning here: the Heisenberg–Weyl algebra $\mathcal{H}$ can not be endowed with a bi-algebra structure as is sometimes tacitly taken in. It is because properties of the co-unit contradict the relation of Eq. (1), *i.e.* it follows that $\varepsilon(I) = \varepsilon(a\,a^\dagger - a^\dagger a) = \varepsilon(a)\,\varepsilon(a^\dagger) - \varepsilon(a^\dagger)\,\varepsilon(a) = 0$ whilst one should have $\varepsilon(I) = 1$. This brings forward the importance of the additional central element $e \neq I$ which saves the day for $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$.

III. ALGEBRA OF DIAGRAMS AND COMPOSITION

In this Section we define the combinatorial class of Heisenberg–Weyl diagrams which is the central object of our study. It shall be equipped with an intuitive notion of composition allowing for construction of an algebra structure providing a combinatorial model of algebras $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_{HW})$.

A. Combinatorial concepts

We start by recalling a few basic notions from graph theory (Diestel, 2005; Lawvere and Schanuel, 1997) needed for a precise definition of the Heisenberg–Weyl diagrams, and then provide an intuitive graphical representation of this structure.

From a set-theoretical point of view a *directed graph* is a collection of *edges* E and *vertices* V with the structure determined by two mappings $h, t : E \longrightarrow V$ prescribing how the *head* and *tail* of an edge are attached to vertices. Here we shall address a slightly more general setting consisting of graphs whose edges may have one of the ends free (but not both), *i.e.* we shall consider finite graphs with *partially defined* mappings h and t such that $dom(h) \cup dom(t) = E$. We shall call a *cycle* in a graph any sequence of edges $e_1, e_2, ..., e_n$ such that $h(e_k) = t(e_{k+1})$ for $k < n$ and $h(e_n) = t(e_1)$. An convenient concept in graph theory concerns the notion of equivalence. Two graphs are said to be *equivalent* if one can be isomorphically transformed into another, *i.e.* both have the same number of vertices and edges and there exist two isomorphisms $\alpha_E : E_1 \longrightarrow E_2$ and $\alpha_V : V_1 \longrightarrow V_2$ faithfully transferring structure of the graphs in the following sense

$$\begin{array}{ccc} E_1 & \overset{h}{\underset{t}{\rightrightarrows}} & V_1 \\ {\scriptstyle \alpha_E}\downarrow & & \downarrow{\scriptstyle \alpha_V} \\ E_2 & \overset{h}{\underset{t}{\rightrightarrows}} & V_2 \end{array}$$

The advantage of such defined *equivalence classes* is that one can liberate himself from specific set-theoretical realizations and think of a graph only in terms of relations between vertices and edges – this is the attitude we shall adopt in the sequel.

In this context we put the following formal definition:

Definition 1 (Heisenberg–Weyl Diagrams)
A Heisenberg–Weyl diagram Γ is a class of partially defined directed graphs without cycles. It consists of three sorts of lines: the inner ones Γ^0 having both head and tail attached to vertices, the ingoing lines Γ^- with free tails, and the outgoing lines Γ^+ with free heads.

A typical modus operandi when working with classes is to invoke to its representatives. Following this practice we shall by default make all statements concerning Heisenberg–Weyl diagrams for its representatives assuming that they are class invariants which can be routinely checked in each case.

The formal Definition 1 gets an intuitive picture in a graphical form, see illustration in Fig. 1. A diagram can be represented as a set of verices • connected by lines each carrying an arrow indicating the direction from the tail to the head. Lines having one of the ends not attached to a vertex will be marked with △ or ▲ at the free head or tail respectively. We conventionally draw all ingoing lines at the bottom and the outgoing lines at the top with all arrows heading upwards which is always possible since the diagrams do not have cycles. It forms a picture of the Heisenberg-Weyl diagram as a sort of process or transformation with vertices playing the role of intermediate steps.

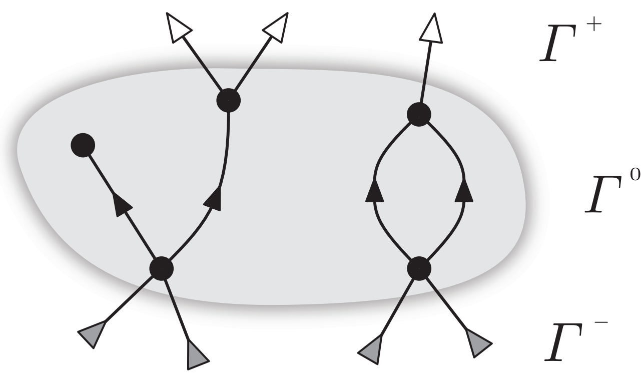


FIG. 1 An example of a Heisenberg–Weyl diagram with distinguished three characteristic sorts of lines: the inner ones $|\Gamma^0| = 4$, the ingoing lines $|\Gamma^-| = 4$ and outgoing lines $|\Gamma^+| = 3$.

An important characteristic of a diagram is the total number of its lines denoted by $|\Gamma|$. In the next sections we shall further restrict this counting to the inner, the ingoing and the outgoing lines, denoting the result by $|\Gamma^0|$, $|\Gamma^-|$ and $|\Gamma^+|$ respectively.

B. Diagram composition

The crucial concept of this paper concerns composition of Heisenberg–Weyl diagrams. It shall have a straightforward interpretation in graphical representation as plugging of free lines one into another, and shall be based on the notion of a matching.

A *matching* m of two sets A and B is a choice of pairs $(a_i, b_i) \in A \times B$ all having different components, *i.e.* if

$a_i = a_j$ or $b_i = b_j$ then $i = j$. We shall denote the collection of all possible matchings by $A \lessdot B$, and its restriction to matchings comprising i pairs only by $A \overset{i}{\lessdot} B$. It is straightforward to check by exact enumeration the formula $|A \overset{i}{\lessdot} B| = \binom{|A|}{i}\binom{|B|}{i}\, i!$ which is valid for any i with the convention $\binom{n}{k} = 0$ for $n < k$.

The concept of diagram composition suggests itself, as:

Definition 2 (Diagram Composition)

Consider two Heisenberg–Weyl diagrams Γ_1 and Γ_2 and a matching $m \in \Gamma_2^- \lessdot \Gamma_1^+$ between the free lines going out from the first one Γ_1^+ and the free lines going into the second one Γ_2^-. The composite diagram, denoted as $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$, is constructed by joining the lines coupled by the matching m.

This descriptive definition can be formalized by referring to representatives in the following way. Given two disjoint graphs, such that $V_{\Gamma_2} \cap V_{\Gamma_1} = \emptyset$ and $E_{\Gamma_2} \cap E_{\Gamma_1} = \emptyset$, we construct the composite graph $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ which consists of vertices $V_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1} = V_{\Gamma_2} \cup V_{\Gamma_1}$ and edges $E_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1} = E_{\Gamma_2} \cup E_{\Gamma_1} \cup m - \big(pr_{V_{\Gamma_2}}(m) \cup pr_{V_{\Gamma_1}}(m)\big)$. The head and tail functions then unambiguously extend on the set $E_{\Gamma_2} \cup E_{\Gamma_1} - \big(pr_{V_{\Gamma_2}}(m) \cup pr_{V_{\Gamma_1}}(m)\big)$ and for $e = (e_{\Gamma_2}, e_{\Gamma_1}) \in m$ we define $h_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1}(e) = h_{\Gamma_2}(e_{\Gamma_2})$ and $t_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1}(e) = t_{\Gamma_1}(e_{\Gamma_1})$. Clearly, choice of the disjoint graphs in a class is always possible and the resulting directed graph does not contain cycles. It remains to check that such defined composition of diagrams making use of representatives is class invariant.

Definition 2 can be straightforwardly seen as if diagrams were put one over another with some of the lines going out from the lower one plugged into some of the lines going into to the upper one in accordance with a given matching $m \in \Gamma_2^- \lessdot \Gamma_1^+$, for illustration see Fig. 2. Observe that in general two graphs can be composed in many ways, *i.e.* as many as there are possible matchings (elements in $\Gamma_2^- \lessdot \Gamma_1^+$). In Section III.C we shall exploit all these possible compositions to endow the diagrams with the structure of an algebra. Note also that that the above construction depends on the order in which diagrams are composed and the reverse order yields different results.

We conclude by two simple remarks concerning compositions of two diagrams Γ_2 and Γ_1 constructed by joining exactly i lines. Firstly, we observe that the these diagrams can be enumerated explicitly by the formula

$$|\Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+| = \binom{|\Gamma_2^-|}{i}\binom{|\Gamma_2^+|}{i}\, i!\,. \tag{12}$$

Secondly, the number of ingoing, outgoing and inner lines in the composed diagram does not depend on the choice of matching the $m \in \Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+$ and reads respectively

$$\begin{aligned} |(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1)^+| &= |\Gamma_2^+| + |\Gamma_1^+| - i\,, \\ |(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1)^-| &= |\Gamma_2^-| + |\Gamma_1^-| - i\,, \\ |(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1)^0| &= |\Gamma_2^0| + |\Gamma_1^0| + i\,. \end{aligned} \tag{13}$$

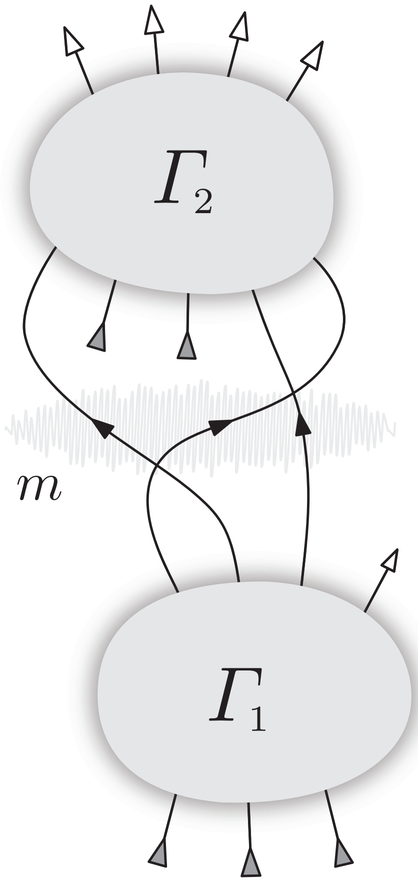


FIG. 2 Composition of two diagrams $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ according to the matching $m \in \Gamma_2^- \lessdot \Gamma_1^+$ consisting of three connections.

C. Algebra of Heisenberg–Weyl Diagrams

Here, we show that the Heisenberg–Weyl diagrams come along with a natural algebraic structure based on diagram composition. It will appear to be a combinatorial refinement of the familiar algebras $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$.

Algebra requires two operations, addition and multiplication, which we construct in the following way. We define $\mathcal{G}$ as a vector space over $\mathbb{K}$ generated by the basis set consisting of all Heisenberg–Weyl diagrams, *i.e.*

$$\mathcal{G} = \Big\{ \textstyle\sum_i \alpha_i\, \Gamma_i : \ \alpha_i \in \mathbb{K},\ \Gamma_i - \text{Heisenberg–Weyl diagram} \Big\}. \tag{14}$$

Addition in $\mathcal{G}$ has the usual form

$$\textstyle\sum_i \alpha_i\, \Gamma_i + \sum_i \beta_i\, \Gamma_i = \sum_i (\alpha_i + \beta_i)\, \Gamma_i. \tag{15}$$

Nontrivial part in the definition of algebra $\mathcal{G}$ concerns multiplication, which by bilinearity

$$\sum_i \alpha_i\ \Gamma_i * \sum_j \beta_j\ \Gamma_j = \sum_{i,j} \alpha_i\beta_j\ \Gamma_i * \Gamma_j, \quad (16)$$

comes down to determining it on the basis set of the Heisenberg–Weyl diagrams. Recalling the notions of Section III.B we define product of two diagrams Γ_2 and Γ_1 as the sum of all possible compositions, *i.e.*

$$\Gamma_2 * \Gamma_1 = \sum_{m\in \Gamma_2^- \lhd\, \Gamma_1^+} \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1\ . \quad (17)$$

Note that all terms in the sum are distinct and coefficients equal to one. Such defined multiplication is noncommutative and possesses unit element which is the void graph Ø (no vertices, no lines). Moreover, the following theorem holds (for the proof of associativity see Appendix A):

Theorem 1 (Algebra of Diagrams)

Heisenberg–Weyl diagrams form a (noncommutative) associative algebra with unit $(\mathcal{G}, +, *, \emptyset)$.

Our objective, now, is to clarify the relation of the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ to the physically relevant algebras $\mathcal{U}(\mathcal{L}_{HW})$ and HW. We shall construct forgetful mappings which give a simple combinatorial prescription how to descend from $\mathcal{G}$ to the later two.

Let us define a linear mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ on the basis elements by

$$\varphi(\Gamma) \overset{df}{=} b^{\dagger\,|\Gamma^+|}\ b^{|\Gamma^-|}\ e^{|\Gamma^0|}\ . \quad (18)$$

This prescription can be intuitively understood by looking at the diagrams as if they were carrying auxiliary labels $b^\dagger$, b and e attached to all the outgoing, ingoing and inner lines respectively. Then the mapping of Eq. (18) just neglects structure of the graph and pays attention to the number of lines only, *i.e.* count them according to the labels. Clearly, φ is onto and it can be proved to be a genuine algebra morphism, *i.e.* it preserves addition and multiplication in $\mathcal{G}$ (for the proof see Appendix B).

Similarly, we define the morphism $\bar{\varphi} : \mathcal{G} \longrightarrow HW$ as

$$\bar{\varphi}(\Gamma) \overset{df}{=} (a^\dagger)^{|\Gamma^+|}\ a^{|\Gamma^-|}\ , \quad (19)$$

which differs from φ by ignoring all inner lines in the diagrams. It can be expressed as $\bar{\varphi} = \varphi \circ \eta$ and hence fulfills all properties of an algebra morphism.

We recapitulate the above discussion in the following theorem:

Theorem 2 (Forgetful mapping)

The mappings $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ *and* $\bar{\varphi} : \mathcal{G} \longrightarrow HW$ *defined in Eqs. (18) and (19) are suriective algebra morphisms, and the following diagram commutes*

$$\begin{array}{ccc} & \mathcal{G} & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\bar{\varphi}} \\ \mathcal{U}(\mathcal{L}_{\mathcal{H}}) & \xrightarrow{\ \eta\ } & \mathcal{H} \end{array} \quad (20)$$

Therefore, the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ can is as a lifting of the algebras $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ and $\mathcal{H}$, and the latter two can be recovered by applying appropriate forgetful mappings φ and $\bar{\varphi}$. As such the algebra $\mathcal{G}$ can be seen fine graining of the abstract algebras $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ and $\mathcal{H}$ which gain a concrete combinatorial interpretation in terms the richer structure of diagrams.

IV. DIAGRAM DECOMPOSITION AND BI-ALGEBRA

We have seen in Section III how the notion of composition allows for combinatorial denition of diagram multiplication opening the doors to the realm of algebra. Here, we shall consider the opposite concept of diagram decomposition which induces a combinatorial co-product in the algebra endowing Heisenberg–Weyl diagrams with the bialgebra structure.

A. Basic concepts: Combinatorial decomposition

Suppose we are given a class of objects which allow for decomposition, *i.e.* split into ordered pairs of pieces from the same class. Without loss of generality one may think of the class of Heisenberg–Weyl diagram and some for the moment unspecified procedure assigning to a given diagram Γ its possible decompositions (Γ', Γ''). In general there might be various ways of splitting an object according to a given rule and, moreover, some of them may yield the same result. We shall denote the collection of all possibilities by $\langle\Gamma\rangle = \{(\Gamma', \Gamma'')\}$ and for short write

$$\Gamma \longrightarrow (\Gamma', \Gamma'') \in \langle\Gamma\rangle\ . \quad (21)$$

Note that in a strict sense $\langle\Gamma\rangle$ is a multiset, *i.e.* it is like a set but with arbitrary repetitions of elements allowed. Hence, in order not to overlook any of the decompositions, some of which may be the same, we should use a more appropriate notation employing the notion of a disjoint union, denoted by $\biguplus$, and write

$$\langle\Gamma\rangle = \biguplus_{\substack{\text{decompositions}\\ \Gamma\to(\Gamma',\Gamma'')}} \{(\Gamma', \Gamma'')\}\ . \quad (22)$$

The notion of decomposition is quite general at this point and its further development obviously depends on the choice of the rule. One usually supplements this concept with additional constraints. Below we discuss some natural conditions expected from a decomposition rule.

(0) Finiteness. It is reasonable to assume that an object decomposes in a finite number of ways, *i.e.* for each Γ the multiset $\langle\Gamma\rangle$ is finite.

(1) Triple decomposition. Decomposition into pairs naturally extends to splitting of an object into three pieces $\Gamma \longrightarrow (\Gamma_3, \Gamma_2, \Gamma_1)$. An obvious way to carry out the multiple splitting is by applying the same procedure repeatedly, *i.e.* decomposing one of the components obtained in the preceding step. However, following this prescription one naturally expects that the result does not depend on the choice of the component it is applied to. In other words, we require to end up with the same collection of triple decompositions when splitting $\Gamma \longrightarrow (\Gamma'', \Gamma_1)$ and then splitting the second component $\Gamma'' \longrightarrow (\Gamma_3, \Gamma_2)$, *i.e.*

$$\Gamma \longrightarrow (\Gamma'', \Gamma_1) \xrightarrow{\Gamma'' \longrightarrow (\Gamma_3, \Gamma_2)} (\Gamma_3, \Gamma_2, \Gamma_1) \ , \quad (23)$$

as in the case when starting with $\Gamma \longrightarrow (\Gamma_3, \Gamma')$ and then splitting the first component $\Gamma' \longrightarrow (\Gamma_2, \Gamma_1)$, *i.e.*

$$\Gamma \longrightarrow (\Gamma_3, \Gamma') \xrightarrow{\Gamma' \longrightarrow (\Gamma_2, \Gamma_1)} (\Gamma_3, \Gamma_2, \Gamma_1) \ . \quad (24)$$

This condition can be seen as the co-associativity property for decomposition, and in explicit form boils down to the following equality:

$$\biguplus_{\substack{(\Gamma'', \Gamma_1) \in \langle \Gamma \rangle \\ (\Gamma_3, \Gamma_2) \in \langle \Gamma'' \rangle}} \{(\Gamma_3, \Gamma_2, \Gamma_1)\} = \biguplus_{\substack{(\Gamma_3, \Gamma') \in \langle \Gamma \rangle \\ (\Gamma_2, \Gamma_1) \in \langle \Gamma' \rangle}} \{(\Gamma_3, \Gamma_2, \Gamma_1)\} \ . \quad (25)$$

(2) Void structure. Often, in a class there exists a sort of a void (or empty) element $\emptyset$, such that objects decompose in a trivial way. It should have the the property that any object Γ split into a pair containing either $\emptyset$ or Γ in a unique way:

$$\Gamma \longrightarrow (\emptyset, \Gamma) \quad \text{and} \quad \Gamma \longrightarrow (\Gamma, \emptyset) \ . \quad (26)$$

(3) Symmetry of decomposition. For some rules the order between components in decompositions is immaterial, *i.e.* the rule allows for an exchange $(\Gamma', \Gamma'') \leftrightarrow (\Gamma'', \Gamma')$. In this case the following symmetry condition holds

$$(\Gamma', \Gamma'') \in \langle \Gamma \rangle \implies (\Gamma'', \Gamma') \in \langle \Gamma \rangle \ . \quad (27)$$

(4) Composition–decomposition compatibility. Suppose that in addition to decomposition we also have a well defined notion of composition of objects in the class. Let the multiset comprising all possible compositions of Γ_2 with Γ_1 be denoted by $\Gamma_2 \blacktriangleleft \Gamma_1$, *e.g.* for the Heisenberg–Weyl diagrams we have

$$\Gamma_2 \blacktriangleleft \Gamma_1 = \biguplus_{m \in \Gamma_2^- \lessdot \Gamma_1^+} \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1 \ . \quad (28)$$

Now, given a pair of objects Γ_2 and Γ_1 we may think of two consistent decomposition schemes which involve composition. We can either start by composing them together $\Gamma_2 \blacktriangleleft \Gamma_1$ and then splitting all resulting objects into pieces, or first decompose each of them separately into $\langle \Gamma_2 \rangle$ and $\langle \Gamma_1 \rangle$ and then compose elements of both sets in a component-wise manner. One may reasonably expect the same outcome either way the procedure goes. Hence, formal description of compatibility comes down to the equality:

$$\biguplus_{\Gamma \in \Gamma_2 \blacktriangleleft \Gamma_1} \langle \Gamma \rangle = \biguplus_{\substack{(\Gamma_2', \Gamma_2'') \in \langle \Gamma_2 \rangle \\ (\Gamma_1', \Gamma_1'') \in \langle \Gamma_1 \rangle}} (\Gamma_2' \blacktriangleleft \Gamma_1') \times (\Gamma_2'' \blacktriangleleft \Gamma_1'') \ . \quad (29)$$

This concept can be rephrased as a separability condition if considered from an operational point of view whereby composition describes possible outcomes of the action of one object on another. In this framework objects of a class are thought of as composite structures consisting of parts given by the notion of decomposition. Separability condition simply asserts that action of objects is independent on whether it is performed on structures taken as the whole or separately part by part.

Having discussed the above quite general conditions expected from a reasonable decomposition rule we are in position to get round to the realm of algebra. Already in Section III.C we have seen how the notion of composition induces multiplication which endows the class of Heisenberg–Weyl diagrams with the structure of an algebra, see Theorem 1. Following this route we shall employ the concept of decomposition to introduce the structure of a bi-algebra.

Let us consider a linear mapping $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ defined on the basis elements as

$$\Delta(\Gamma) = \sum_{(\Gamma', \Gamma'') \in \langle \Gamma \rangle} \Gamma' \otimes \Gamma'' \ . \quad (30)$$

Note, that although all coefficients in Eq. (30) are equal to one, some terms in the sum may appear several times. It is because elements in the multiset $\langle \Gamma \rangle$ may repeat and the numbers counting their multiplicities are sometimes section coefficients (Joni and Rota, 1979). Observe that the sum is well defined as long the number of decompositions is finite, *i.e.* condition *(0)* holds.

We shall also need a linear mapping $\varepsilon : \mathcal{G} \longrightarrow \mathbb{K}$ which picks out the void element $\emptyset$. It shall be defined in a canonical way

$$\varepsilon(\Gamma) = \begin{cases} 1 & \text{if} \quad \Gamma = \emptyset \ , \\ 0 & \text{otherwise} \ . \end{cases} \quad (31)$$

as the projection on the subspace generated by $\emptyset$.

The mappings Δ and ε build upon a reasonable decomposition procedure make $\mathcal{G}$ into a bi-algebra as summarized in the following lemma (for the proofs see Appendix C):

Lemma 1 (Decomposition and Bi-algebra)

(i) *If the conditions (0), (1) and (2) are satisfied, the mappings $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ and $\varepsilon : \mathcal{G} \longrightarrow \mathbb{K}$ defined in Eqs. (30) and (31) are the co-product and co-unit in the algebra $\mathcal{G}$. Such defined co-algebra $(\mathcal{G}, \Delta, \varepsilon)$ is co-commutative provided the condition (3) is fulfilled.*

(ii) *In addition, if the condition (4) holds we have a genuine bi-algebra structure $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon)$.*

We remark that the above discussion is applicable to wide range of combinatorial classes and decomposition rules which thus far were left unspecified. Below we shall employ these concepts to the class of Heisenberg–Weyl diagrams.

B. Bi-algebra of Heisenberg–Weyl diagrams

In this Section we shall provide explicit decomposition rule for the Heisenberg–Weyl diagrams satisfying all conditions discussed in Section IV.A. In this way we shall complete the whole picture by introducing a bi-algebra structure in $\mathcal{G}$.

We start by observing that for a given Heisenberg–Weyl graph Γ, each subset of its edges $L \subset E_\Gamma$ induces a subgraph $\Gamma|_L$ which is defined by restriction of the head and tail functions to the subset L. Likewise, the remaining part of edges $R = E_\Gamma - L$ gives rise to a subgraph $\Gamma|_R$. Clearly, the results are again Heisenberg–Weyl graphs. Thus, by considering ordered partitions of the set of edges into two subsets $L + R = E_\Gamma$, *i.e.* $L \cup R = E_\Gamma$ and $L \cap R = \emptyset$, we end up with pairs of disjoint graphs $(\Gamma|_L, \Gamma|_R)$. This suggests the following definition:

Definition 3 (Diagram Decomposition)
We shall consider decomposition of a Heisenberg–Weyl diagram Γ to be any splitting (Γ_L, Γ_R) induced by an ordered partition of its lines $L + R = E_\Gamma$. Hence, the multiset $\langle \Gamma \rangle$ comprising all possible decompositions can be indexed by the set of ordered double partitions $\{(L, R) : L + R = E_\Gamma\}$, and we have

$$\langle \Gamma \rangle = \biguplus_{L+R=E_\Gamma} \{(\Gamma|_L, \Gamma|_R)\} . \tag{32}$$

Graphical picture is clear: decomposition of a diagram $\Gamma \longrightarrow (\Gamma|_L, \Gamma|_R)$ comes down to the choice of lines $L \subset E_\Gamma$, which taken out make up the first component of the pair whilst the reminder induced by $R = E_\Gamma - L$ constitutes the second one. See illustration in Fig. 3.

Enumeration of all decompositions of a diagram Γ is straightforward as the multiset $\langle \Gamma \rangle$ can be indexed by subsets of E_Γ. Since $|E_\Gamma| = |\Gamma|$ the explicit counting gives $|\langle \Gamma \rangle| = \sum_i \binom{|\Gamma|}{i} = 2^{|\Gamma|}$. This simple observation can be generalized to calculate the number of decompositions $(\Gamma|_L, \Gamma|_R) \in \langle \Gamma \rangle$ in which the first component has i outgoing, j ingoing and k inner lines, *i.e.* $|\,\Gamma|_L^+| = i, |\,\Gamma|_L^-| = j, |\,\Gamma|_L^0| = k$. Accordingly, enumeration boils down the choice of i, j and k lines out of the sets Γ^+, Γ^- and Γ^0 respectively, which gives

$$\left|\left\{(\Gamma|_L, \Gamma|_R) \in \langle \Gamma \rangle : \substack{|\Gamma|_L^+|=i \\ |\Gamma|_L^-|=j \\ |\Gamma|_L^0|=k}\right\}\right| = \binom{|\Gamma^+|}{i}\binom{|\Gamma^-|}{j}\binom{|\Gamma^0|}{k}. \tag{33}$$

Observe that the second component $\Gamma|_R$ is always determined by the first one $\Gamma|_L$ and hence the number of its outgoing, ingoing and inner lines is given by

$$\begin{aligned} |\,\Gamma|_R^+| &= |\Gamma^+| - i , \\ |\,\Gamma|_R^-| &= |\Gamma^-| - j , \\ |\,\Gamma|_R^0| &= |\Gamma^0| - k . \end{aligned} \tag{34}$$

We note that the numbers in Eq. (33) count multiplicities of elements in $\langle \Gamma \rangle$ which are called section coefficients (Joni and Rota, 1979).

Having explicitly defined the notion of diagram decomposition one may check that it satisfies conditions *(1) - (4)* of Section IV.A, for the proofs see Appendix D. In this context Eq. (30) defining the co-product in the algebra $\mathcal{G}$ takes the form

$$\Delta(\Gamma) = \sum_{L+R=E_\Gamma} \Gamma|_L \otimes \Gamma|_R . \tag{35}$$

Referring to Lemma 1 we supplement Theorem 1 by the following result

Theorem 3 (Bi-algebra of Diagrams)
The algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ has a bi-algebra structure $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon)$ with the (co-commutative) co-product and co-unit defined in Eqs. (35) and (31) respectively.

The algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ was shown to be directly related to the algebra $\mathcal{U}(\mathcal{L}_{HW})$ through the forgetful mapping φ which preserves algebraic operations as explained in Theorem 2. Here, however, in the context of Theorem 3 the algebra $\mathcal{G}$ is additionally equipped with the co-product and co-unit. Since $\mathcal{U}(\mathcal{L}_{HW})$ is the bi-algebra as well it is natural to ask weather this extra structure carries over with φ. It turns up that it is also preserved, and we augment Theorem 2 by the following proposition (for the proof see Appendix B)

Proposition 1 (Bi-algebra morphism φ)
The forgetful mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ defined in Eq. (18) is a bi-algebra morphism.

In this way, we have extended results of Section III to encompass the bi-algebra structure of the enveloping algebr $\mathcal{U}(\mathcal{L}_\mathcal{H})$. This completes the picture of the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ as a combinatorial model capturing all relevant properties of the algebras $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_\mathcal{H})$.

V. CONCLUSIONS

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Conclusions: TO DO ...
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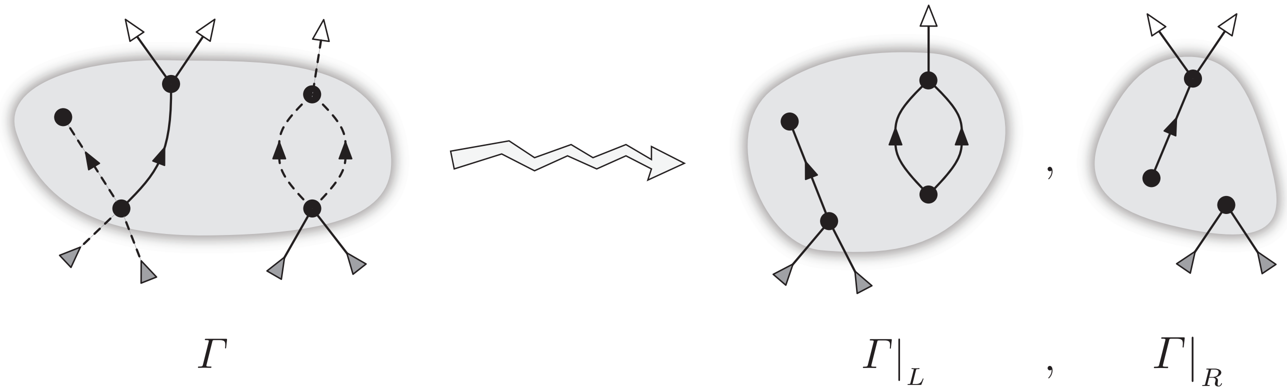

FIG. 3 An example of diagram decomposition $\Gamma \longrightarrow (\Gamma|_L\,, \Gamma|_R)$. The choice of edges $L \subset E_\Gamma$ inducing the diagram $\Gamma|_L$ is depicted on the left diagram as dashed lines.

Possible Journals: *Rev. Mod. Phys., Phys. Rev. A, J. Phys. A, Phys. Lett. A, ...*
Possible referees: *Severini, Louck, Vourdas, Katriel, Burdik, Foata, Spekkens, Coecke, ...*
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Appendix A: Associativity of multiplication in $\mathcal{G}$

We shall prove associativity of multiplication defined in Eq. (17). From bilinearity we only need to check it for the basis elements, *i.e.*

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) = (\Gamma_3 * \Gamma_2) * \Gamma_1\ . \tag{A1}$$

Written explicitly, the left and right hand sides of this equation take the form

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) = \sum_{m'} \sum_{m_{21}} \Gamma_3 \overset{m'}{\blacktriangleleft} (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1) \tag{A2}$$

where $m' \in \Gamma_3^- \lhd\!\!\lhd (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1)^+$ and $m_{21} \in \Gamma_2^- \lhd\!\!\lhd \Gamma_1^+$, whilst

$$(\Gamma_3 * \Gamma_2) * \Gamma_1 = \sum_{m_{32}} \sum_{m''} (\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2) \overset{m''}{\blacktriangleleft} \Gamma_1 \tag{A3}$$

where $m_{32} \in \Gamma_3^- \lhd\!\!\lhd \Gamma_2^+$ and $m'' \in (\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2)^- \lhd\!\!\lhd \Gamma_1^+$.

Let us take a look at the double sums in the above equations, indexed by (m', m_{21}) and (m_{32}, m'') respectively, and observe that there exists a one-to-one correspondence between theirs elements. We construct it by fine graining of the matchings, see Fig. 4, and define the following two mappings. The first one as

$$(m', m_{21}) \ \longrightarrow\ (m_{32}, m'')\ , \tag{A4}$$

where $m_{32} = m' \cap (\Gamma_3^- \times \Gamma_2^+)$ and $m'' = m_{21} \cup (m' \cap (\Gamma_3^- \times \Gamma_1^+))$, and similarly the second one

$$(m_{32}, m'') \ \longrightarrow\ (m', m_{21})\ , \tag{A5}$$

with $m' = m_{32} \cup (m'' \cap (\Gamma_3^- \times \Gamma_1^+))$ and $m_{21} = m'' \cap (\Gamma_2^- \times \Gamma_1^+)$. Clearly, the mappings are inverses one of another, which assures a one-to-one correspondence between elements of the double sums in Eqs. (A2) and (A3). Moreover, the summands that are mapped one onto another are equal, *i.e.* the corresponding diagrams $\Gamma_3 \overset{m'}{\blacktriangleleft} (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1)$ and $(\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2) \overset{m''}{\blacktriangleleft} \Gamma_1$ are exactly the same. This ends the proof by showing equality of the right hand sides of Eqs. (A2) and (A3).

Appendix B: Forgetful morphism φ

In Theorem 2 and Proposition 1 we have declared that the linear mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ defined in Eq. (18) is a bi-algebra morphism. Now, we prove this statement.

We start by showing that φ preserves multiplication in $\mathcal{G}$. From linearity it is enough to check for the basis elements that $\varphi(\Gamma_2 * \Gamma_1) = \varphi(\Gamma_2)\, \varphi(\Gamma_1)$, which is justified in the following sequence of equalities:

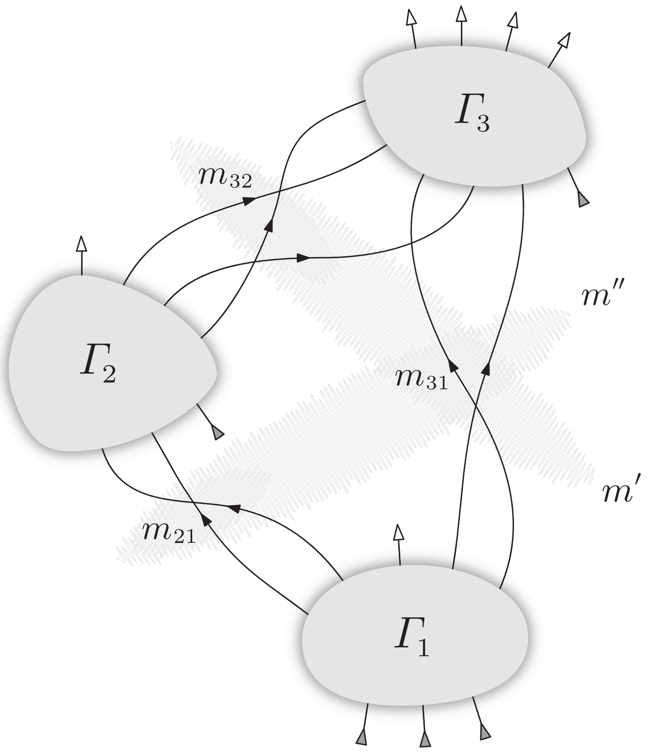


FIG. 4 Fine graining of the matchings $m' \in \Gamma_3^- \blacktriangleleft\!\!\lhd (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1)^+$ and $m'' \in (\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2)^- \blacktriangleleft\!\!\lhd \Gamma_1^+$ used in the proof of associativity of multiplication.

$$\varphi(\Gamma_2 * \Gamma_1) \overset{(17)}{=} \sum_{m \in \Gamma_2^- \blacktriangleleft\!\!\lhd \Gamma_1^+} \varphi(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1) = \sum_i \sum_{m \in \Gamma_2 \overset{i}{\blacktriangleleft\!\!\lhd} \Gamma_1} \varphi(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1) \tag{B1}$$

$$\overset{(13)}{=} \sum_i \sum_{m \in \Gamma_2^- \overset{i}{\blacktriangleleft\!\!\lhd} \Gamma_1^+} (b^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\, b^{|\Gamma_2^-|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i}$$

$$= \sum_i (b^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\, b^{|\Gamma_2^-|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i} \sum_{m \in \Gamma_2^- \overset{i}{\blacktriangleleft\!\!\lhd} \Gamma_1^+} 1 \tag{B2}$$

$$\overset{(12)}{=} \sum_i \binom{|\Gamma_2^-|}{i} \binom{|\Gamma_1^+|}{i}\, i!\, (b^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\, b^{|\Gamma_2^-|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i}$$

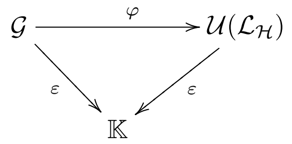
$$\overset{(8)}{=} \left((b^\dagger)^{|\Gamma_2^+|}\, b^{|\Gamma_2^-|}\, e^{|\Gamma_2^0|}\right) \left((b^\dagger)^{|\Gamma_1^+|}\, b^{|\Gamma_1^-|}\, e^{|\Gamma_1^0|}\right) = \varphi(\Gamma_2)\, \varphi(\Gamma_1)\ .$$

In the above derivation the main trick in Eq. (B1) consists in splitting of the set of diagram matchings into disjoint subsets according to the number of connected lines, *i.e.* $\Gamma_2^- \blacktriangleleft\!\!\lhd \Gamma_1^+ = \bigcup_i \Gamma_2^- \overset{i}{\blacktriangleleft\!\!\lhd} \Gamma_1^+$. Then upon observation that the summands in Eq. (B2) do not depend on $m \in \Gamma_2^- \overset{i}{\blacktriangleleft\!\!\lhd} \Gamma_1^+$ we may execute explicitly one of the sums counting elements in $\Gamma_2^- \overset{i}{\blacktriangleleft\!\!\lhd} \Gamma_1^+$ with the help of Eq. (12).

We also need to show that the co-product and co-unit carry over with φ, *i.e.* the following diagrams commute

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\ \varphi\ } & \mathcal{U}(\mathcal{L}_\mathcal{H}) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ \mathcal{G} \otimes \mathcal{G} & \xrightarrow{\ \varphi\otimes\varphi\ } & \mathcal{U}(\mathcal{L}_\mathcal{H}) \otimes \mathcal{U}(\mathcal{L}_\mathcal{H}) \end{array}$$

and



It means that when proceeding via mapping φ from $\mathcal{G}$ to $\mathcal{U}(\mathcal{L}_{HW})$ one can use co-product and co-unit in either of the algebras and the same result obtains, *i.e.*

$$(\varphi \otimes \varphi) \circ \Delta \;=\; \Delta \circ \varphi \;, \tag{B3}$$
$$\varepsilon \;=\; \varepsilon \circ \varphi \;, \tag{B4}$$

where Δ and ε on the left-hand-sides act in $\mathcal{G}$ whilst on the right-hand-sides in $\mathcal{U}(\mathcal{L}_{HW})$. The proof of Eq. (B3) rests upon the counting formula in Eq. (33) and observation of Eq. (34), which justify the following equalities

$$\begin{aligned}(\varphi \otimes \varphi) \circ \Delta\,(\Gamma) \quad &= \sum_{L+R=E_\Gamma} \varphi\,(\Gamma|_L) \otimes \varphi\,(\Gamma|_R) = \sum_{L\subset E_\Gamma} \varphi\,(\Gamma|_L) \otimes \varphi\,\big(\Gamma|_{E_\Gamma - L}\big) \\ &\overset{(33),(34)}{=} \sum_{i,j,k} \binom{|\Gamma^+|}{i}\binom{|\Gamma^-|}{j}\binom{|\Gamma^0|}{k}\; b^{\dagger\, i}\; b^j\; e^k \otimes b^{\dagger\,|\Gamma^+|-i}\; b^{|\Gamma^-|-j}\; e^{|\Gamma^0|-k} \overset{(10)}{=} \Delta \circ \varphi\,(\Gamma)\;.\end{aligned}$$

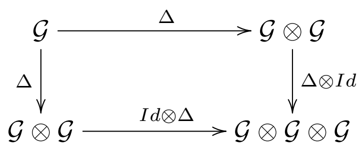
Check of Eq. (B4) readily obtains by comparing Eqs. (11) and (31) .

Appendix C: From decomposition to bi-algebra

In order to prove Lemma 1 we should check in part *(i)* co-associativity of the co-product Δ and properties of the co-unit ε, whilst for part *(ii)* show that the mappings Δ and ε preserve multiplication in $\mathcal{G}$.

(i) Co-algebra

Co-product $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ is co-associative if the following diagram commutes



and we need to verify the equality

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta\;. \tag{C1}$$

Since Δ defined in Eq. (30) is linear it is enough to check it for the basis elements Γ. Accordingly, the left-hand side takes the form

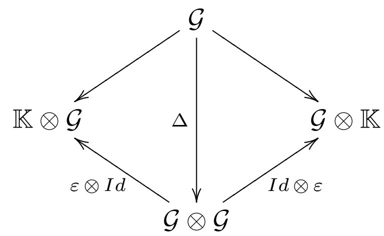
$$\begin{aligned}(\Delta \otimes Id) \circ \Delta\,(\Gamma) \;&=\; (Id \otimes \Delta) \sum_{(\Gamma_1,\Gamma'')\in\langle\Gamma\rangle} \Gamma_1 \otimes \Gamma'' \\ &=\; \sum_{\substack{(\Gamma_1,\Gamma'')\in\langle\Gamma\rangle \\ (\Gamma_2,\Gamma_3)\in\langle\Gamma''\rangle}} \Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3\;, \end{aligned} \tag{C2}$$

whereas the right-hand-side reads

$$\begin{aligned}(Id \otimes \Delta) \circ \Delta\,(\Gamma) \;&=\; (\Delta \otimes Id) \sum_{(\Gamma',\Gamma_3)\in\langle\Gamma\rangle} \Gamma' \otimes \Gamma_3 \\ &=\; \sum_{\substack{(\Gamma',\Gamma_3)\in\langle\Gamma\rangle \\ (\Gamma_1,\Gamma_2)\in\langle\Gamma'\rangle}} \Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3\;. \end{aligned} \tag{C3}$$

If condition *(1)* holds the property of Eq. (25) asserts equality of the right-hand-sides of Eqs. (C2) and (C3) and the co-product defined in Eq. (30) is co-associative.

Co-unit $\varepsilon : \mathcal{G} \longrightarrow \mathbb{K}$ by definition should assert commutativity of the diagram



which upon identification $\mathbb{K}\otimes\mathcal{G} = \mathcal{G}\otimes\mathbb{K} = \mathcal{G}$ boils down to the equalities

$$(\varepsilon \otimes Id) \circ \Delta = Id = (Id \otimes \varepsilon) \circ \Delta\;. \tag{C4}$$

We shall check first of them for the basis elements Γ by direct calculation

$$\begin{aligned}(\varepsilon \otimes Id) \circ \Delta\,(\Gamma) \;&=\; (\varepsilon \otimes Id) \sum_{(\Gamma_1,\Gamma_2)\in\langle\Gamma\rangle} \Gamma_1 \otimes \Gamma_2 \\ &=\; \sum_{(\Gamma_1,\Gamma_2)\in\langle\Gamma\rangle} \varepsilon(\Gamma_1) \otimes \Gamma_2 \qquad \text{(C5)} \\ &=\; 1 \otimes \Gamma = \Gamma = Id\,(\Gamma)\;.\end{aligned}$$

Note that we have used condition *(2)* by taking all terms in the sum Eq. (C5) equal to zero except the unique decomposition $(\emptyset, \Gamma)$ picked up by ε as defined in Eq. (31).

Identification $1 \otimes \Gamma = \Gamma$ ends the proof of the first equality in Eq. (C4); verification of the second one is analogous.

Co-commutativity of the co-product Δ under the condition *(3)* is straightforward since from Eq. (27) we have

$$\Delta(\Gamma) = \sum_{(\Gamma',\Gamma'')\in\langle\Gamma\rangle} \Gamma' \otimes \Gamma'' = \sum_{(\Gamma',\Gamma'')\in\langle\Gamma\rangle} \Gamma'' \otimes \Gamma' \ .$$

(ii) Bi-algebra

Structure of a bi-algebra obtains whenever the co-product $\Delta : \mathcal{G} \otimes \mathcal{G} \longrightarrow \mathcal{G}$ and co-unit $\varepsilon : \mathcal{G} \longrightarrow \mathbb{K}$ of the co-algebra are compatible with multiplication in $\mathcal{G}$. Thus, we need to verify for basis elements Γ_1 and Γ_2 that

$$\Delta\left(\Gamma_2 * \Gamma_1\right) \ = \ \Delta\left(\Gamma_2\right) * \Delta\left(\Gamma_1\right) \ , \tag{C6}$$

with component-wise multiplication in the tensor product $\mathcal{G} \otimes \mathcal{G}$ on the right-hand-side, and

$$\varepsilon\left(\Gamma_2 * \Gamma_1\right) \ = \ \varepsilon\left(\Gamma_2\right) \varepsilon\left(\Gamma_1\right) \ , \tag{C7}$$

with terms on the right-hand-side multiplied in $\mathbb{K}$.

We check Eq. (C6) directly by expanding both hand sides using definitions of Eqs. (17) and (30). Accordingly, the left-hand-side takes the form

$$\begin{aligned}\Delta\left(\Gamma_2 * \Gamma_1\right) \ &= \sum_{m\in\Gamma_2^- \lhd \Gamma_1^+} \Delta\left(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1\right) \\ &= \sum_{m\in\Gamma_2^- \lhd \Gamma_1^+} \ \sum_{(\Gamma',\Gamma'')\in\langle\Gamma_2 \overset{m}{\blacktriangleleft}\Gamma_1\rangle} \Gamma' \otimes \Gamma''\end{aligned} \tag{C8}$$

whilst the right-hand-side reads

$$\begin{aligned}\Delta\left(\Gamma_2\right) * \Delta\left(\Gamma_1\right) &= \sum_{\substack{(\Gamma_1',\Gamma_1'')\in\langle\Gamma_1\rangle \\ (\Gamma_2',\Gamma_2'')\in\langle\Gamma_2\rangle}} \underbrace{\left(\Gamma_2' \otimes \Gamma_2''\right) * \left(\Gamma_1' \otimes \Gamma_1''\right)}_{(\Gamma_2' * \Gamma_1')\otimes(\Gamma_2'' * \Gamma_1'')} \\ = \sum_{\substack{(\Gamma_1',\Gamma_1'')\in\langle\Gamma_1\rangle \\ (\Gamma_2',\Gamma_2'')\in\langle\Gamma_2\rangle}} & \sum_{\substack{m'\in\Gamma_2' \lhd \Gamma_1' \\ m''\in\Gamma_2'' \lhd \Gamma_1''}} \left(\Gamma_2' \overset{m'}{\blacktriangleleft} \Gamma_1'\right) \otimes \left(\Gamma_2'' \overset{m''}{\blacktriangleleft} \Gamma_1''\right)\end{aligned} \tag{C9}$$

A closer look at condition *(4)* and Eq. (29) shows a one-to-one correspondence between terms in the sums on the right-hand-sides of Eqs. (C8) and (C9) verifying validity of Eq. (C6).

Check of Eq. (C7) rests upon simple observation that composition of diagrams $\Gamma_2 * \Gamma_1$ yields the void diagram only if both of them are void. Then, both hand sides are equal to 1 if $\Gamma_1 = \Gamma_2 = \emptyset$ and 0 otherwise, which entails Eq. (C7).

Appendix D: Properties of diagram decomposition

We shall verify that decomposition of Definition 3 satisfies conditions *(0) - (4)* of Section IV.A.

Condition *(0)* follows directly from the construction as we consider finite diagrams only.

Proof of condition *(1)* consists in providing a one-to-one correspondence between schemes (23) and (24) decomposing a diagram Γ into triples. Accordingly, one easily checks that each triple $(\Gamma_L, \Gamma_M, \Gamma_R)$ obtained by

$$\Gamma \ \longrightarrow \ (\Gamma_L, \Gamma_{\bar{R}}) \ \xrightarrow{\Gamma_{\bar{R}} \longrightarrow (\Gamma_M, \Gamma_R)} \ (\Gamma_L, \Gamma_M, \Gamma_R) \tag{D1}$$

also turns up as the decomposition

$$\Gamma \ \longrightarrow \ (\Gamma_{\bar{L}}, \Gamma_R) \ \xrightarrow{\Gamma_{\bar{L}} \longrightarrow (\Gamma_L, \Gamma_M)} \ (\Gamma_L, \Gamma_M, \Gamma_R) \tag{D2}$$

for $\bar{L} = L + M$. Vice versa as well, triples obtained by scheme (D2) coincide with the results of (D1) for the choice $\bar{R} = M + R$. Therefore, the multisets of triple decompositions are equal and Eq. (25) holds.

Condition *(2)* is straightforward since the void graph $\emptyset$ is given by empty set of lines, and hence the decompositions $\Gamma \longrightarrow (\Gamma, \emptyset)$ and $\Gamma \longrightarrow (\emptyset, \Gamma)$ are uniquely defined by the partitions $E_\Gamma + \emptyset = E_\Gamma$ and $\emptyset + E_\Gamma = E_\Gamma$ respectively.

Symmetry condition *(3)* results from swapping subsets $L \leftrightarrow R$ in the partition $L + R = E_\Gamma$ which readily yields Eq. (27).

In order to check property *(4)* we need to construct a one-to-one correspondence between elements of both sides of Eq. (29). First, we observe that elements of the left-hand-side are decompositions of $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ for all $m \in \Gamma_2 \lhd \Gamma_1$, *i.e.*

$$\left(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1|_L \, , \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1|_R\right) \tag{D3}$$

where $L + R = E_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1}$. On the other hand, the right-hand-side consists of component-wise compositions of pairs $(\Gamma_2|_{L_2}, \Gamma_2|_{R_2}) \in \langle\Gamma_2\rangle$ and $(\Gamma_1|_{L_1}, \Gamma_1|_{R_1}) \in \langle\Gamma_1\rangle$ for $L_2 + R_2 = E_{\Gamma_2}$ and $L_1 + R_1 = E_{\Gamma_1}$, which written explicitly are of the form

$$\left(\Gamma_2|_{L_2} \overset{m_L}{\blacktriangleleft} \Gamma_1|_{L_1} \, , \Gamma_2|_{R_2} \overset{m_R}{\blacktriangleleft} \Gamma_1|_{R_1}\right) \tag{D4}$$

with $m_L \in \Gamma_2|_{L_2} \lhd \Gamma_1|_{L_1}$ and $m_R \in \Gamma_2|_{R_2} \lhd \Gamma_1|_{R_1}$. We construct two mappings between elements of type (D3) and (D4) by the following assignments, see Fig. 5 for schematic illustration. The first one is defined as:

$$(m, L, R) \longrightarrow (L_1, R_1, L_2, R_2, m_L, m_R) \ ,$$

where $L_i = E_{\Gamma_i} \cap L$, $R_i = E_{\Gamma_i} \cap R$ for $i = 1, 2$ and $m_L = m \cap L$, $m_R = m \cap R$. The second one is given by:

$$(L_1, R_1, L_2, R_2, m_L, m_R) \longrightarrow (m, L, R) \ ,$$

with $m = m_L \cup m_R$ and $L = L_2 \cup L_1$, $R = R_2 \cup R_2$. One checks that these mappings are inverses one of another and, moreover, the corresponding pairs of diagrams (D3) and (D4) are the same. This verifies that the multisets on the left and right hand side of Eq. (29) are equal and condition *(4)* obtains.

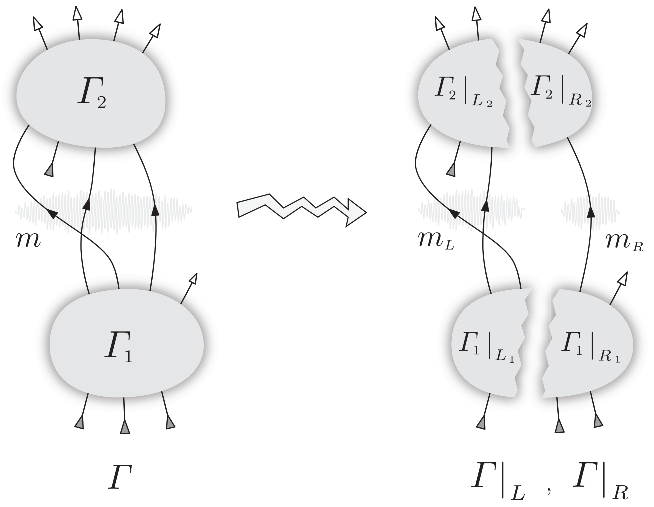


FIG. 5 Decompositions of $\Gamma = \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ for some $m \in \Gamma_2 \lessdot \Gamma_1$

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