Heisenberg–Weyl Diagrams: Combinatorial Algebra with Decomposition

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I. INTRODUCTION

Comprehension of abstract mathematical concepts always goes through concrete models. Oftentimes, convenient representations are attained in terms of combinatorial objects. Their advantage comes from simplicity based on intuitive notions of enumeration, composition and decomposition which allow for insightful interpretations and neat pictorial arguments. This makes combinatorial perspective particularly attractive to quantum physics in its active pursuit of proper outlook and better understanding of fundamental phenomena, *e.g.* see (Baez and Dolan, 2001; Louck, 2008; Spekkens, 2007; **?**) for a few recent developments in this direction. In the present paper we take up an algebraic structure of Quantum Theory which shall be considered from a combinatorial point of view.

The present-day formalism and structure of Quantum Theory is founded on the theory of operators acting in a Hilbert space. According to a few basic postulates the physical concepts of a system, observables and transformations, find their representation as operators which account for experimental results. An important role in this abstract description is played by the notions of addition, multiplication and tensor product which are responsible for peculiar quantum properties such as interference, non-compatibility of measurements and entanglement in composite systems (Hughes, 1989; Isham, 1995; Peres, 2002). From the algebraic point of view the appropriate structure capturing these features is a *bi-algebra*. This consists of a vector space with two operations, multiplication and co-multiplication, describing how operators compose and decompose. In the following we shall be concerned with a combinatorial model providing an intuitive picture of this abstract structure.

The bare formalism by itself is, however, not enough for a description of real quantum phenomena. One has yet to associate operators with physical quantities. This in turn will involve the association of some algebraic structure to physical concepts related to the system. In practice the most common correspondence rules are based on the Heisenberg–Weyl algebra. This mainly derives from the analogy with classical mechanics whose Poissonian structure is reflected in the quantum-mechanical commutator of position and momentum observables $[x,p]=i\hbar$ (Dirac, 1982); this commutator immediately brings a Lie algebra structure into play. Another important instance of the use of an equivalent commutator is that of the creation–annihilation operators $[a,a^\dagger]=1$, employed in the occupation number representation in quantum mechanics or the second quantization formalism of quantum field theory. Accordingly, we take the Heisenberg–Weyl algebra as the basis for our combinatorial approach.

In this paper we are interested in the development of a combinatorial perspective on the Heisenberg–Weyl algebra and present a comprehensive model of this algebra in terms of diagrams. We shall discuss natural notions of diagram composition and decomposition which provide a straightforward interpretation of abstract operations of multiplication and co-multiplication. Such a constructed combinatorial algebra $\mathcal{G}$ can be seen as a lifting of the Heisenberg–Weyl algebra $\mathcal{H}$ to a richer structure of diagrams, capturing all the properties of the latter. Moreover, it will be shown to have a natural bi-algebra structure providing a concrete model for the enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ as well. Schematically, these relationships

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can be pictured as follows

$$
\begin{tikzcd}
& \mathcal{G} \arrow[dl, "\varphi"'] \arrow[dr, "\bar{\varphi}"] & \\
\mathcal{U}(\mathcal{L}_{\mathcal{H}}) \arrow[rr, two heads, "\eta"] & & \mathcal{H} \\
& \mathcal{L}_{\mathcal{H}} \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Combinatorial Algebra

Algebra

Lie Algebra

where all the arrows are (bi-)algebra morphisms. Whilst the lower part of the diagram is standard, the upper part and the construction of the combinatorial algebra $\mathcal{G}$ brings forth a genuine combinatorial underpinning of these abstract algebraic structures.

(Flajolet and Sedgewick, 2008)(Bergeron *et al.*, 1998)(Hall, 2004)

II. HEISENBERG–WEYL ALGEBRA

The objective of this paper is to develop a combinatorial model of the Heisenberg–Weyl algebra. In order to fully appreciate the versatility of the following construction we start by briefly recalling some common algebraic structures and clarifying their relation to the Heisenberg–Weyl algebra.

A. Algebraic setting

An associative algebra with unit is one of the most basic structures used in the theoretical description of physical phenomena. It consists of a *vector space* $\mathcal{A}$ over a field $\mathbb{K}$ which is equipped with a bilinear *multiplication* law $\mathcal{A} \times \mathcal{A} \ni (x, y) \longrightarrow x\, y \in \mathcal{A}$ which is associative and possesses a *unit* element I.[1] Important notions in this framework are a *basis* of an algebra, by which is meant a basis for its underlying vector space structure, and the associated *structure constants*. For each basis $\{x_i\}$ the latter are defined as the coefficients $\gamma_{ij}^k \in \mathbb{K}$ in the expansion of the product $x_i\, x_j = \sum_k \gamma_{ij}^k\, x_k$. We note that the structure constants uniquely determine the multiplication law in the algebra.[2] When the underlying vector space is finite dimensional of dimension N, that is each vector-space element has a unique expansion in terms of N basis elements, then there is only a finite number N^3 of non-vanishing γ_{ij}^k's. A canonical example of the (non-commutative) associative algebra with unit is a matrix algebra, or more generally an algebra of linear operators acting in a vector space.

A description of composite systems is obtained through the construction of a tensor product. Of particular importance for physical applications is how the transformations distribute among the components. A canonical example is the algebra of angular momentum and its representation on composite systems. In general, this issue is properly captured by the notion of a *bi-algebra* which consists of an associative algebra with unit $\mathcal{A}$ which is additionally equipped with a *co-product* and a *co-unit*. The co-product is defined as a co-associative linear mapping $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ prescribing the action of an algebra in a tensor product, whilst the co-unit $\varepsilon : \mathcal{A} \longrightarrow \mathbb{K}$ gives the representation in a trivial subsystem $\mathbb{K}$. Furthermore, the bi-algebra axioms require Δ and ε to be algebra morphisms, *i.e.* preserve multiplication in the algebra, which asserts the correct transfer of algebraic structure of $\mathcal{A}$ into tensor product (see [] for a complete set of bi-algebra axioms).

It is instructive in this context to discuss the difference between Lie algebras and associative algebras which is often misconstrued. A *Lie algebra* is a vector space $\mathcal{L}$ over a field $\mathbb{K}$ with a bilinear law $\mathcal{L} \times \mathcal{L} \ni (x, y) \rightarrow [x, y] \in \mathcal{L}$, called the Lie bracket, which is antisymmetric $[x, y] = -[y, x]$ and satisfies the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$. As evident from the definition, Lie algebras are not associative and lack an identity element. A standard remedy for these deficiencies consist in passing to its *enveloping algebra* $\mathcal{U}(\mathcal{L})$ which has the more familiar structure of an associative algebra with unit and at the same time captures all the relevant properties of $\mathcal{L}$. An important step in its realization is the Poincaré-Birkhoff-Witt theorem which provides an explicit construction of $\mathcal{U}(\mathcal{L})$ in terms of ordered monomials in the basis elements of $\mathcal{L}$. As such, the enveloping algebras can be seen as giving faithful models of Lie algebras in terms of a structure with an associative law.

Below we shall illustrate these abstract algebraic constructions to explain the structure of the Heisenberg–Weyl algebra These abstract algebraic concepts gain on a concrete example.

B. Heisenberg–Weyl algebra revisited

In this paper we shall consider the *Heisenberg–Weyl algebra*, denoted by $\mathcal{H}$, which is an associative algebra with unit generated by two elements a and $a^\dagger$ subject to the relation

$$a\, a^\dagger = a^\dagger a + I \; . \tag{1}$$

This means that the algebra consists of elements $A \in \mathcal{H}$ which are linear combinations of finite products of the generators, *i.e.*

$$A = \sum_{\substack{r_k,\ldots,r_1 \\ s_k,\ldots,s_1}} A_{\substack{r_k,\ldots,r_1 \\ s_k,\ldots,s_1}} \;\; a^{\dagger\, r_k}\, a^{s_k} \ldots a^{\dagger\, r_2}\, a^{s_2}\; a^{\dagger\, r_1}\, a^{s_1}, \tag{2}$$

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[1] A full list of axioms may be found in any standard text on algebra, such as Bourbaki, N *Algebra I*, Springer (1989)

[2] The structure constants must of course satisfy the constraints provided by the associative law

where the sum ranges over a finite set of multi-indexes $r_k, ..., r_1 \in \mathbb{N}$ and $s_k, ..., s_1 \in \mathbb{N}$ (with the convention $a^0 = a^{\dagger\, 0} = I$). Throughout the paper we stick to the notation used in the occupation number representation in which a and $a^\dagger$ are interpreted as *annihilation* and *creation* operators. We note, however, that one should not attach too much weight to this choice as we consider algebraic properties only, so particular realizations are irrelevant and the crux of the study is the sole relation of Eq. (1). For example, one could equally well use the multiplication X and derivative operators $D = \partial_x$ acting in the space of polynomials or square integrable functions which also satisfy the relation $[D, X] = I$.

Observe that the representation given by Eq. (2) is ambiguous in so far as the rewrite rule of Eq. (1) allows different representations of the same element of the algebra, *e.g.* $aa^\dagger = a^\dagger a + I$. The remedy for this situation consists in fixing a preferred order of the generators. Conventionally, this is done by choosing the *normally ordered* form in which all annihilators stand to the right of creators. As a result, each element of the algebra $\mathcal{H}$ can be uniquely written in normally ordered form as

$$A = \sum_{k,l} \alpha_{kl}\ a^{\dagger\, k}\, a^l\,. \tag{3}$$

In this way, we find that the normally ordered monomials constitute a natural basis for the Heisenberg–Weyl algebra, *i.e.*

$$\text{Basis of } \mathcal{H}\ : \qquad \left\{ a^{\dagger\, k} a^l \right\}_{k,l\in\mathbb{N}}\,,$$

indexed by pairs of integers $k, l = 0, 1, 2, ...$, and Eq. (3) is the expansion of the element A in this basis. We should note that the normally ordered representation of the elements of the algebra suggests itself not only as the simplest one but is also of practical use and importance in applications in quantum optics (Glauber, 1963; Klauder and Skagerstam, 1985; Schleich, 2001) and quantum field theory (Bjorken and Drell, 1993; Mattuck, 1992). In the sequel we choose to work in this particular basis; for the complete algebraic description of $\mathcal{H}$ we still need the structure constants of the algebra. They can be readily read off from the formula for the expansion of the product of basis elements

$$a^{\dagger\, p} a^q\, a^{\dagger\, k} a^l = \sum_i^q \binom{q}{i}\binom{k}{i}\, i!\ a^{\dagger\, p+k-i} a^{q+l-i}\ . \tag{4}$$

We note that working in a fixed basis is in general a nontrivial task. In our case, the problem comes down to rearranging a and $a^\dagger$ to normally ordered form which may often be achieved by insightful combinatorial methodology (Blasiak *et al.*, 2007; Wilcox, 1967).

C. Enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$

We recall that the *Heisenberg–Weyl Lie algebra*, denoted by $\mathcal{L}_\mathcal{H}$, is a 3-dimensional vector space with basis $\{a^\dagger, a, e\}$ and Lie bracket defined by $[a, a^\dagger] = e$, $[a^\dagger, e] = [a, e] = 0$. Passing to the associative algebra consists of imposing the linear order $a^\dagger \succ a \succ e$ and constructing the *enveloping algebra* $\mathcal{U}(\mathcal{L}_\mathcal{H})$ with basis given by the family

$$\text{Basis of } \mathcal{U}(\mathcal{L}_\mathcal{H})\ : \qquad \left\{ a^{\dagger\, k} a^l\, e^m \right\}_{k,l,m\in\mathbb{N}}\,,$$

which is indexed by triples of integers $k, l, m = 0, 1, 2, ...$. Hence, elements $B \in \mathcal{U}(\mathcal{L}_\mathcal{H})$ are of the form

$$B = \sum_{k,l,m} \beta_{klm}\ a^{\dagger\, k} a^l\, e^m\,. \tag{5}$$

According to the the Poincaré-Birkhoff-Witt theorem, the associative multiplication law in the enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ is defined by concatenation subject to the rewrite rules

$$\begin{aligned} a\, a^\dagger &= a^\dagger a + e\,, \\ e\, a^\dagger &= a^\dagger e\,, \\ e\, a &= a\, e\ . \end{aligned} \tag{6}$$

One checks that the formula for multiplication of basis elements in $\mathcal{U}(\mathcal{L}_\mathcal{H})$ is a slight generalization of Eq. (4) and reads

$$\begin{aligned} &a^{\dagger\, p} a^q\, e^r\, a^{\dagger\, k} a^l\, e^m = \\ &\qquad = \sum_i^q \binom{q}{i}\binom{k}{i}\, i!\ a^{\dagger\, p+k-i}\ a^{q+l-i}\ e^{r+l+i}\ . \end{aligned} \tag{7}$$

Note that the algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ differs from $\mathcal{H}$ by the additional central element e which should not be confused with the unity I.[3] This distinction plays an important role in some applications as explained below. In situations when this difference is insubstantial one may set $e \to I$ recovering the Heisenberg–Weyl algebra $\mathcal{H}$, *i.e.* we have the surjective morphism $\eta : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathcal{H}$ given by

$$\eta\left(a^{\dagger\, i} a^j\, e^k\right) = a^{\dagger\, i} a^j. \tag{8}$$

This completes the algebraic picture which can be subsumed in the following diagram

$$\begin{tikzcd} \mathcal{U}(\mathcal{L}_\mathcal{H}) \arrow[rr, two heads, "\eta"] & & \mathcal{H} \\ & \mathcal{L}_\mathcal{H} \arrow[ul, hook, "\iota"] \arrow[ur, hook', "\kappa"'] & \end{tikzcd}$$

We emphasize that the inclusions $\iota : \mathcal{L}_\mathcal{H} \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$ and $\kappa = \eta \circ \iota : \mathcal{L}_\mathcal{H} \longrightarrow \mathcal{H}$ are Lie algebra morphisms, while the surjection $\eta : \mathcal{U}(\mathcal{L}_\mathcal{H}) \longrightarrow \mathcal{H}$ is a morphism of associative algebras with unit. Note that different structures are carried over by these morphisms.

[3] As usual, we write $a^0 = a^{\dagger\, 0} = e^0 = I$

Finally, we observe that the enveloping algebra $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ is equipped with a Hopf algebra structure. It is constructed in a standard way by determining the co-product $\Delta : \mathcal{U}(\mathcal{L}_{\mathcal{H}}) \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}}) \otimes \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ on the generators $x = a^\dagger$, a, e setting $\Delta(x) = x \otimes I + I \otimes x$, which further extends to

$$\Delta\left(a^{\dagger\,p} a^q\, e^r\right) = \sum_{i,j,k} \binom{p}{i}\binom{q}{j}\binom{r}{k}\; a^{\dagger\,i} a^j\, e^k \otimes a^{\dagger\,p-i} a^{q-j}\, e^{r-k}\,. \quad (9)$$

Similarly, the antipode $S : \mathcal{U}(\mathcal{L}_{HW}) \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ is given on generators by $S(x) = -x$, and hence from the anti-morpfism property yields

$$S\left(a^{\dagger\,p} a^q\, e^r\right) = (-1)^{p+q+r}\; e^r\, a^q\, a^{\dagger\,p}\,. \quad (10)$$

Finally, the co-unit $\varepsilon : \mathcal{U}(\mathcal{L}_{HW}) \longrightarrow \mathbb{K}$ is defined in the following way

$$\varepsilon\left(a^{\dagger\,p} a^q\, e^r\right) \;=\; \begin{cases} 1 & \text{if} \quad p, q, r = 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \quad (11)$$

A word of warning here: the Heisenberg–Weyl algebra $\mathcal{H}$ can not be endowed with a bi-algebra structure as is sometimes tacitly assumed. It is because properties of the co-unit contradict the relation of Eq. (1), *i.e.* it follows that $\varepsilon(I) = \varepsilon(a\,a^\dagger - a^\dagger a) = \varepsilon(a)\,\varepsilon(a^\dagger) - \varepsilon(a^\dagger)\,\varepsilon(a) = 0$ whilst one should have $\varepsilon(I) = 1$. This brings out the importance of the additional central element $e \neq I$ which saves the day for $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$.

III. ALGEBRA OF DIAGRAMS AND COMPOSITION

In this Section we define the combinatorial class of Heisenberg–Weyl diagrams which is the central object of our study. We shall equip this class with an intuitive notion of *composition*, permitting the construction of an algebra structure, and thus providing a combinatorial model of the algebras $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$.

A. Combinatorial concepts

We start by recalling a few basic notions from graph theory (Diestel, 2005) needed for a precise definition of the Heisenberg–Weyl diagrams, and then provide an intuitive graphical representation of this structure.

From a set-theoretical point of view, a *directed graph* is a collection of *edges* E and *vertices* V with the structure determined by two mappings $h, t : E \longrightarrow V$ prescribing how the *head* and *tail* of an edge are attached to vertices. Here we shall address a slightly more general setting consisting of graphs whose edges may have one of the ends free (but not both), *i.e.* we shall consider finite graphs with *partially defined* mappings h and t such that $dom(h) \cup dom(t) = E$. We shall call a *cycle* in a graph any sequence of edges $e_1, e_2, ..., e_n$ such that $h(e_k) = t(e_{k+1})$ for $k < n$ and $h(e_n) = t(e_1)$. A convenient concept in graph theory concerns the notion of equivalence. Two graphs are said to be *equivalent* if one can be isomorphically transformed into another, *i.e.* both have the same number of vertices and edges and there exist two isomorphisms $\alpha_E : E_1 \longrightarrow E_2$ and $\alpha_V : V_1 \longrightarrow V_2$ faithfully transferring the structure of the graphs in the following sense

$$\begin{tikzcd} E_1 \arrow[r, shift left, "h"] \arrow[r, shift right, "t"'] \arrow[d, "\alpha_E"'] & V_1 \arrow[d, "\alpha_V"] \\ E_2 \arrow[r, shift left, "h"] \arrow[r, shift right, "t"'] & V_2 \end{tikzcd}$$

The advantage of such defined *equivalence classes* is that we can liberate ourselves from specific set-theoretical realizations and think of a graph only in terms of relations between vertices and edges – this is the attitude we shall adopt in the sequel.

In this context we propose the following formal definition:

Definition 1 (Heisenberg–Weyl Diagrams)
A Heisenberg–Weyl diagram Γ is a class of partially defined directed graphs without cycles. It consists of three sorts of lines: the inner ones Γ^0 having both head and tail attached to vertices, the incoming lines Γ^- with free tails, and the outgoing lines Γ^+ with free heads.

A typical *modus operandi* when working with classes is to invoke representatives. Following this practice we shall by default make all statements concerning Heisenberg–Weyl diagrams with reference to its representatives, assuming that they are class invariants, which assumption can be routinely checked in each case.

The formal Definition 1 gives an intuitive picture in graphical form - see the illustration Fig. 1. A diagram can be represented as a set of vertices $\bullet$ connected by lines each carrying an arrow indicating the direction from the tail to the head. Lines having one of the ends not attached to a vertex will be marked with $\vartriangle$ or $\blacktriangle$ at the free head or tail respectively. We conventionally draw all incoming lines at the bottom and the outgoing lines at the top with all arrows heading upwards; this is always possible since the diagrams do not have cycles. This pictures the Heisenberg-Weyl diagram as a sort of process or transformation with vertices playing the role of intermediate steps.

An important characteristic of a diagram is the total number of its lines denoted by $|\Gamma|$. In the next sections we shall further restrict this counting to the inner, the incoming and the outgoing lines, denoting the result by $|\Gamma^0|$, $|\Gamma^-|$ and $|\Gamma^+|$ respectively.

B. Diagram composition

A crucial concept of this paper concerns *composition* of Heisenberg–Weyl diagrams. This has a straightforward

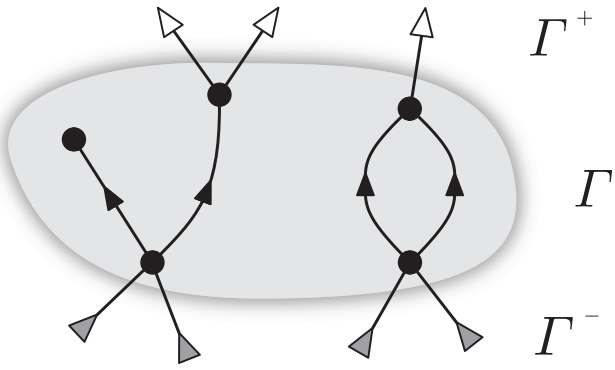

FIG. 1 An example of a Heisenberg–Weyl diagram with three distinguished characteristic sorts of lines: the inner ones $|\Gamma^0| = 4$, the incoming lines $|\Gamma^-| = 4$ and outgoing lines $|\Gamma^+| = 3$.

graphical representation as the attaching of free lines one to another, and shall be based on the notion of a matching.

A *matching* m of two sets A and B is a choice of pairs $(a_i, b_i) \in A \times B$ all having different components, *i.e.* if $a_i = a_j$ or $b_i = b_j$ then $i = j$. We shall denote the collection of all possible matchings by $A {<}\!\!\lhd B$, and its restriction to matchings comprising i pairs only by $A \overset{i}{{<}\!\!\lhd} B$. It is straightforward to check by exact enumeration the formula $|A \overset{i}{{<}\!\!\lhd} B| = \binom{|A|}{i}\binom{|B|}{i}\, i!$ which is valid for any i with the convention $\binom{n}{k} = 0$ for $n < k$.

The concept of diagram composition suggests itself, as:

Definition 2 (Diagram Composition)

Consider two Heisenberg–Weyl diagrams Γ_2 and Γ_1 and a matching $m \in \Gamma_2^- {<}\!\!\lhd \Gamma_1^+$ between the free lines going out from the first one Γ_1^+ and the free lines going into the second one Γ_2^-. The composite diagram, denoted as $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$, is constructed by joining the lines coupled by the matching m.

This descriptive definition can be formalized by referring to representatives in the following way. Given two disjoint graphs, such that $V_{\Gamma_2} \cap V_{\Gamma_1} = \emptyset$ and $E_{\Gamma_2} \cap E_{\Gamma_1} = \emptyset$, we construct the composite graph $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ which consists of vertices $V_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1} = V_{\Gamma_2} \cup V_{\Gamma_1}$ and edges $E_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1} = E_{\Gamma_2} \cup E_{\Gamma_1} \cup m - (pr_2(m) \cup pr_1(m))$. The head and tail functions then unambiguously extend to the set $E_{\Gamma_2} \cup E_{\Gamma_1} - (pr_2(m) \cup pr_1(m))$ and for $e = (e_{\Gamma_2}, e_{\Gamma_1}) \in m$ we define $h_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1}(e) = h_{\Gamma_2}(e_{\Gamma_2})$ and $t_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1}(e) = t_{\Gamma_1}(e_{\Gamma_1})$. Clearly, choice of the disjoint graphs in a class is always possible and the resulting directed graph does not contain cycles. It remains to check that the composition of diagrams so defined, making use of representatives, is class invariant.

Definition 2 can be straightforwardly seen as if diagrams were put one over another with some of the lines going out from the lower one plugged into some of the lines going into to the upper one in accordance with a given matching $m \in \Gamma_2^- {<}\!\!\lhd \Gamma_1^+$, for illustration see Fig. 2. Observe that in general two graphs can be composed in many ways, *i.e.* as many as there are possible matchings (elements in $\Gamma_2^- {<}\!\!\lhd \Gamma_1^+$). In Section III.C we shall exploit all these possible compositions to endow the diagrams with the structure of an algebra. Note also that the above construction depends on the order in which diagrams are composed and the reverse order yields different results.

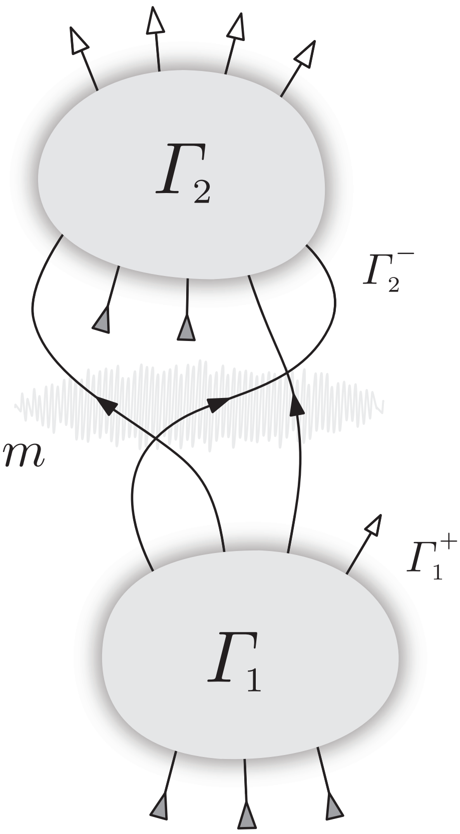

FIG. 2 Composition of two diagrams $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ according to the matching $m \in \Gamma_2^- {<}\!\!\lhd \Gamma_1^+$ consisting of three connections.

We conclude by two simple remarks concerning the composition of two diagrams Γ_2 and Γ_1 constructed by joining exactly i lines. Firstly, we observe that possible compositions can be enumerated explicitly by the formula

$$|\Gamma_2^- \overset{i}{{<}\!\!\lhd} \Gamma_1^+| = \binom{|\Gamma_2^-|}{i}\binom{|\Gamma_2^+|}{i}\, i!\ . \qquad (12)$$

Secondly, the number of incoming, outgoing and inner lines in the composed diagram does not depend on the choice of a matching $m \in \Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+$ and reads respectively

$$
\begin{aligned}
|(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1)^+| &= |\Gamma_2^+| + |\Gamma_1^+| - i \ , \\
|(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1)^-| &= |\Gamma_2^-| + |\Gamma_1^-| - i \ , \\
|(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1)^0| &= |\Gamma_2^0| + |\Gamma_1^0| + i \ . \qquad (13)
\end{aligned}
$$

C. Algebra of Heisenberg–Weyl Diagrams

Here we show that the Heisenberg–Weyl diagrams come equipped with a natural algebraic structure based on diagram composition. It will appear to be a combinatorial refinement of the familiar algebras $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$.

An algebra requires two operations, addition and multiplication, which we construct in the following way. We define $\mathcal{G}$ as a vector space over $\mathbb{K}$ generated by the basis set consisting of all Heisenberg–Weyl diagrams, *i.e.*

$$
\mathcal{G} = \Big\{ \sum\nolimits_i \alpha_i \ \Gamma_i : \ \alpha_i \in \mathbb{K}, \ \Gamma_i - {\scriptstyle \text{Heisenberg–Weyl} \atop \text{diagram}} \Big\} . \quad (14)
$$

Addition in $\mathcal{G}$ has the usual form

$$
\sum\nolimits_i \alpha_i \ \Gamma_i + \sum\nolimits_i \beta_i \ \Gamma_i = \sum\nolimits_i \ (\alpha_i + \beta_i) \ \Gamma_i. \quad (15)
$$

The nontrivial part in the definition of the algebra $\mathcal{G}$ concerns multiplication, which by bilinearity

$$
\sum\nolimits_i \alpha_i \ \Gamma_i * \sum\nolimits_j \beta_j \ \Gamma_j = \sum\nolimits_{i,j} \alpha_i \beta_j \ \Gamma_i * \Gamma_j, \quad (16)
$$

reduces to determining it on the basis set of the Heisenberg–Weyl diagrams. Recalling the notions of Section III.B, we define the product of two diagrams Γ_2 and Γ_1 as the sum of all possible compositions, *i.e.*

$$
\Gamma_2 * \Gamma_1 = \sum_{m \in \Gamma_2^- \lessdot \Gamma_1^+} \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1 \ . \quad (17)
$$

Note that all terms in the sum are distinct and have coefficients equal to one. The multiplication thus defined is noncommutative and possesses a unit element which is the void graph Ø (no vertices, no lines). Moreover, the following theorem holds (for the proof of associativity see Appendix A):

Theorem 1 (Algebra of Diagrams)

Heisenberg–Weyl diagrams form a (noncommutative) associative algebra with unit $(\mathcal{G}, +, *, \emptyset)$.

Our objective, now, is to clarify the relation of the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ to the physically relevant algebras $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ and $\mathcal{H}$. We shall construct forgetful mappings which give a simple combinatorial prescription of how to descend from $\mathcal{G}$ to the two latter structures.

We define a linear mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ on the basis elements by

$$
\varphi(\Gamma) = a^{\dagger \, |\Gamma^+|} \ a^{\,|\Gamma^-|} \ e^{\,|\Gamma^0|} \ . \quad (18)
$$

This prescription can be intuitively understood by looking at the diagrams as if they were carrying auxiliary labels $a^\dagger$, a and e attached to all the outgoing, incoming and inner lines respectively. Then the mapping of Eq. (18) just neglects the structure of the graph and only pays attention to the number of lines, *i.e.* counting them according to the labels. Clearly, φ is onto and it can be proved to be a genuine algebra morphism, *i.e.* it preserves addition and multiplication in $\mathcal{G}$ (for the proof see Appendix B).

Similarly, we define the morphism $\bar{\varphi} : \mathcal{G} \longrightarrow \mathcal{H}$ as

$$
\bar{\varphi}(\Gamma) = (a^\dagger)^{|\Gamma^+|} \ a^{\,|\Gamma^-|} \ , \quad (19)
$$

which differs from φ by ignoring all inner lines in the diagrams. It can be expressed as $\bar{\varphi} = \varphi \circ \eta$ and hence satisfies all the properties of an algebra morphism.

We recapitulate the above discussion in the following theorem:

Theorem 2 (Forgetful mapping)

The mappings $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{\mathcal{H}})$ *and* $\bar{\varphi} : \mathcal{G} \longrightarrow \mathcal{H}$ *defined in Eqs. (18) and (19) are surjective algebra morphisms, and the following diagram commutes*

$$
\begin{tikzcd}
 & \mathcal{G} \arrow[dl, "\varphi"'] \arrow[dr, "\bar{\varphi}"] & \\
\mathcal{U}(\mathcal{L}_{\mathcal{H}}) \arrow[rr, "\eta"] & & \mathcal{H}
\end{tikzcd} \quad (20)
$$

Therefore, the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ is a lifting of the algebras $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ and $\mathcal{H}$, and the latter two can be recovered by applying appropriate forgetful mappings φ and $\bar{\varphi}$. As such, the algebra $\mathcal{G}$ can be seen as a fine graining of the abstract algebras $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$ and $\mathcal{H}$. Thus these latter algebras gain a concrete combinatorial interpretation in terms the richer structure of diagrams.

IV. DIAGRAM DECOMPOSITION AND BI-ALGEBRA

We have seen in Section III how the notion of composition allows for a combinatorial definition of diagram multiplication, opening the door to the realm of algebra. Here, we shall consider the opposite concept of diagram *decomposition* which induces a combinatorial co-product in the algebra endowing Heisenberg–Weyl diagrams with a bi-algebra structure.

A. Basic concepts: Combinatorial decomposition

Suppose we are given a class of objects which allow for decomposition, *i.e.* split into ordered pairs of pieces from the same class. Without loss of generality one may think of the class of Heisenberg–Weyl diagrams and some, for the moment unspecified, procedure assigning to a given diagram Γ its possible decompositions (Γ'', Γ'). In general there might be various ways of splitting an object according to a given rule and, moreover, some of them may yield the same result. We shall denote the collection of all possibilities by $\langle\Gamma\rangle = \{(\Gamma'',\Gamma')\}$ and for brevity write

$$\Gamma \longrightarrow (\Gamma'',\Gamma') \in \langle\Gamma\rangle\ . \tag{21}$$

Note that in a strict sense $\langle\Gamma\rangle$ is a multiset, *i.e.* it is like a set but with arbitrary repetitions of elements allowed. Hence, in order not to overlook any of the decompositions, some of which may be the same, we should use a more appropriate notation employing the notion of a disjoint union, denoted by $\biguplus$, and write

$$\langle\Gamma\rangle = \biguplus_{\substack{\text{decompositions}\\ \Gamma\to(\Gamma'',\Gamma')}} \{(\Gamma'',\Gamma')\}\ . \tag{22}$$

The concept of decomposition is quite general at this point and its further development obviously depends on the choice of the rule. One usually supplements this construction with additional constraints. Below we discuss some natural conditions one might expect from a decomposition rule.

(0) Finiteness. It is reasonable to assume that an object decomposes in a finite number of ways, *i.e.* for each Γ the multiset $\langle\Gamma\rangle$ is finite.

(1) Triple decomposition. Decomposition into pairs naturally extends to splitting an object into three pieces $\Gamma \longrightarrow (\Gamma_3,\Gamma_2,\Gamma_1)$. An obvious way to carry out the multiple splitting is by applying the same procedure repeatedly, *i.e.* decomposing one of the components obtained in the preceding step. However, following this prescription one usually expects that the result does not depend on the choice of the component it is applied to. In other words, we require that we end up with the same collection of triple decompositions when splitting $\Gamma \longrightarrow (\Gamma'',\Gamma_1)$ and then splitting the left component $\Gamma'' \longrightarrow (\Gamma_3,\Gamma_2)$, *i.e.*

$$\Gamma \longrightarrow (\Gamma'',\Gamma_1) \xrightarrow{\Gamma''\longrightarrow(\Gamma_3,\Gamma_2)} (\Gamma_3,\Gamma_2,\Gamma_1)\ , \tag{23}$$

as in the case when starting with $\Gamma \longrightarrow (\Gamma_3,\Gamma')$ and then splitting the right component $\Gamma' \longrightarrow (\Gamma_2,\Gamma_1)$, *i.e.*

$$\Gamma \longrightarrow (\Gamma_3,\Gamma') \xrightarrow{\Gamma'\longrightarrow(\Gamma_2,\Gamma_1)} (\Gamma_3,\Gamma_2,\Gamma_1)\ . \tag{24}$$

This condition can be seen as the co-associativity property for decomposition, and in explicit form boils down to the following equality:

$$\biguplus_{\substack{(\Gamma'',\Gamma_1)\in\langle\Gamma\rangle\\ (\Gamma_3,\Gamma_2)\in\langle\Gamma''\rangle}} \{(\Gamma_3,\Gamma_2,\Gamma_1)\} = \biguplus_{\substack{(\Gamma_3,\Gamma')\in\langle\Gamma\rangle\\ (\Gamma_2,\Gamma_1)\in\langle\Gamma'\rangle}} \{(\Gamma_3,\Gamma_2,\Gamma_1)\}\ . \tag{25}$$

The above procedure straightforwardly extends to splitting into multiple pieces $\Gamma \longrightarrow (\Gamma_n,...\Gamma_1)$. Clearly, the condition of Eq. (25) entails analogous property for multiple decompositions.

(2) Void structure. Often, in a class there exists a sort of a void (or empty) element $\emptyset$, such that objects decompose in a trivial way. It should have the the property that any object $\Gamma \neq \emptyset$ splits into a pair containing either $\emptyset$ or Γ in two ways only:

$$\Gamma \longrightarrow (\emptyset,\Gamma) \quad \text{and} \quad \Gamma \longrightarrow (\Gamma,\emptyset)\ , \tag{26}$$

and $\emptyset \longrightarrow (\emptyset,\emptyset)$. Note that $\emptyset$ is unique.

(3) Symmetry of decomposition. For some rules the order between components in decompositions is immaterial, *i.e.* the rule allows for an exchange $(\Gamma',\Gamma'') \longleftrightarrow (\Gamma'',\Gamma')$. In this case the following symmetry condition holds

$$(\Gamma',\Gamma'') \in \langle\Gamma\rangle \iff (\Gamma'',\Gamma') \in \langle\Gamma\rangle\ . \tag{27}$$

(4) Composition–decomposition compatibility. Suppose that in addition to decomposition we also have a well-defined notion of composition of objects in the class. Let the multiset comprising all possible compositions of Γ_2 with Γ_1 be denoted by $\Gamma_2 \blacktriangleleft \Gamma_1$, *e.g.* for the Heisenberg–Weyl diagrams we have

$$\Gamma_2 \blacktriangleleft \Gamma_1 = \biguplus_{m\in\Gamma_2^-\lessdot\Gamma_1^+} \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1\ . \tag{28}$$

Now, given a pair of objects Γ_2 and Γ_1, we may think of two consistent decomposition schemes which involve composition. We can either start by composing them together $\Gamma_2 \blacktriangleleft \Gamma_1$ and then splitting all resulting objects into pieces, or first decompose each of them separately into $\langle\Gamma_2\rangle$ and $\langle\Gamma_1\rangle$ and then compose elements of both sets in a component-wise manner. One may reasonably expect the same outcome no matter which way the procedure goes. Hence, a formal description of compatibility comes down to the equality:

$$\biguplus_{\Gamma\in\Gamma_2\blacktriangleleft\Gamma_1} \langle\Gamma\rangle = \biguplus_{\substack{(\Gamma_2'',\Gamma_2')\in\langle\Gamma_2\rangle\\ (\Gamma_1'',\Gamma_1')\in\langle\Gamma_1\rangle}} (\Gamma_2'' \blacktriangleleft \Gamma_1'') \times (\Gamma_2' \blacktriangleleft \Gamma_1')\ . \tag{29}$$

We remark that this property implies that the void element $\emptyset$of condition *(2)* is the same as the neutral element for composition.

(5) Finiteness of multiple decmpositions. Recall the process of multiple decompositions $\Gamma \longrightarrow (\Gamma_n,...\Gamma_1)$ constructed in the condition *(1)* and observe that one may go with the number of components to any $n \in \mathbb{N}$. However, if one considers only nontrivial decompositions which do not contain void $\emptyset$ components it is often the case that the process terminates after a finite number of steps. In other words, for each Γ there exists $N \in \mathbb{N}$ such that

$$\{\Gamma \longrightarrow (\Gamma_n,...\Gamma_1) : \Gamma_n,...,\Gamma_1 \neq \emptyset\} = \emptyset \tag{30}$$

for $n > N$. In practice, objects usually carry various characteristics counted by natural numbers, *e.g.* the number of elements they are build of. Then, if the decomposition rule decreases such a characteristic in each of the components in a nontrivial splitting, it inevitably uses up and the condition of Eq. (30) is automatically fulfilled.

Having discussed the above quite general conditions expected from a reasonable decomposition rule we are now in a position to return to the realm of algebra. Already in Section III.C we have seen how the notion of composition induces multiplication which endows the class of Heisenberg–Weyl diagrams with the structure of an algebra, see Theorem 1. Following this route we shall employ the concept of decomposition to introduce the structure of a Hopf algebra in $\mathcal{G}$. A central role in the construction will play three mappings given below.

Let us consider a linear mapping $\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ defined on the basis elements as

$$\Delta(\mathit{\Gamma}) = \sum_{(\mathit{\Gamma}', \mathit{\Gamma}'') \in \langle \mathit{\Gamma} \rangle} \mathit{\Gamma}' \otimes \mathit{\Gamma}'' \ . \tag{31}$$

Note, that although all coefficients in Eq. (31) are equal to one, some terms in the sum may appear several times. This is because elements in the multiset $\langle \mathit{\Gamma} \rangle$ may repeat and the numbers counting their multiplicities are sometimes called section coefficients (Joni and Rota, 1979). Observe that the sum is well defined as long the number of decompositions is finite, *i.e.* condition *(0)* obtains.

We shall also make use of a linear mapping $\varepsilon : \mathcal{G} \longrightarrow \mathbb{K}$ which picks out the void element $\emptyset$. It shall be defined in a canonical way

$$\varepsilon(\mathit{\Gamma}) = \begin{cases} 1 & \text{if} \quad \mathit{\Gamma} = \emptyset \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{32}$$

i.e. simply extracting the expansion coefficient standing at the void $\emptyset$.

Finally, we shall need a linear mapping $S : \mathcal{G} \longrightarrow \mathcal{G}$ defined by the formula

$$S(\mathit{\Gamma}) = \sum_{\substack{\mathit{\Gamma} \to (\mathit{\Gamma}_n, \dots, \mathit{\Gamma}_1) \\ \mathit{\Gamma}_n, \dots, \mathit{\Gamma}_1 \neq \emptyset}} (-1)^n \ \mathit{\Gamma}_n * \dots * \mathit{\Gamma}_1 \ , \tag{33}$$

for $\mathit{\Gamma} \neq \emptyset$ and $S(\emptyset) = \emptyset$. Note that it is an alternating sum over products of nontrivial multiple decompositions of an object. Clearly, if the condition *(5)* holds the sum is finite and S is well defined.

The mappings Δ, ε and S, built upon a reasonable decomposition procedure, provide $\mathcal{G}$ with a rich algebraic structure as summarized in the following lemma (for the proofs see Appendix C):

Lemma 1 (Decomposition and Bi-algebra)

(i) *If the conditions (0), (1) and (2) are satisfied, the mappings Δ and ε defined in Eqs. (31) and (32) are the co-product and co-unit in the algebra $\mathcal{G}$. The co-algebra $(\mathcal{G}, \Delta, \varepsilon)$ thus defined is co-commutative, provided condition (3) is fulfilled.*

(ii) *In addition, if condition (4) holds we have a genuine bi-algebra structure $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon)$.*

(iii) *Finally, under condition (5) we establish a Hopf algebra structure $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon, S)$ with the antipode S defined in Eq. (33).*

We remark that the above discussion is applicable to a wide range of combinatorial classes and decomposition rules which we have thus far left unspecified. Below we shall apply these concepts to the class of Heisenberg–Weyl diagrams.

B. Hopf algebra of Heisenberg–Weyl diagrams

In this Section we provide an explicit decomposition rule for the Heisenberg–Weyl diagrams satisfying all the conditions discussed in Section IV.A. In this way we complete the whole picture by introducing a Hopf algebra structure on $\mathcal{G}$.

We start by observing that for a given Heisenberg–Weyl graph $\mathit{\Gamma}$, each subset of its edges $L \subset E_{\mathit{\Gamma}}$ induces a subgraph $\mathit{\Gamma}|_L$ which is defined by restriction of the head and tail functions to the subset L. Likewise, the remaining part of the edges $R = E_{\mathit{\Gamma}} - L$ gives rise to a subgraph $\mathit{\Gamma}|_R$. Clearly, the results are again Heisenberg–Weyl graphs. Thus, by considering ordered partitions of the set of edges into two subsets $L + R = E_{\mathit{\Gamma}}$, *i.e.* $L \cup R = E_{\mathit{\Gamma}}$ and $L \cap R = \emptyset$, we end up with pairs of disjoint graphs $(\mathit{\Gamma}|_L \, , \mathit{\Gamma}|_R)$. This suggests the following definition:

Definition 3 (Diagram Decomposition)
We shall consider a decomposition of a Heisenberg–Weyl diagram $\mathit{\Gamma}$ to be any splitting $(\mathit{\Gamma}_L, \mathit{\Gamma}_R)$ induced by an ordered partition of its lines $L + R = E_{\mathit{\Gamma}}$. Hence, the multiset $\langle \mathit{\Gamma} \rangle$ comprising all possible decompositions can be indexed by the set of ordered double partitions $\{(L, R) : \ L + R = E_{\mathit{\Gamma}}\}$, and we have

$$\langle \mathit{\Gamma} \rangle = \biguplus_{L+R=E_{\mathit{\Gamma}}} \{(\mathit{\Gamma}|_L \, , \mathit{\Gamma}|_R)\} \ . \tag{34}$$

The graphical picture is clear: the decomposition of a diagram $\mathit{\Gamma} \longrightarrow (\mathit{\Gamma}|_L \, , \mathit{\Gamma}|_R)$ is defined by the choice of lines $L \subset E_{\mathit{\Gamma}}$, which taken out make up the first component of the pair whilst the reminder induced by $R = E_{\mathit{\Gamma}} - L$ constitutes the second one. (See the illustration in Fig. 3.)

The enumeration of all decompositions of a diagram $\mathit{\Gamma}$ is straightforward since the multiset $\langle \mathit{\Gamma} \rangle$ can be indexed by subsets of $E_{\mathit{\Gamma}}$. Because $|E_{\mathit{\Gamma}}| = |\mathit{\Gamma}|$, explicit counting gives $|\langle \mathit{\Gamma} \rangle| = \sum_i \binom{|\mathit{\Gamma}|}{i} = 2^{|\mathit{\Gamma}|}$. This simple observation can be generalized to calculate the number of decompositions $(\mathit{\Gamma}|_L \, , \mathit{\Gamma}|_R) \in \langle \mathit{\Gamma} \rangle$ in which the first component has i outgoing, j incoming and k inner lines, *i.e.* $|\, \mathit{\Gamma}|_L^+ \,| = i, |\, \mathit{\Gamma}|_L^- \,| = j, |\, \mathit{\Gamma}|_L^0 \,| = k$. Accordingly, the enumeration boils down to the choice of i, j and k lines out

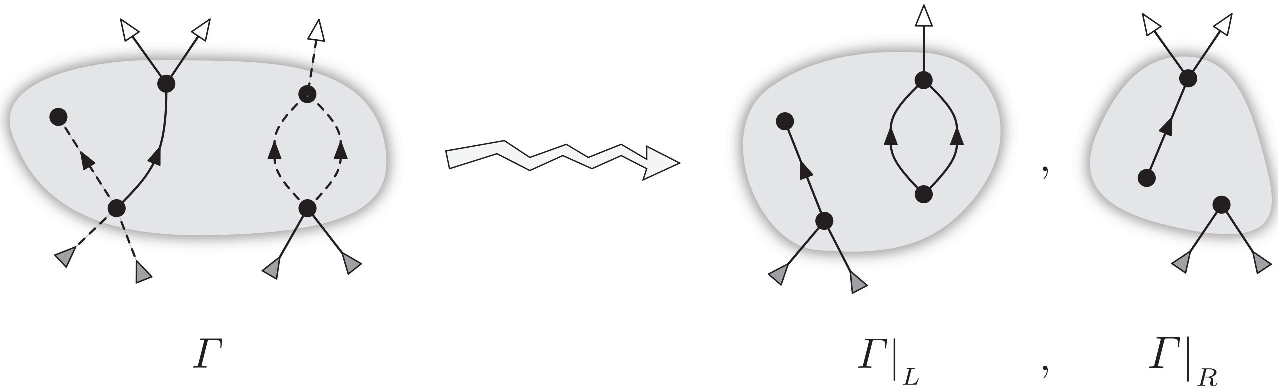

FIG. 3 An example of diagram decomposition $\Gamma \longrightarrow (\Gamma|_L\,, \Gamma|_R)$. The choice of edges $L \subset E_\Gamma$ inducing the diagram $\Gamma|_L$ is depicted on the left diagram as dashed lines.

of the sets Γ^+, Γ^- and Γ^0 respectively, which gives

$$\left|\left\{ (\Gamma|_L\,, \Gamma|_R) \in \langle\Gamma\rangle : \begin{smallmatrix} |\,\Gamma|_L^+| = i \\ |\,\Gamma|_L^-| = j \\ |\,\Gamma|_L^0| = k \end{smallmatrix} \right\}\right| = \binom{|\Gamma^+|}{i}\binom{|\Gamma^-|}{j}\binom{|\Gamma^0|}{k}. \tag{35}$$

Observe that the second component $\Gamma|_R$ is always determined by the first one $\Gamma|_L$ and hence the number of its outgoing, incoming and inner lines is given by

$$\begin{aligned} |\,\Gamma|_R^+| &= |\Gamma^+| - i\,, \\ |\,\Gamma|_R^-| &= |\Gamma^-| - j\,, \\ |\,\Gamma|_R^0| &= |\Gamma^0| - k\,. \end{aligned} \tag{36}$$

Having explicitly defined the notion of diagram decomposition, one may check that it satisfies conditions *(1) - (5)* of Section IV.A; for the proofs see Appendix D. In this context Eq. (31) defining the co-product in the algebra $\mathcal{G}$ takes the form

$$\Delta(\Gamma) = \sum_{L+R=E_\Gamma} \Gamma|_L \otimes \Gamma|_R\,, \tag{37}$$

and the antipode of Eq. (33) rewrites as

$$S(\Gamma) = \sum_{\substack{A_n+\ldots+A_1=E_\Gamma \\ A_n,\ldots,A_1\neq\emptyset}} (-1)^n\;\; \Gamma|_{A_n} * \ldots * \Gamma|_{A_1}\,. \tag{38}$$

for $\Gamma \neq \emptyset$ and $S(\emptyset) = \emptyset$. Therefore, referring to Lemma 1 we supplement Theorem 1 by the following result

Theorem 3 (Hopf algebra of Diagrams)
The algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ has a Hopf algebra structure $(\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon, S)$ with (co-commutative) co-product, co-unit and antipode defined in Eqs. (37), (32) and (38) respectively.

The algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ was shown to be directly related to the algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$ through the forgetful mapping φ which preserves algebraic operations as explained in Theorem 2. Here, however, in the context of Theorem 3 the algebra $\mathcal{G}$ is additionally equipped with a co-product, co-unit and antipode. Since $\mathcal{U}(\mathcal{L}_\mathcal{H})$ is also a Hopf algebra, it is natural to ask whether this extra structure is preserved by the morphism φ of Eq. (18). It turns up that indeed it is also preserved, and we augment Theorem 2 by the following proposition (for the proof see Appendix B):

Theorem 4 (Hopf algebra morphism φ)
The forgetful mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$ defined in Eq. (18) is a Hopf algebra morphism.

In this way, we have extended the results of Section III to encompass the Hopf algebra structure of the enveloping algebra $\mathcal{U}(\mathcal{L}_\mathcal{H})$. This completes the picture of the algebra of Heisenberg–Weyl diagrams $\mathcal{G}$ as a combinatorial model which captures all the relevant properties of the algebras $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_\mathcal{H})$.

V. CONCLUSIONS

(Sweedler, 1969) (Cartier, 2007)
****

Conclusions: TO DO ...
****

Possible Journals: *Rev. Mod. Phys., Phys. Rev. A, J. Phys. A, Phys. Lett. A, ...*
Possible referees: *Severini, Louck, Vourdas, Katriel, Burdik, Foata, Coecke, ...*
****

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Appendix A: Associativity of multiplication in $\mathcal{G}$

We shall prove associativity of the multiplication defined in Eq. (17). From bilinearity we only need to check it for the basis elements, *i.e.*

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) = (\Gamma_3 * \Gamma_2) * \Gamma_1 \ . \tag{A1}$$

Written explicitly, the left and right hand sides of this equation take the form

$$\Gamma_3 * (\Gamma_2 * \Gamma_1) = \sum_{m'} \sum_{m_{21}} \Gamma_3 \overset{m'}{\blacktriangleleft} (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1) \tag{A2}$$

where $m' \in \Gamma_3^- \lessdot (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1)^+$ and $m_{21} \in \Gamma_2^- \lessdot \Gamma_1^+$, whilst

$$(\Gamma_3 * \Gamma_2) * \Gamma_1 = \sum_{m_{32}} \sum_{m''} (\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2) \overset{m''}{\blacktriangleleft} \Gamma_1 \tag{A3}$$

where $m_{32} \in \Gamma_3^- \lessdot \Gamma_2^+$ and $m'' \in (\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2)^- \lessdot \Gamma_1^+$.

Consider the double sums in the above equations, indexed by (m', m_{21}) and (m_{32}, m'') respectively, and observe that there exists a one-to-one correspondence between their elements. We construct it by a fine graining of the matchings, see Fig. 4, and define the following two mappings. The first one is

$$(m', m_{21}) \longrightarrow (m_{32}, m'') \ , \tag{A4}$$

where $m_{32} = m' \cap (\Gamma_3^- \times \Gamma_2^+)$ and $m'' = m_{21} \cup (m' \cap (\Gamma_3^- \times \Gamma_1^+))$, and similarly the second one

$$(m_{32}, m'') \longrightarrow (m', m_{21}) \ , \tag{A5}$$

with $m' = m_{32} \cup (m'' \cap (\Gamma_3^- \times \Gamma_1^+))$ and $m_{21} = m'' \cap (\Gamma_2^- \times \Gamma_1^+)$. Clearly, the mappings are inverses of each other, which assures a one-to-one correspondence between elements of the double sums in Eqs. (A2) and (A3). Moreover, the summands that are mapped one onto another are equal, *i.e.* the corresponding diagrams $\Gamma_3 \overset{m'}{\blacktriangleleft} (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1)$ and $(\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2) \overset{m''}{\blacktriangleleft} \Gamma_1$ are exactly the same. This ends the proof by showing equality of the right hand sides of Eqs. (A2) and (A3).

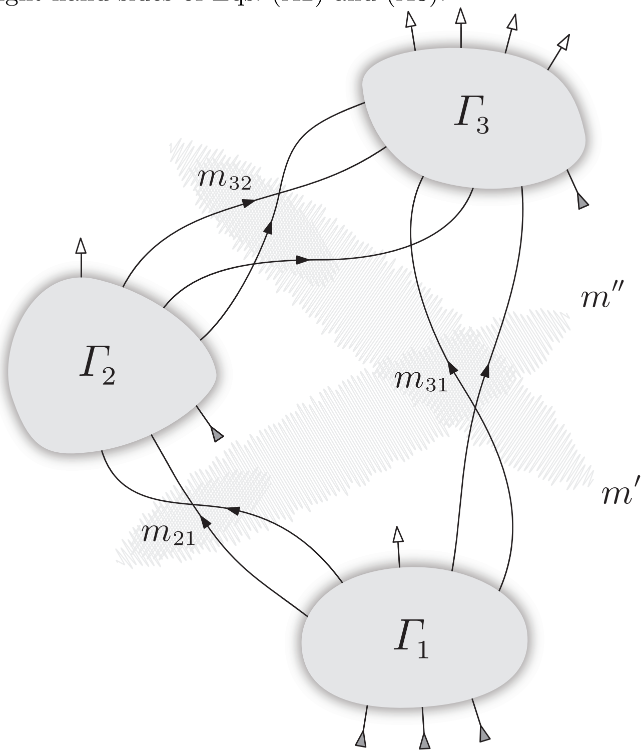

FIG. 4 Fine graining of the matchings $m' \in \Gamma_3^- \lessdot (\Gamma_2 \overset{m_{21}}{\blacktriangleleft} \Gamma_1)^+$ and $m'' \in (\Gamma_3 \overset{m_{32}}{\blacktriangleleft} \Gamma_2)^- \lessdot \Gamma_1^+$ used in the proof of associativity of multiplication.

Appendix B: Forgetful morphism φ

In Theorems 2 and 4 we have declared that the linear mapping $\varphi : \mathcal{G} \longrightarrow \mathcal{U}(\mathcal{L}_{HW})$ defined in Eq. (18) is a Hopf algebra morphism. Now, we prove this statement.

We start by showing that φ preserves multiplication in $\mathcal{G}$. From linearity it is enough to check for the basis elements that $\varphi(\Gamma_2 * \Gamma_1) = \varphi(\Gamma_2)\, \varphi(\Gamma_1)$, which is verifieded in the following sequence of equalities:

$$
\begin{aligned}
\varphi(\Gamma_2 * \Gamma_1) &\overset{(17)}{=} \sum_{m\in\Gamma_2^- \lessdot \Gamma_1^+} \varphi(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1) = \sum_i \sum_{m\in\Gamma_2 \overset{i}{\lessdot} \Gamma_1} \varphi(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1) && \text{(B1)}\\
&\overset{(13)}{=} \sum_i \sum_{m\in\Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+} (a^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\, a^{|\Gamma_2^-|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i}\\
&= \sum_i (a^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\, a^{|\Gamma_2^-|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i} \sum_{m\in\Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+} 1 && \text{(B2)}\\
&\overset{(12)}{=} \sum_i \binom{|\Gamma_2^-|}{i}\binom{|\Gamma_1^+|}{i}\, i!\, (a^\dagger)^{|\Gamma_2^+|+|\Gamma_1^+|-i}\, a^{|\Gamma_2^-|+|\Gamma_1^-|-i}\, e^{|\Gamma_2^0|+|\Gamma_1^0|+i}\\
&\overset{(7)}{=} \left((a^\dagger)^{|\Gamma_2^+|}\, a^{|\Gamma_2^-|}\, e^{|\Gamma_2^0|}\right)\left((a^\dagger)^{|\Gamma_1^+|}\, a^{|\Gamma_1^-|}\, e^{|\Gamma_1^0|}\right) = \varphi(\Gamma_2)\,\varphi(\Gamma_1)\ .
\end{aligned}
$$

In the above derivation the main trick in Eq. (B1) consists in splitting the set of diagram matchings into disjoint subsets according to the number of connected lines, *i.e.* $\Gamma_2^- \lessdot \Gamma_1^+ = \bigcup_i \Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+$. Then upon observation that the summands in Eq. (B2) do not depend on $m \in \Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+$ we may execute explicitly one of the sums counting elements in $\Gamma_2^- \overset{i}{\lessdot} \Gamma_1^+$ with the help of Eq. (12).

We also need to show that the co-product, co-unit and antipode are preserved by φ. This means that when proceeding via mapping φ from $\mathcal{G}$ to $\mathcal{U}(\mathcal{L}_{HW})$ one can use co-product, co-unit and antipode in either of the algebras and obtain the same result *i.e.*

$$
\begin{aligned}
(\varphi\otimes\varphi)\circ\Delta &= \Delta\circ\varphi\ , && \text{(B3)}\\
\varepsilon &= \varepsilon\circ\varphi\ , && \text{(B4)}\\
\varphi\circ S &= S\circ\varphi\ , && \text{(B5)}
\end{aligned}
$$

where Δ, ε and S on the left-hand-sides act in $\mathcal{G}$ whilst on the right-hand-sides in $\mathcal{U}(\mathcal{L}_{HW})$. The proof of Eq. (B3) rests upon the counting formula in Eq. (35) and the observation of Eq. (36), which justify the following equalities

$$
\begin{aligned}
(\varphi\otimes\varphi)\circ\Delta\,(\Gamma) &= \sum_{L+R=E_\Gamma} \varphi\,(\Gamma|_L)\otimes\varphi\,(\Gamma|_R) = \sum_{L\subset E_\Gamma} \varphi\,(\Gamma|_L)\otimes\varphi\left(\Gamma|_{E_\Gamma - L}\right)\\
&\overset{(35),(36)}{=} \sum_{i,j,k} \binom{|\Gamma^+|}{i}\binom{|\Gamma^-|}{j}\binom{|\Gamma^0|}{k}\, a^{\dagger\, i}\, a^j\, e^k \otimes a^{\dagger\,|\Gamma^+|-i}\, a^{|\Gamma^-|-j}\, e^{|\Gamma^0|-k} \overset{(9)}{=} \Delta\circ\varphi\,(\Gamma)\ .
\end{aligned}
$$

Eq. (B4) is readily checked by comparing Eqs. (11) and (32).

....

ϕ preserves S: TO BE COMPLETED

......

Appendix C: From decomposition to Hopf algebra

In order to prove Lemma 1 we should check in part *(i)* co-associativity of the co-product Δ and properties of the co-unit ε, in part *(ii)* show that the mappings Δ and ε preserve multiplication in $\mathcal{G}$ whilst for part *(iii)* verify the defining properties of the antipode S.

(i) Co-algebra

The co-product $\Delta : \mathcal{G} \longrightarrow \mathcal{G}\otimes\mathcal{G}$ is co-associative if the following equality holds

$$
(\Delta\otimes Id)\circ\Delta = (Id\otimes\Delta)\circ\Delta\ . \qquad \text{(C1)}
$$

Since Δ defined in Eq. (31) is linear it is enough to check it for the basis elements Γ. Accordingly, the left-hand side takes the form

$$
\begin{aligned}
(\Delta\otimes Id)\circ\Delta\,(\Gamma) &= (Id\otimes\Delta) \sum_{(\Gamma_1,\Gamma'')\in\langle\Gamma\rangle} \Gamma_1\otimes\Gamma''\\
&= \sum_{\substack{(\Gamma_1,\Gamma'')\in\langle\Gamma\rangle\\ (\Gamma_2,\Gamma_3)\in\langle\Gamma''\rangle}} \Gamma_1\otimes\Gamma_2\otimes\Gamma_3\ , && \text{(C2)}
\end{aligned}
$$

whereas the right-hand-side reads

$$(Id\otimes\Delta)\circ\Delta\,(\Gamma) \;=\; (\Delta\otimes Id)\sum_{(\Gamma',\Gamma_3)\in\langle\Gamma\rangle}\Gamma'\otimes\Gamma_3$$
$$=\;\sum_{\substack{(\Gamma',\Gamma_3)\in\langle\Gamma\rangle\\(\Gamma_1,\Gamma_2)\in\langle\Gamma'\rangle}}\Gamma_1\otimes\Gamma_2\otimes\Gamma_3\ . \quad \text{(C3)}$$

If condition *(1)* of Section IV.A holds the property of Eq. (25) asserts equality of the right-hand-sides of Eqs. (C2) and (C3) and the co-product defined in Eq. (31) is co-associative.

The co-unit $\varepsilon:\mathcal{G}\longrightarrow\mathbb{K}$ by definition should satisfy the equalities

$$(\varepsilon\otimes Id)\circ\Delta = Id = (Id\otimes\varepsilon)\circ\Delta\ , \quad \text{(C4)}$$

where the identification $\mathbb{K}\otimes\mathcal{G}=\mathcal{G}\otimes\mathbb{K}=\mathcal{G}$ is implied. We shall check the first one for the basis elements Γ by direct calculation

$$(\varepsilon\otimes Id)\circ\Delta\,(\Gamma) \;=\; (\varepsilon\otimes Id)\sum_{(\Gamma_1,\Gamma_2)\in\langle\Gamma\rangle}\Gamma_1\otimes\Gamma_2$$
$$=\;\sum_{(\Gamma_1,\Gamma_2)\in\langle\Gamma\rangle}\varepsilon(\Gamma_1)\otimes\Gamma_2 \quad \text{(C5)}$$
$$=\;1\otimes\Gamma=\Gamma=Id\,(\Gamma)\ .$$

Note that we have applied condition *(2)* of Section IV.A by taking all terms in the sum Eq. (C5) equal to zero except the unique decomposition $(\emptyset,\Gamma)$ picked up by ε as defined in Eq. (32). The identification $1\otimes\Gamma=\Gamma$ completes the proof of the first equality in Eq. (C4); verification of the second one is analogous.

Co-commutativity of the co-product Δ under the condition *(3)* is straightforward since from Eq. (27) we have

$$\Delta(\Gamma)=\sum_{(\Gamma',\Gamma'')\in\langle\Gamma\rangle}\Gamma'\otimes\Gamma''=\sum_{(\Gamma',\Gamma'')\in\langle\Gamma\rangle}\Gamma''\otimes\Gamma'\ .$$

(ii) Bi-algebra

The structure of a bi-algebra results whenever the co-product $\Delta:\mathcal{G}\otimes\mathcal{G}\longrightarrow\mathcal{G}$ and co-unit $\varepsilon:\mathcal{G}\longrightarrow\mathbb{K}$ of the co-algebra are compatible with multiplication in $\mathcal{G}$. Thus, we need to verify for basis elements Γ_1 and Γ_2 that

$$\Delta\,(\Gamma_2*\Gamma_1)\;=\;\Delta\,(\Gamma_2)*\Delta\,(\Gamma_1)\ , \quad \text{(C6)}$$

with component-wise multiplication in the tensor product $\mathcal{G}\otimes\mathcal{G}$ on the right-hand-side, and

$$\varepsilon\,(\Gamma_2*\Gamma_1)\;=\;\varepsilon\,(\Gamma_2)\,\varepsilon\,(\Gamma_1)\ , \quad \text{(C7)}$$

with terms on the right-hand-side multiplied in $\mathbb{K}$.

We check Eq. (C6) directly by expanding both sides using definitions of Eqs. (17) and (31). Accordingly, the left-hand-side takes the form

$$\Delta\,(\Gamma_2*\Gamma_1)\;=\;\sum_{\Gamma\in\Gamma_2\blacktriangleleft\Gamma_1}\Delta\,(\Gamma)$$
$$=\;\sum_{\Gamma\in\Gamma_2\blacktriangleleft\Gamma_1}\ \sum_{(\Gamma'',\Gamma')\in\langle\Gamma\rangle}\Gamma''\otimes\Gamma' \quad \text{(C8)}$$

while the right-hand-side reads

$$\Delta\,(\Gamma_2)*\Delta\,(\Gamma_1)\;=\;\sum_{\substack{(\Gamma_2'',\Gamma_2')\in\langle\Gamma_2\rangle\\(\Gamma_1'',\Gamma_1')\in\langle\Gamma_1\rangle}}\underbrace{(\Gamma_2''\otimes\Gamma_2')*(\Gamma_1''\otimes\Gamma_1')}_{(\Gamma_2''*\Gamma_1'')\otimes(\Gamma_2'*\Gamma_1')}$$
$$=\;\sum_{\substack{(\Gamma_2'',\Gamma_2')\in\langle\Gamma_2\rangle\\(\Gamma_1'',\Gamma_1')\in\langle\Gamma_1\rangle}}\ \sum_{\substack{\Gamma''\in\Gamma_2''\blacktriangleleft\Gamma_1''\\\Gamma'\in\Gamma_2'\blacktriangleleft\Gamma_1'}}\Gamma''\otimes\Gamma' \quad \text{(C9)}$$

A closer look at condition *(4)* and Eq. (29) shows a one-to-one correspondence between terms in the sums on the right-hand-sides of Eqs. (C8) and (C9), verifying the validity of Eq. (C6).

Verification of Eq. (C7) rests upon the simple observation that composition of diagrams $\Gamma_2*\Gamma_1$ yields the void diagram only if both of them are void. Then, both sides are equal to 1 if $\Gamma_1=\Gamma_2=\emptyset$ and 0 otherwise, which confirms Eq. (C7).

(iii) Hopf algebra

A Hopf algebra structure consists of a bi-algebra $(\mathcal{G},+,*,\emptyset,\Delta,\varepsilon)$ equipped with an antipode $S:\mathcal{G}\longrightarrow\mathcal{G}$ which is an endomorphism satisfying the property

$$\mu\circ(Id\otimes S)\circ\Delta=\epsilon=\mu\circ(S\otimes Id)\circ\Delta\ , \quad \text{(C10)}$$

where $\mu:\mathcal{G}\otimes\mathcal{G}\longrightarrow\mathcal{G}$ is the multiplication $\mu(\Gamma_2\otimes\Gamma_1)=\Gamma_2*\Gamma_1$, and the mapping $\epsilon:\mathcal{G}\longrightarrow\mathcal{G}$ defined as $\epsilon=\emptyset\,\varepsilon$ is the projection on the subspace spanned by $\emptyset$, i.e.

$$\epsilon(\Gamma)=\begin{cases}\Gamma & \text{if}\quad \Gamma=\emptyset\ ,\\ 0 & \text{otherwise}\ .\end{cases} \quad \text{(C11)}$$

Here, we shall prove that S given in Eq. (33) satisfies the condition of Eq. (C10). We start by considering an auxiliary linear mapping $\Phi:End(\mathcal{G})\longrightarrow End(\mathcal{G})$ defined as

$$\Phi(f)=\mu\circ(Id\otimes f)\circ\Delta,\qquad f\in End(\mathcal{G}). \quad \text{(C12)}$$

Observe that under the assumption that Φ is invertible the first equality in Eq. (C10) can be rephrased into the condition

$$S=\Phi^{-1}(\epsilon)\ . \quad \text{(C13)}$$

Now, our objective is to show that Φ is invertible and calculate its inverse explicitly. By extracting identity we get $\Phi = Id + \Phi^+$ and observe that such defined Φ^+ can be written in the form

$$\Phi^+(f) = \mu \circ (\bar{\epsilon} \otimes f) \circ \Delta, \qquad f \in End(\mathcal{G}) \ , \tag{C14}$$

where $\bar{\epsilon} = Id - \epsilon$ is the complement of ϵ projecting on the subspace spanned by $\Gamma \neq \emptyset$, *i.e.*

$$\bar{\epsilon}(\Gamma) = \begin{cases} 0 & \text{if} \quad \Gamma = \emptyset \ , \\ \Gamma & \text{otherwise} \ . \end{cases} \tag{C15}$$

We claim that he mapping Φ is invertible with the inverse given by[4]

$$\Phi^{-1} = \sum_{n=0}^{\infty} (-\Phi^+)^n \ . \tag{C16}$$

In order to check that the above sum is well defined we shall analyze the sum term by term. It is not difficult to calculate powers of Φ^+ explicitly

$$\left(\Phi^+\right)^n (f)(\Gamma) = \sum_{\substack{\Gamma \to (\Gamma_n, \ldots, \Gamma_1, \Gamma_0) \\ \Gamma_n, \ldots, \Gamma_1 \neq \emptyset}} \Gamma_n * \ldots * \Gamma_1 * f(\Gamma_0) \, . \tag{C17}$$

We note that in the above formula products of multiple decompositions arise from repeated use of the property of Eq. (C6) and the exclusion of empty components in the decompositions (except the single one on the right hand side) comes from the definition of $\bar{\epsilon}$ in Eq. (C15). The latter constraint together with condition *(5)* asserts that the number of non-vanishing terms in Eq. (C16) is always finite proving that Φ^{-1} is well defined. Finally, using Eqs. (C16) and (C17) one explicitly calculates S from Eq. (C13) obtaining the formula of Eq. (33).

In conclusion, from the construction the linear mapping S of Eq. (33) satisfies the first equality in Eq. (C10); the second equality can be checked analogously. Therefore we have proved S to be an antipode thus making $\mathcal{G}$ into a Hopf algebra. We remark that, by a general theory of Hopf algebras (Sweedler, 1969), the property of Eq. (C10) implies that: S is unique, it is an antimorphism and if G is co-commutative it is an involution $S \circ S = Id$.

Appendix D: Properties of diagram decomposition

We shall verify that the decomposition of Definition 3 satisfies conditions *(0)* - *(5)* of Section IV.A.

Condition *(0)* follows directly from the construction, as we consider finite diagrams only.

[4] For a linear mapping $L = Id + L^+ : V \longrightarrow V$ its inverse can be constructed as $L^{-1} = \sum_{n=0}^{\infty}(-L^+)^n$ provided the sum is well defined. Indeed, one readily checks that $L \circ L^{-1} = (Id + L^+) \circ \sum_{n=0}^{\infty}(-L^+)^n = \sum_{n=0}^{\infty}(-L^+)^n + \sum_{n=0}^{\infty}(-L^+)^{n+1} = Id$ and similarly $L^{-1} \circ L = Id$.

Proof of condition *(1)* consists in providing a one-to-one correspondence between schemes (23) and (24) decomposing a diagram Γ into triples. Accordingly, one easily checks (see illustration Fig. 5) that each triple $(\Gamma|_L\,, \Gamma|_M\,, \Gamma|_R)$ obtained by

$$\Gamma \longrightarrow (\Gamma|_L\,, \Gamma|_{\bar{R}}) \xrightarrow{\Gamma|_{\bar{R}} \longrightarrow (\Gamma|_M\,, \Gamma|_R)} (\Gamma|_L\,, \Gamma|_M\,, \Gamma|_R) \tag{D1}$$

also turns up as the decomposition

$$\Gamma \longrightarrow (\Gamma|_{\bar{L}}\,, \Gamma|_R) \xrightarrow{\Gamma|_{\bar{L}} \longrightarrow (\Gamma|_L\,, \Gamma|_M)} (\Gamma|_L\,, \Gamma|_M\,, \Gamma|_R) \tag{D2}$$

for $\bar{L} = L + M$. Conversely, triples obtained by the scheme (D2) coincide with the results of (D1) for the choice $\bar{R} = M + R$. Therefore, the multisets of triple decompositions are equal and Eq. (25) holds.

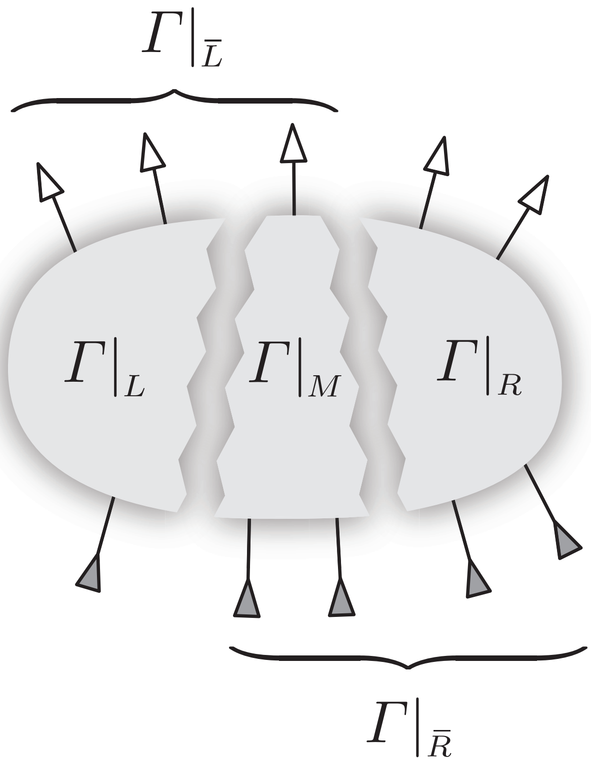

FIG. 5 Triple decomposition of a Heisenberg–Weyl diagram used in the proof of condition *(1)*.

Condition *(2)* is straightforward since the void graph $\emptyset$ is given by empty set of lines, and hence the decompositions $\Gamma \longrightarrow (\Gamma, \emptyset)$ and $\Gamma \longrightarrow (\emptyset, \Gamma)$ are uniquely defined by the partitions $E_\Gamma + \emptyset = E_\Gamma$ and $\emptyset + E_\Gamma = E_\Gamma$ respectively.

The symmetry condition *(3)* results from swapping subsets $L \leftrightarrow R$ in the partition $L + R = E_\Gamma$ which readily yields Eq. (27).

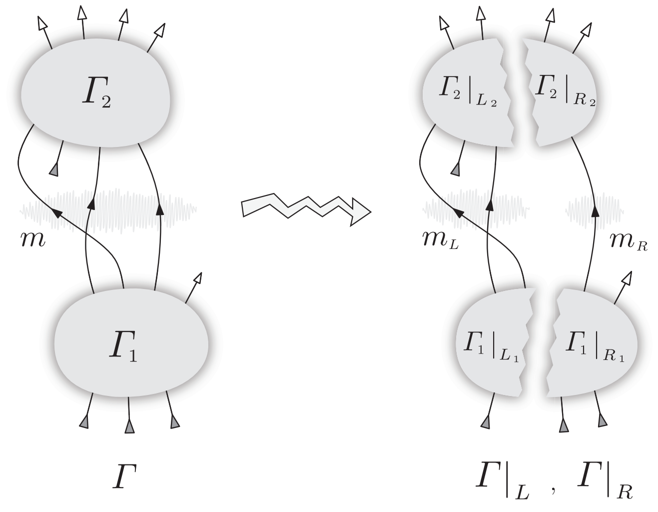

FIG. 6 Decompositions of a composite diagram $\Gamma = \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ for some $m \in \Gamma_2 \lessdot \Gamma_1$ used in the proof of condition *(4)*.

In order to check property *(4)* we need to construct a one-to-one correspondence between elements of both sides of Eq. (29). First, we observe that elements of the left-hand-side are decompositions of $\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1$ for all $m \in \Gamma_2 \lessdot \Gamma_1$, *i.e.*

$$(\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1|_L\,, \Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1|_R) \tag{D3}$$

where $L + R = E_{\Gamma_2 \overset{m}{\blacktriangleleft} \Gamma_1}$. On the other hand, the right-hand-side consists of component-wise compositions of pairs $(\Gamma_2|_{L_2}\,, \Gamma_2|_{R_2}) \in \langle \Gamma_2 \rangle$ and $(\Gamma_1|_{L_1}\,, \Gamma_1|_{R_1}) \in \langle \Gamma_1 \rangle$ for $L_2 + R_2 = E_{\Gamma_2}$ and $L_1 + R_1 = E_{\Gamma_1}$, which written explicitly are of the form

$$(\Gamma_2|_{L_2} \overset{m_L}{\blacktriangleleft} \Gamma_1|_{L_1}\,, \Gamma_2|_{R_2} \overset{m_R}{\blacktriangleleft} \Gamma_1|_{R_1}) \tag{D4}$$

with $m_L \in \Gamma_2|_{L_2} \lessdot \Gamma_1|_{L_1}$ and $m_R \in \Gamma_2|_{R_2} \lessdot \Gamma_1|_{R_1}$. We construct two mappings between elements of type (D3) and (D4) by the following assignments, see Fig. 6 for a schematic illustration. The first one is defined as:

$$(m, L, R) \longrightarrow (L_1, R_1, L_2, R_2, m_L, m_R)\ ,$$

where $L_i = E_{\Gamma_i} \cap L$, $R_i = E_{\Gamma_i} \cap R$ for $i = 1, 2$ and $m_L = m \cap L$, $m_R = m \cap R$. The second one is given by:

$$(L_1, R_1, L_2, R_2, m_L, m_R) \longrightarrow (m, L, R)\ ,$$

with $m = m_L \cup m_R$ and $L = L_2 \cup L_1$, $R = R_2 \cup R_2$. One checks that these mappings are inverses of each other and, moreover, the corresponding pairs of diagrams (D3) and (D4) are the same. This verifies that the multisets on the left- and right-hand side of Eq. (29) are equal and condition *(4)* obtains.

Condition *(5)* is straightforward from the construction since the edges of a diagram Γ can be nontrivially partitioned at most into $|\Gamma|$ subsets (each comprising one edge only).

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