Dynamic combinatorics, complex systems and applications to physics

Gérard H. E. Duchamp (a)

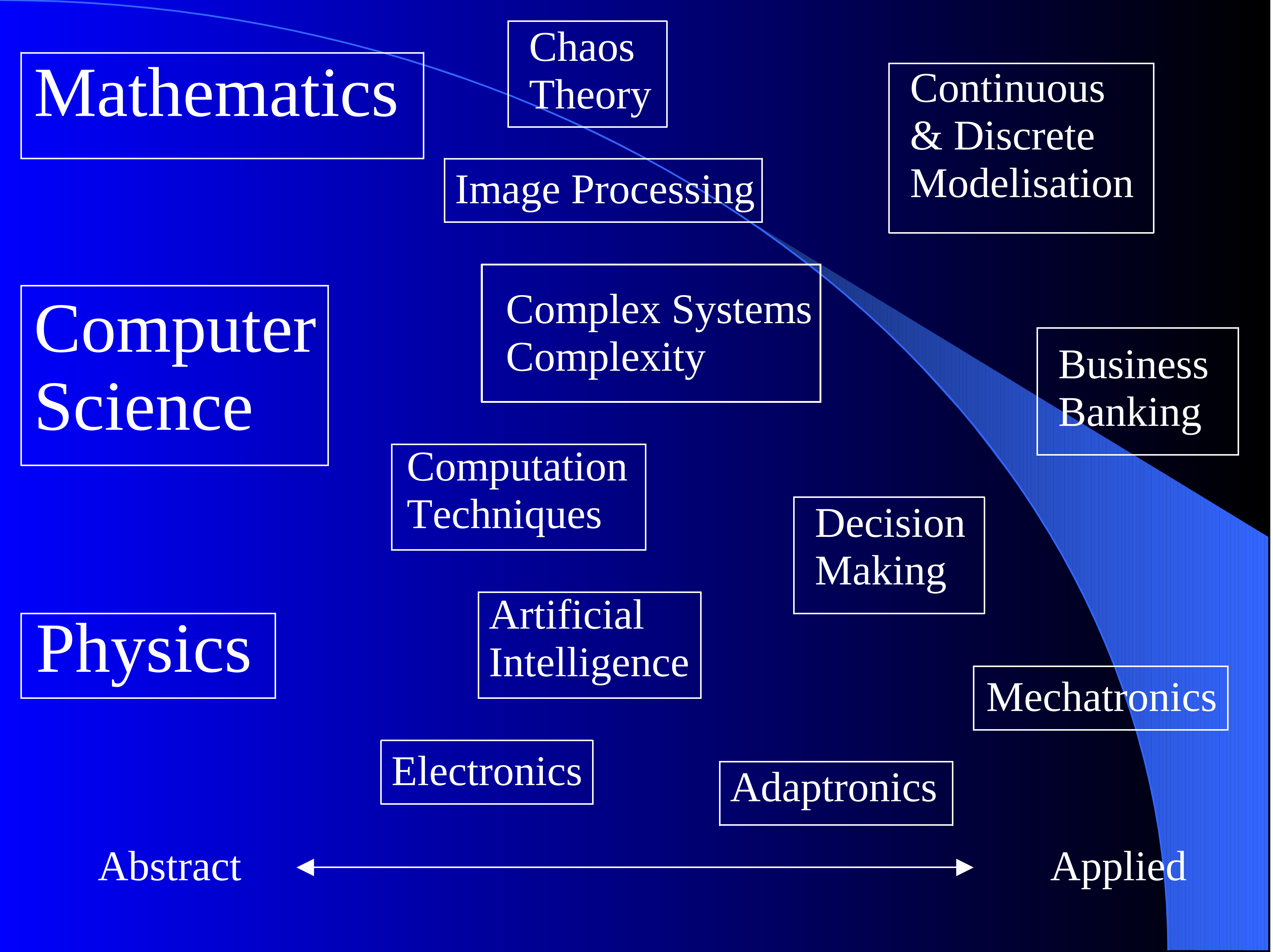
Cyrille Bertelle (b)

Rawan Ghnemat (b)

(a) LIPN, *Université de Paris XIII*, France

(b) LITIS, *Université du Havre*, France

Seminar cycle & Projects management
19-23 August 2007
Jordan


Mathematics
Chaos Theory
Continuous & Discrete Modelisation
Image Processing
Computer Science
Complex Systems Complexity
Business Banking
Computation Techniques
Decision Making
Physics
Artificial Intelligence
Mechatronics
Electronics
Adaptronics
Abstract
Applied

Combinatorics (mathematics)

Information
(comp. sci.)

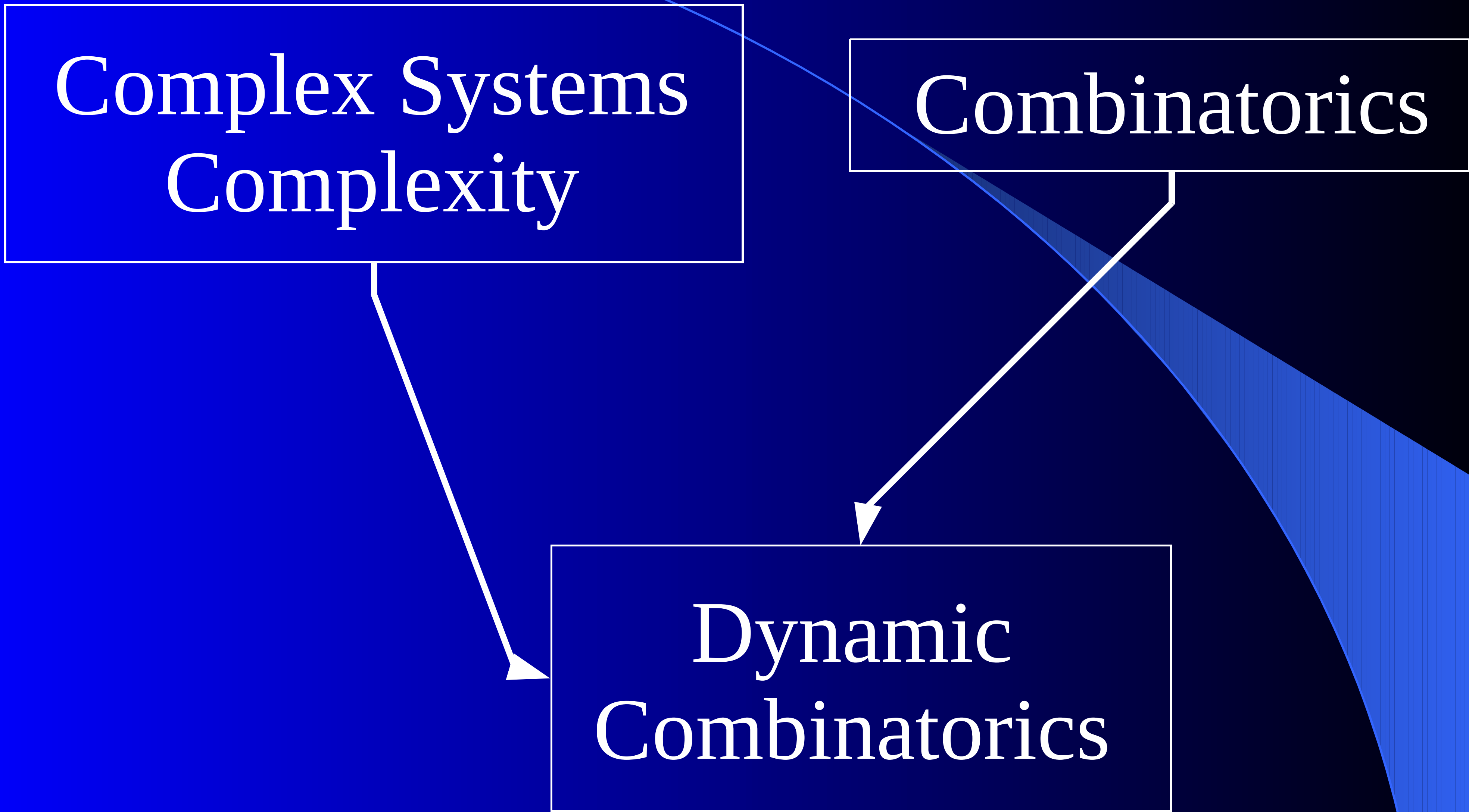
Physics
(class. quant.)

Combinatorics (mathematics)

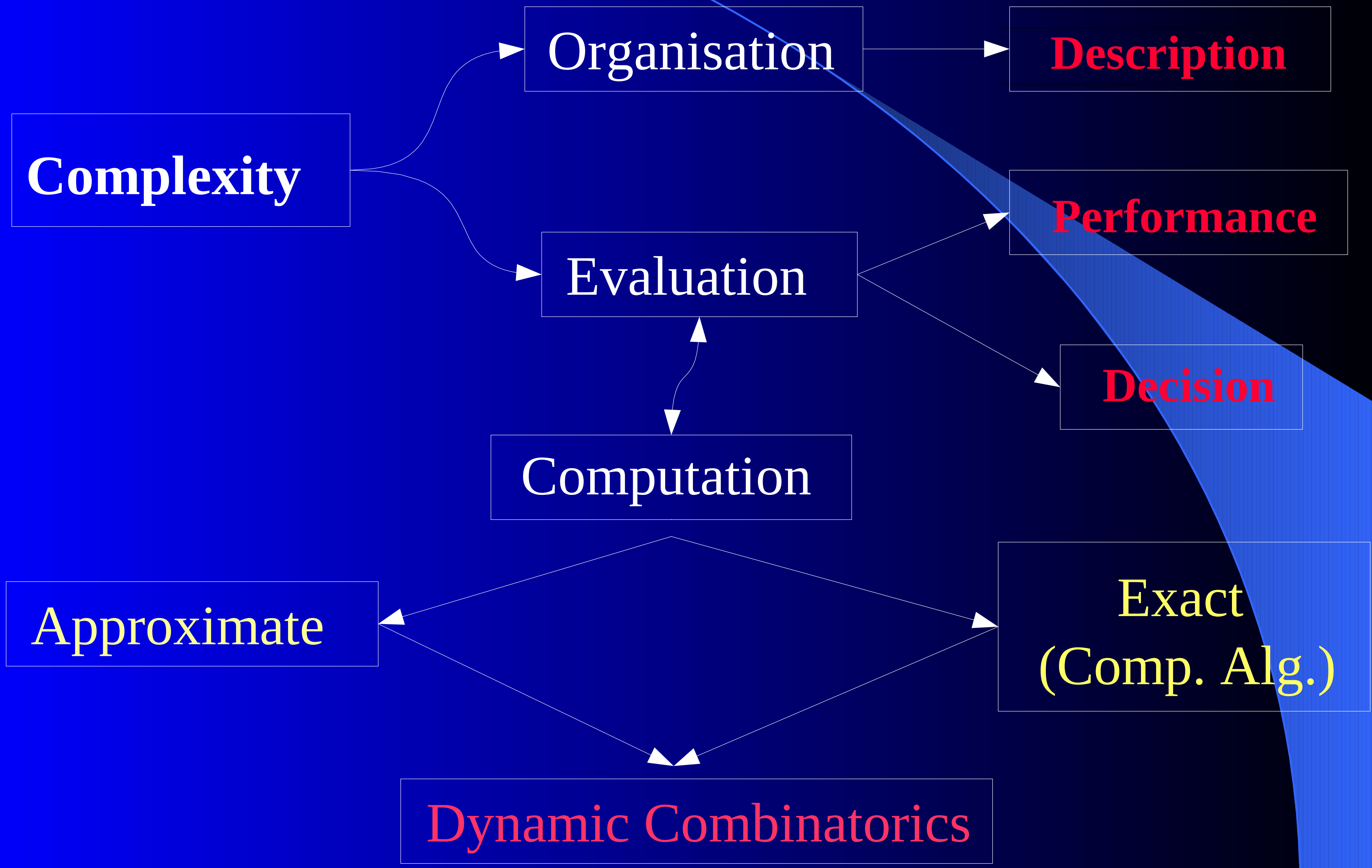
Complex
Systems

Information
(comp. sci.)

Physics
(class. quant.)


Complex Systems Complexity
Combinatorics
Dynamic Combinatorics

Problematics of Dynamic Combinatorics



What is the Legacy ?

Mathematics

- Noncommutative
- Representations
- Formulas, Universal Algebra
- Deformations

Comp. Sciences

- Words
- Automata Transition Structures
- Trees with Operateurs
- q-analogues

Physics

- Strings operators
- Fields, Flows, Dynamic Systems (Chaos, Catastrophes)
- Diagrams
- Quantum Groups

Combinatorics

Combinatorics

… on words

- Langages
- Theory of codes
- Automata
- Transition structures
- Grammars
- Transducers Rationnal and algebraic expressions
- …

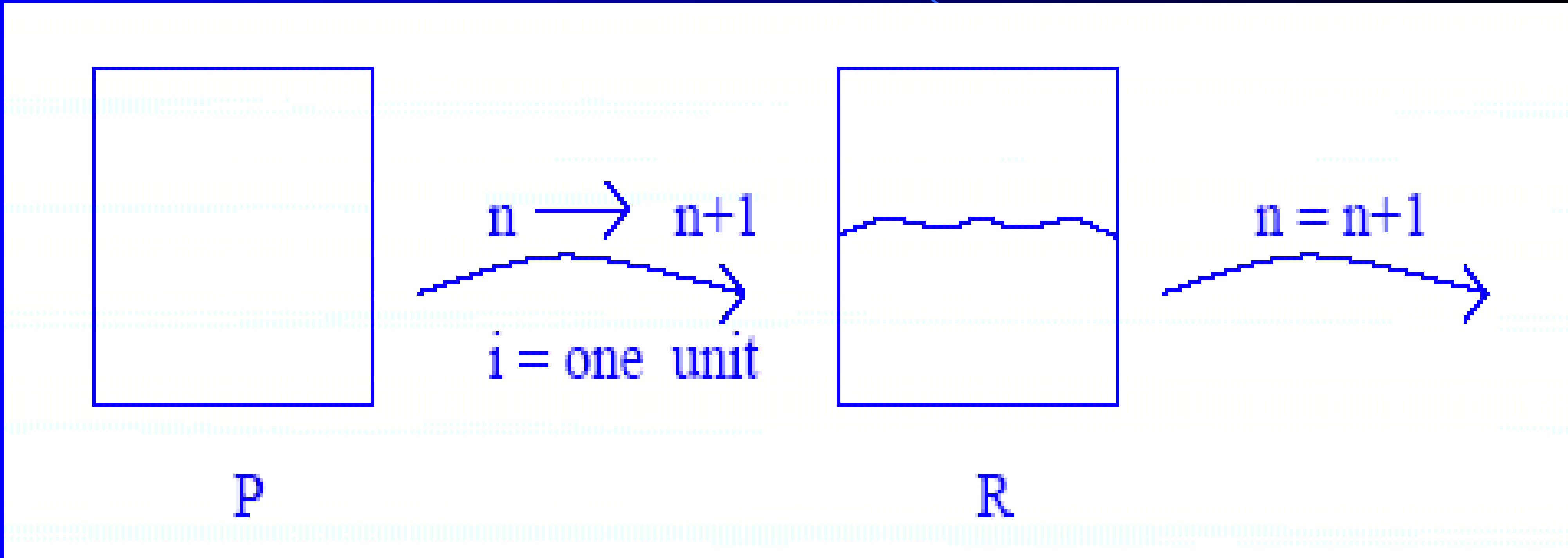
enumerative, analytic

- Polyominos
- Pathes (Dyck,…)
- Configurations
- q-grammars
- Generating Functions
- Continued Fractions (mono, multivariate,.)
- Orthogonal Polynomials
- …

algebraic

- Non commutative Continued fractions
- Representations of groups and deformations
- Quantum Groups Functors
- Characters
- Special Functions
- …

A first example . . .



n → n+1
i = one unit
n = n+1
P
R

- $a_1 = 0$
- $a_{i+1} \leq a_i + 1$

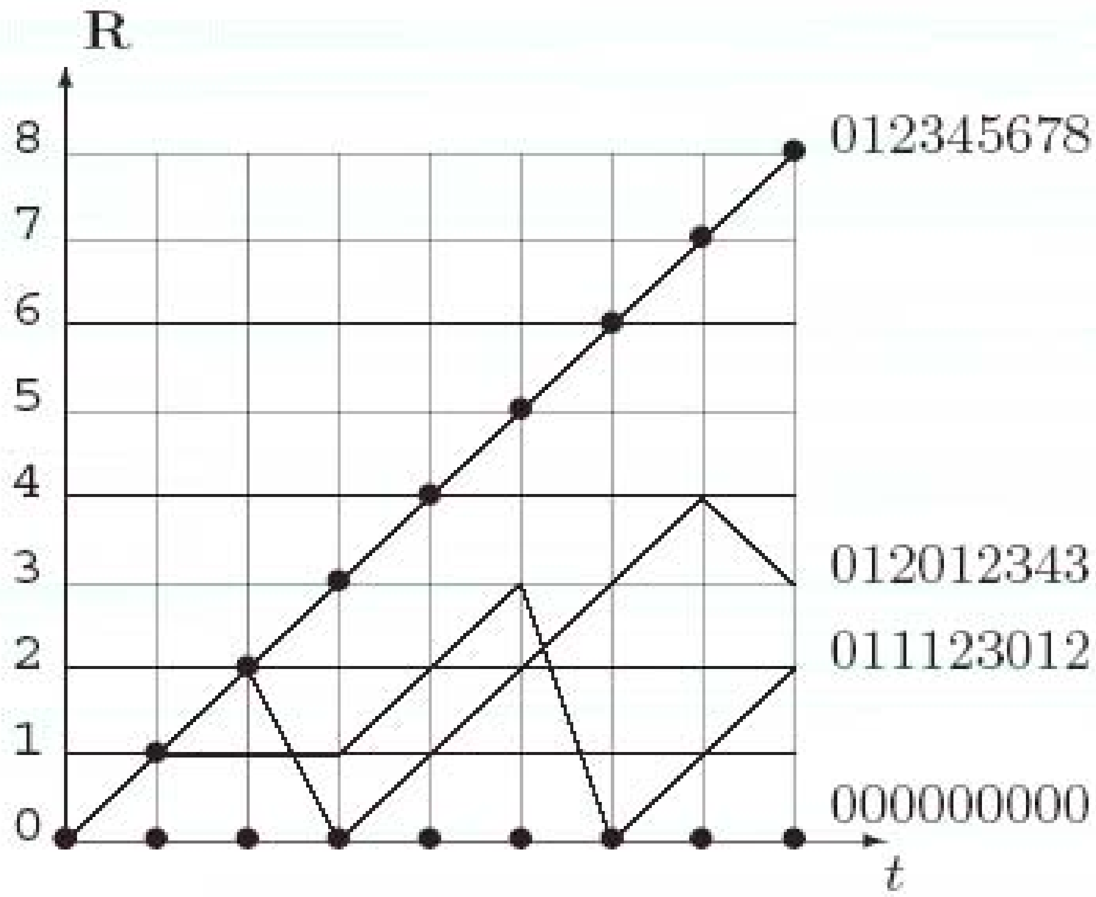


Figure 4.2: Maximal, minimal (dotted) and two intermediate trajectories. Their codes are on the right.

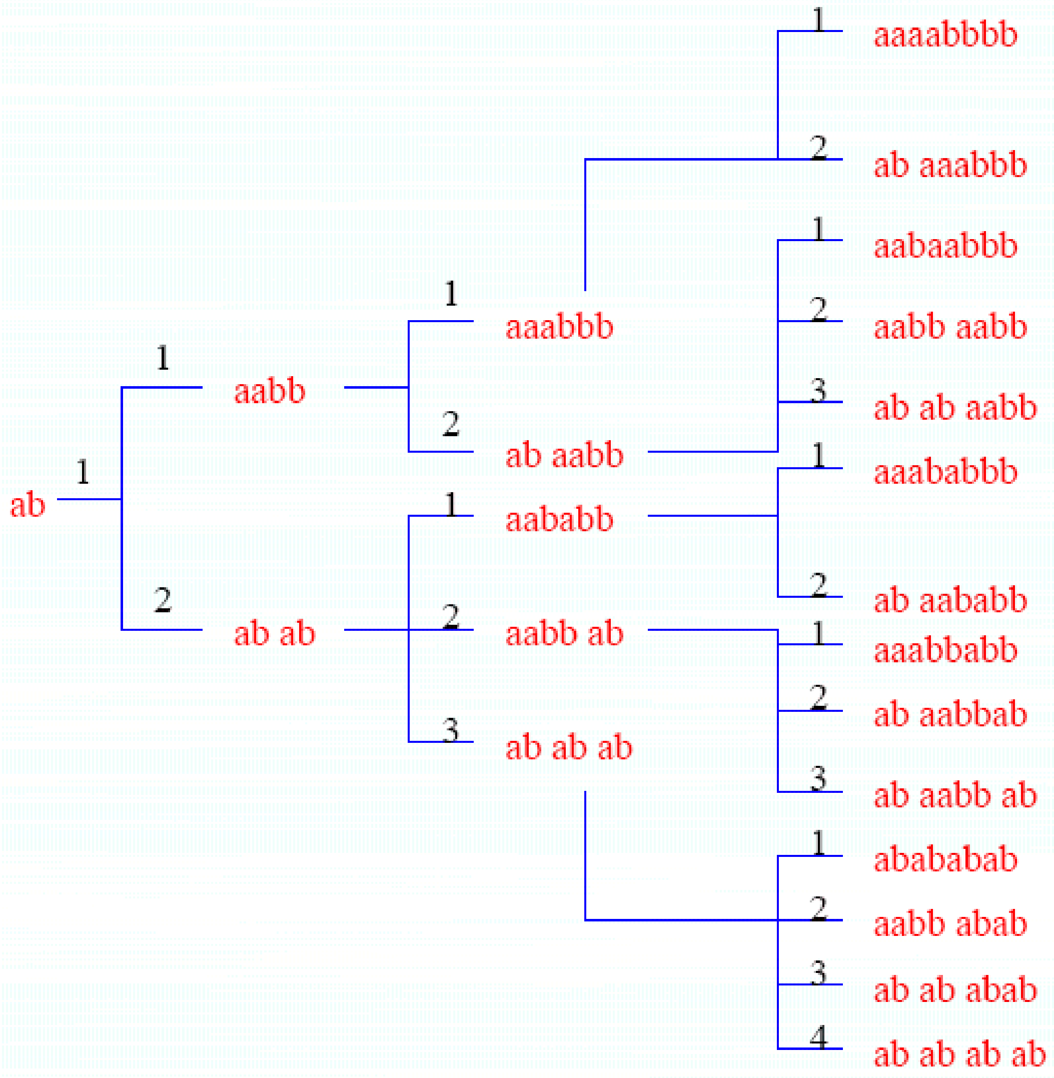


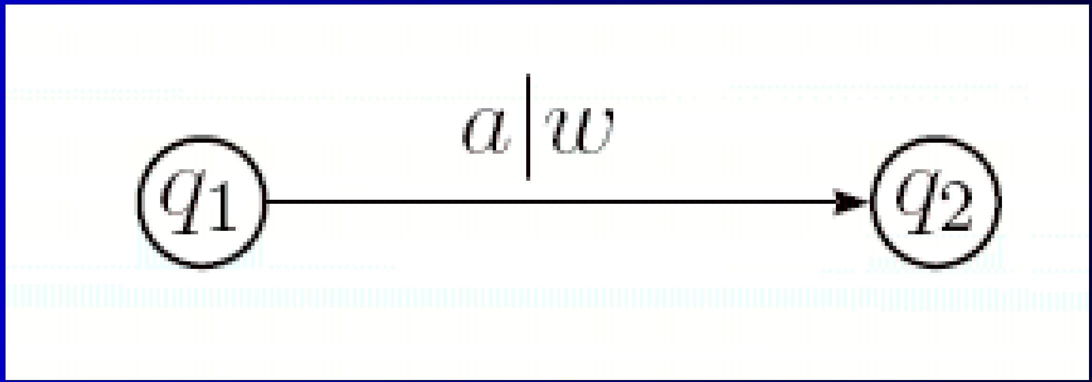
Figure 2.9: Dyck words of length 2n with k factors

We will return to the Dyck paths later on. For the moment let us define what is a transition structure.

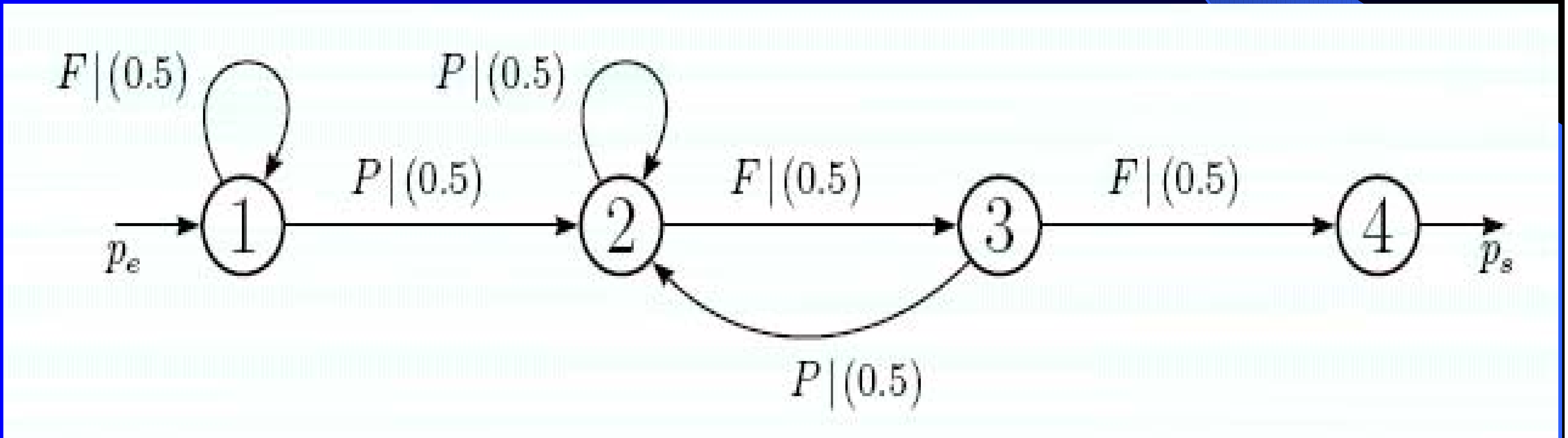
Definition (transition structure) : It is a graph (finite of infinite) with its arcs marked with pairs

(command letter | coefficient)

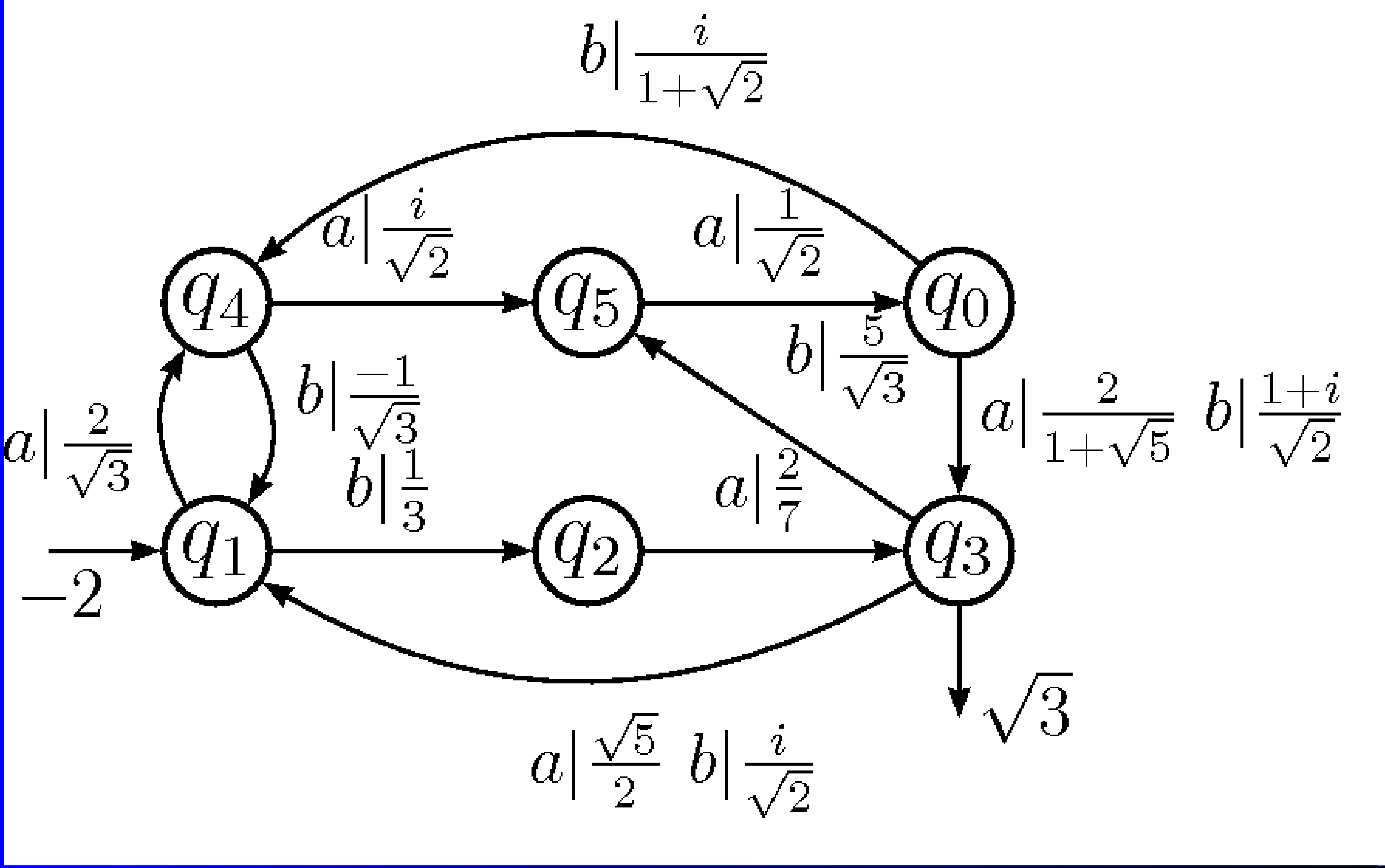
Examples : Prisoner's dilemma, Markov chains, classical engineering.



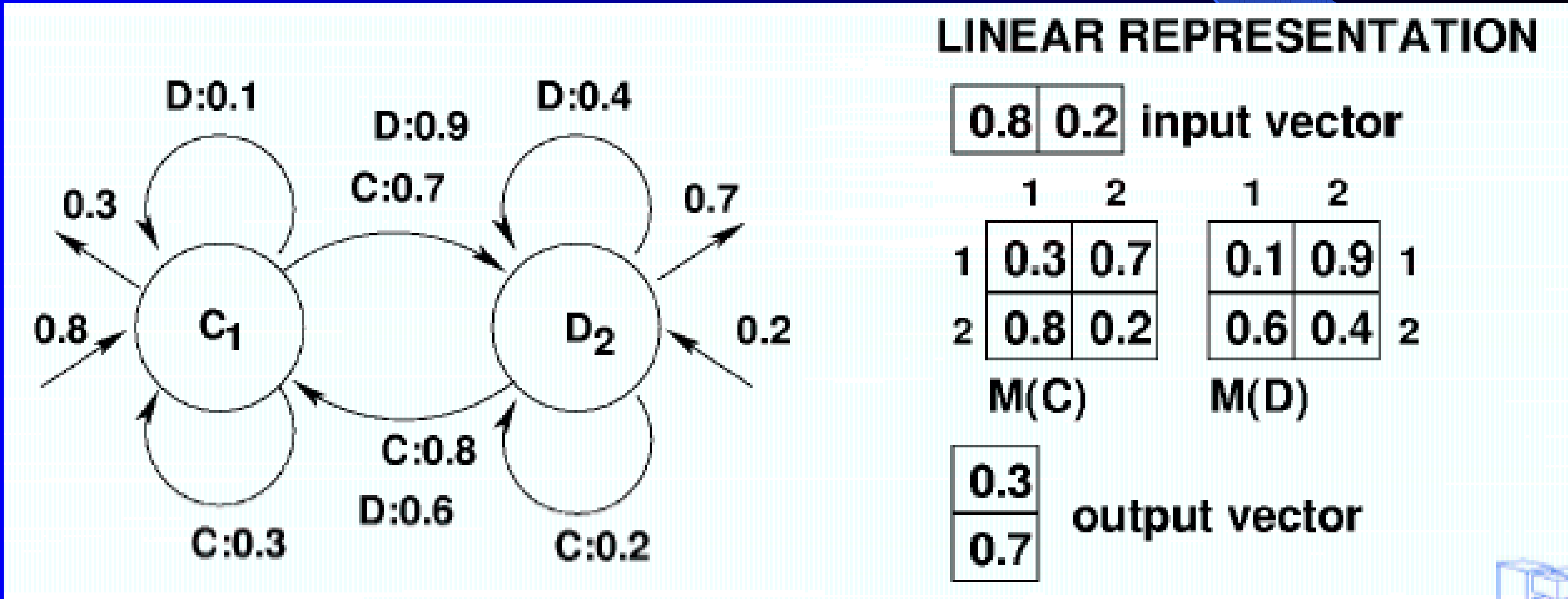
Example : A Markov chain generated by a game.



Example : An automaton generated by arbitrary transition coefficients.



Example : Automaton in Rawan's presentation.



Behaviour of an Automaton and how to compute it effectively

An automaton is a machine which takes a string (sequence of letters) and returns a value.

This value is computed as follows :

1) The weight of a path is the product of the weights (or coefficients) of its edges

2) The label of a path is the product (concatenation) of the labels of its edges

Behaviour ... (cont'd)

3) The behaviour between two states « r,s » w.r.t. A word « w » is the product of

3a) the ingoing coefficient of the first state (here « r ») by

3b) the sum of the weights of the paths going from « r » to « s » with label « w » by

3c) the outgoing coefficient of the second state (here « s »)

Behaviour … (cont'd)

4) The behaviour of the automaton under consideration w.r.t. a word « w » is then the sum of all the behaviours of the automaton between two states « r,s » for all possible pairs of states.

Behaviour … (cont'd)

There is a simple formula using the linear representation. The behaviour of an automaton with linear representation (I,M,T) is the product

$$IM(w)T$$

where M(w) is the canonical exention of M to the strings.

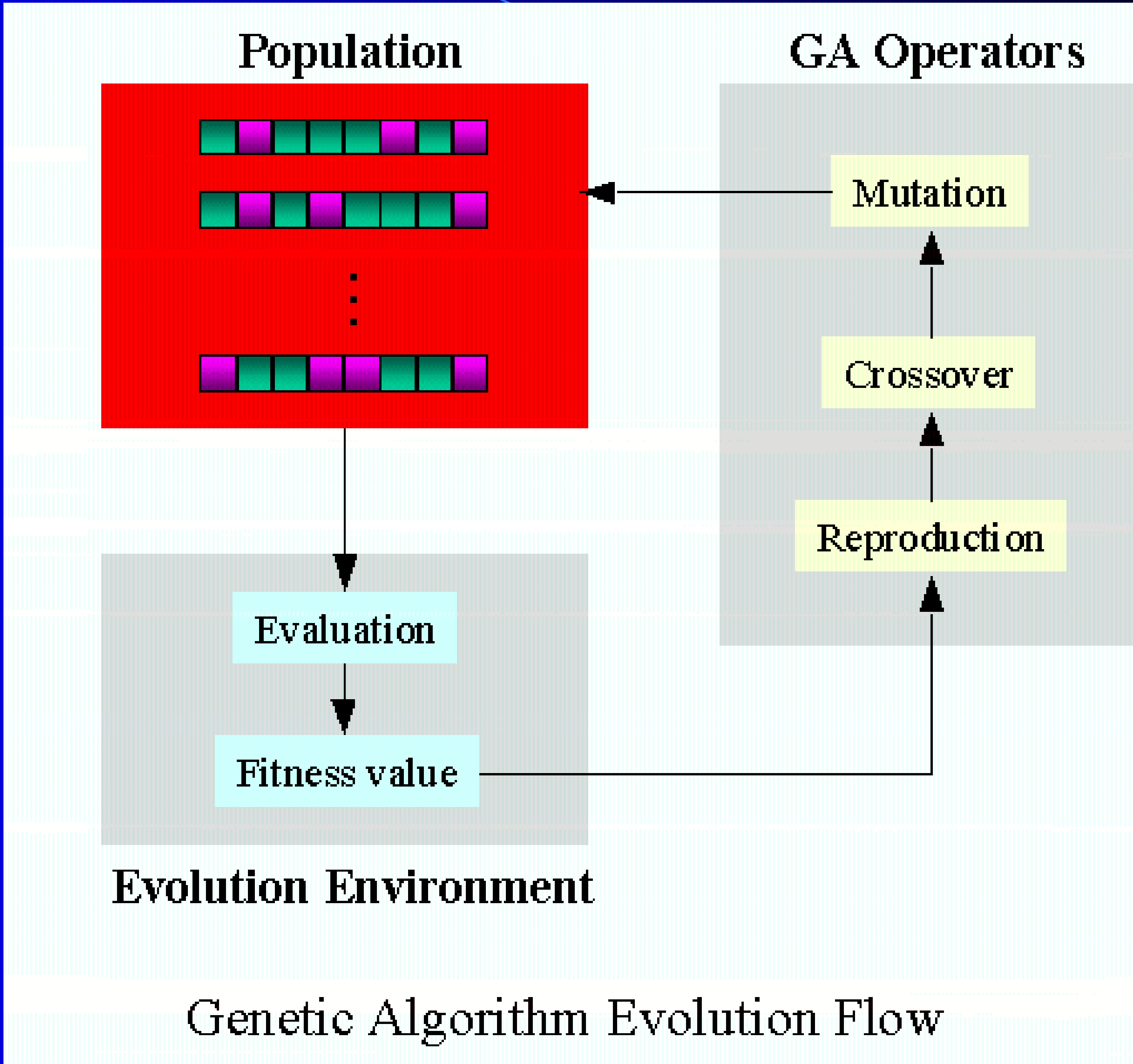
$$M(a_1 a_2 \ldots a_n) = M(a_1)M(a_2)\ldots M(a_n)$$

Behaviour … (end)

The behaviour, as a function on words belongs to the rational class. If time permits, we will return to its complete calculation as a rational expression and the problem of its algorithmic evaluation by means of special cancellation operators. Linear representations can also be used to compute distances between automata.

Example -> use of genetic algorithms to control indirect (set of) parameters : the spectrum of a matrix.

Genetic algorithms : general pattern



Genetic algorithms : implementation

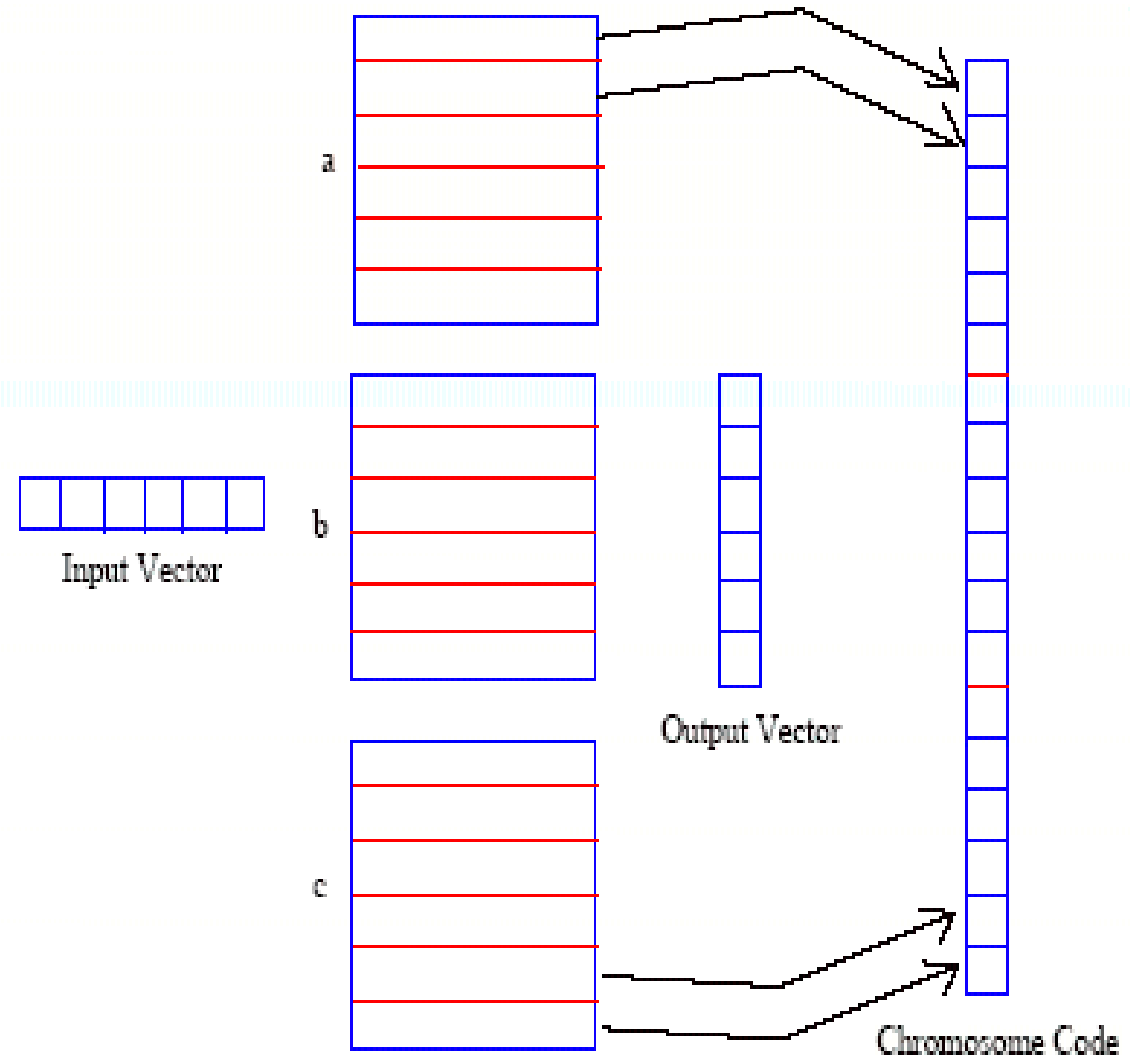


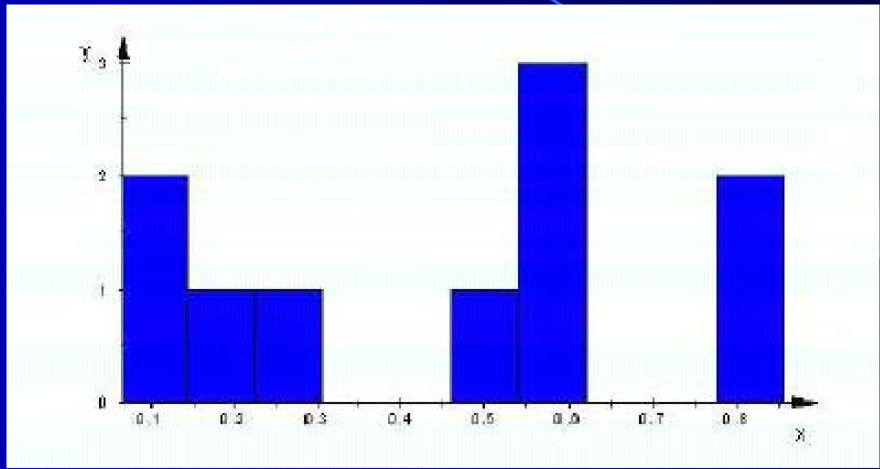
Figure 4.13: Chromosome code

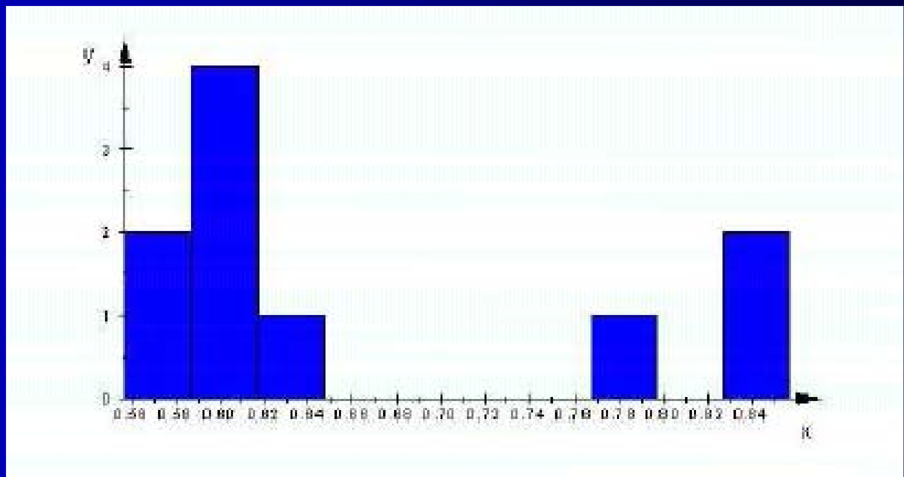
Genetic algorithms : implementation

Below, the results of an experiment aiming to control the second greatest eigenvalue of the transfer matrix of a population of probabilistic automata.

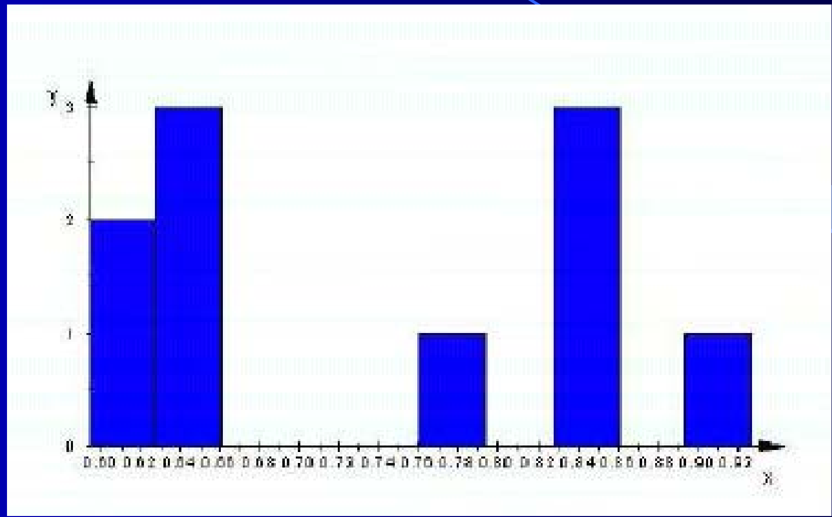
- The fitness function of each automaton corresponds to the second greatest eigenvalue (in module). The first being, of course, of value 1.

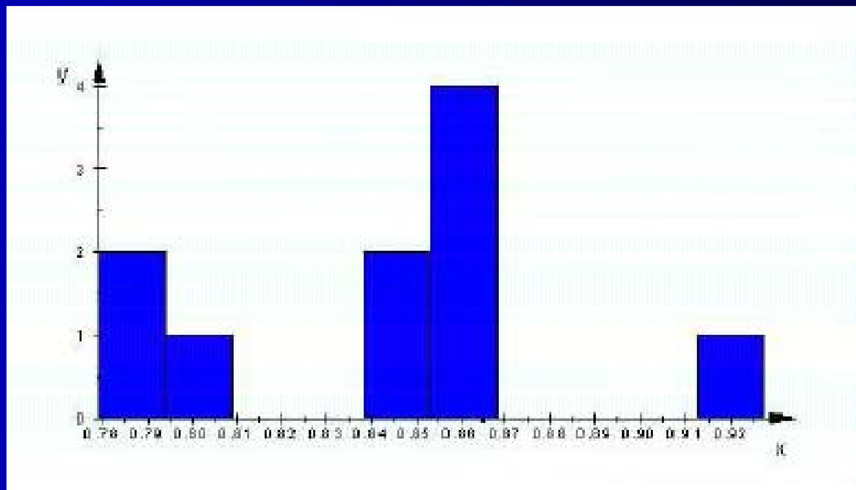
Genetic algorithms ; results



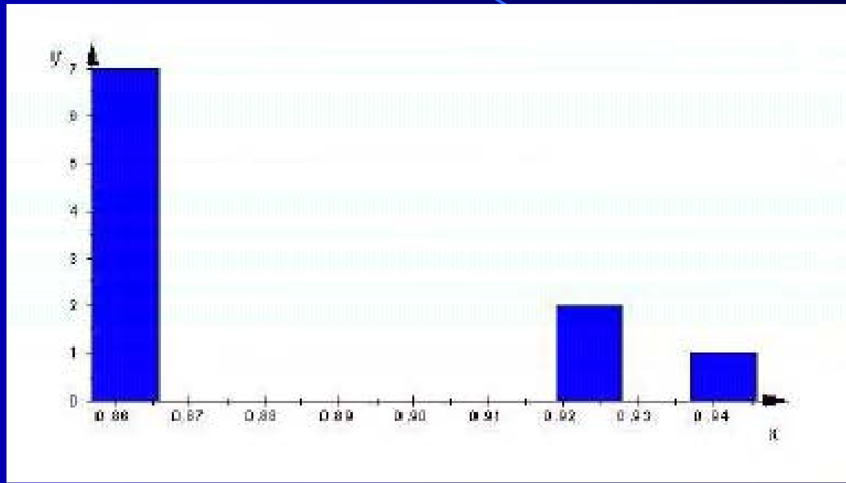


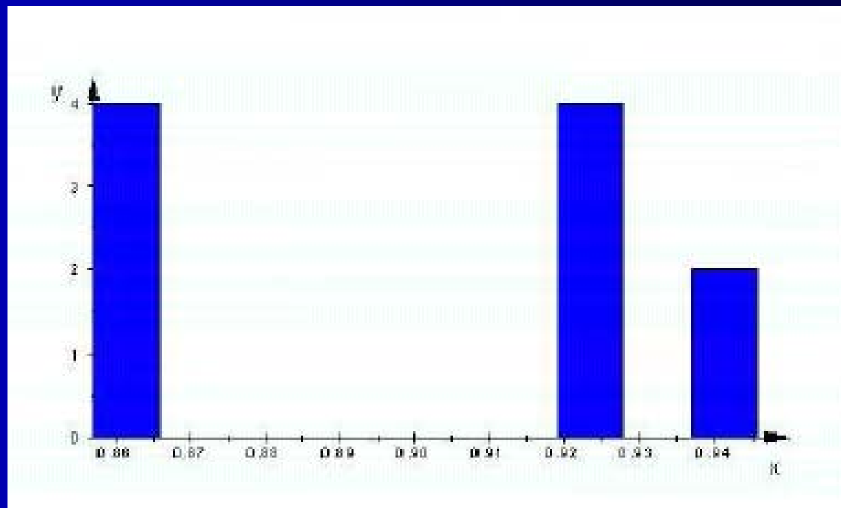
Genetic algorithms ; results



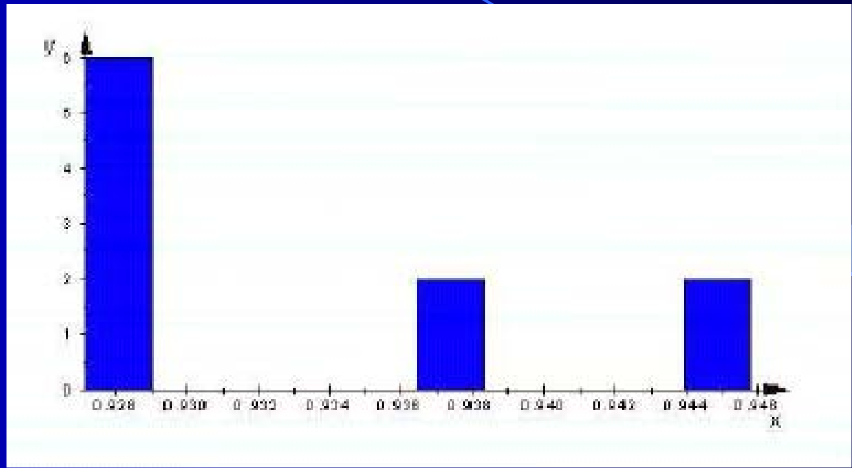


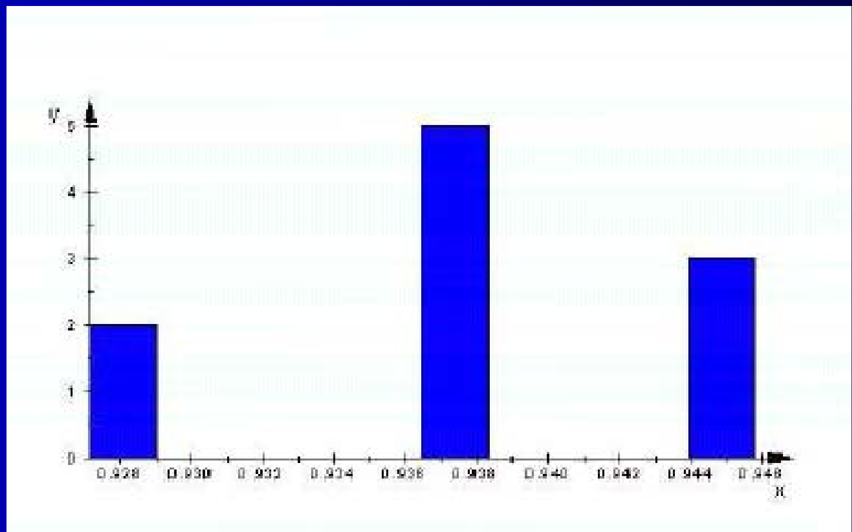
Genetic algorithms ; results



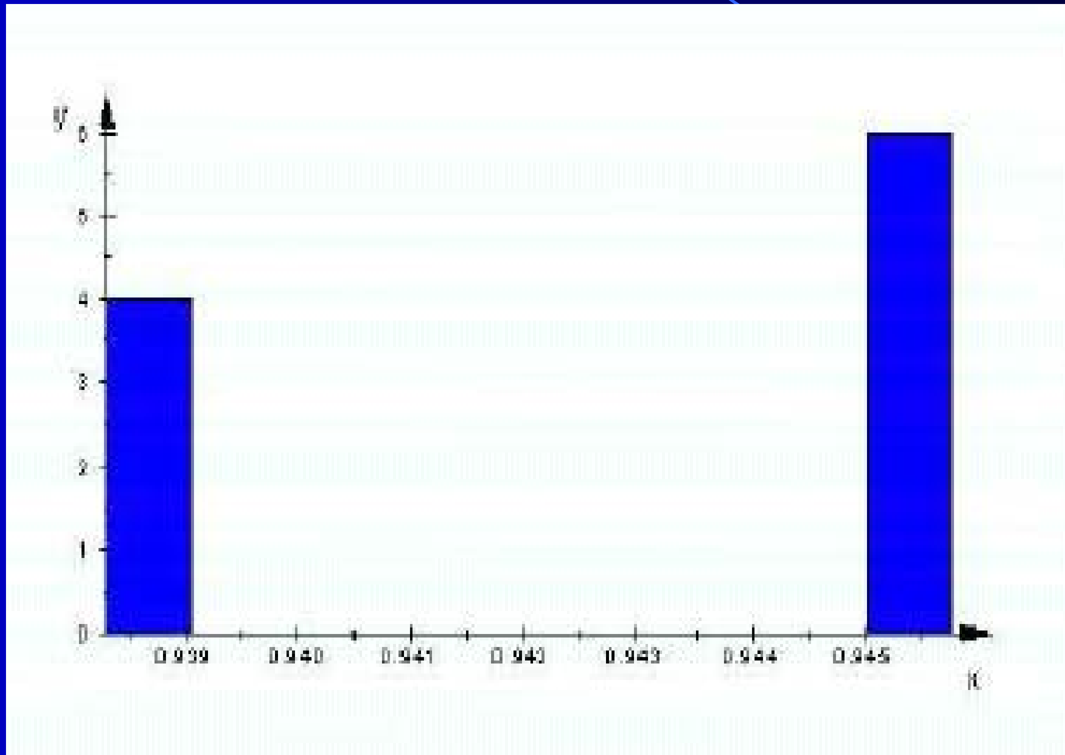


Genetic algorithms ; results





Genetic algorithms ; results

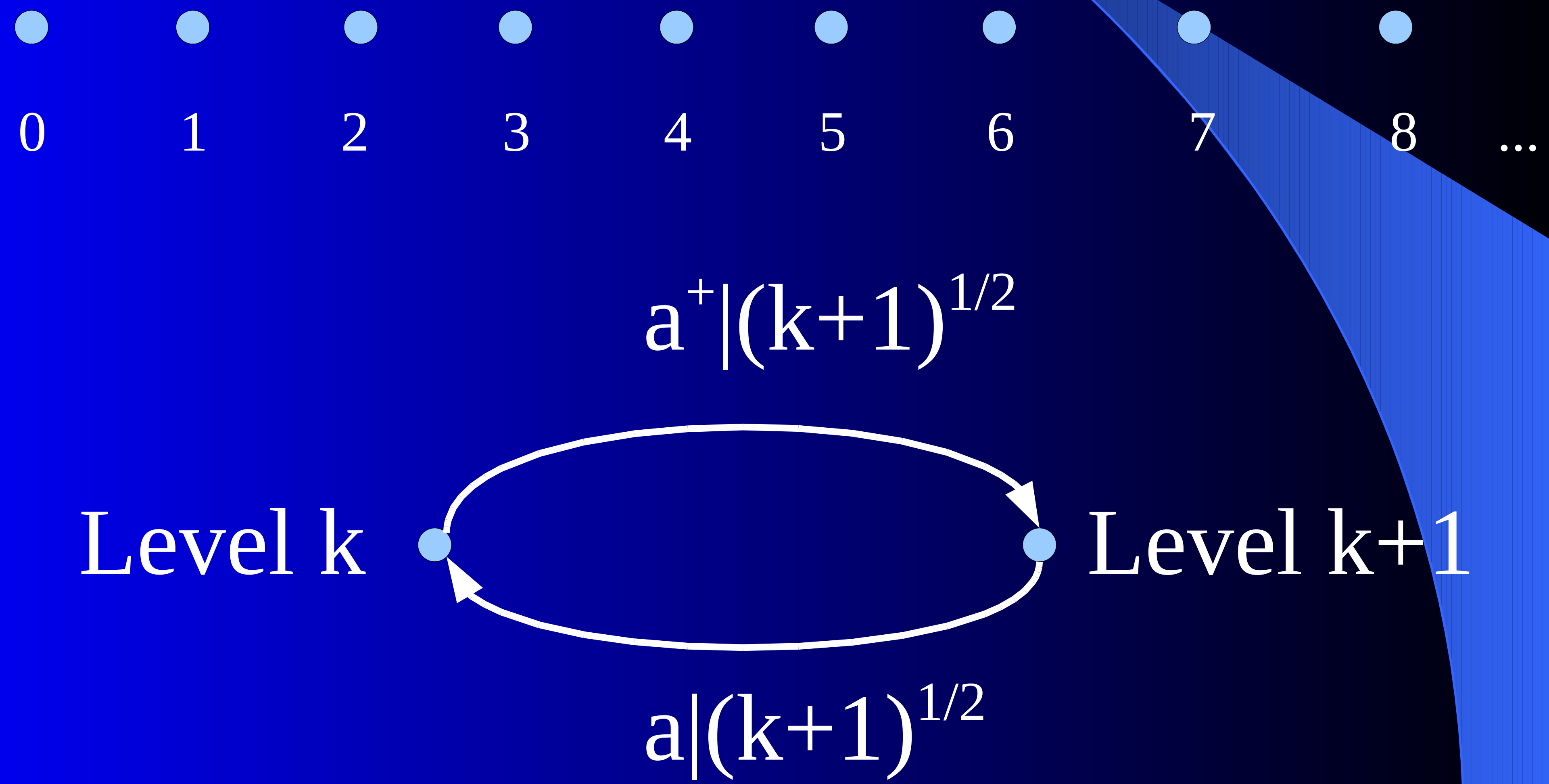


Final result : the population is rendered homogeneous

General transition systems

- Automata (finite number of edges)
- Sweedler's duals (physics, finite number of states)
- Representations
- Level systems (Quantum Physics)
- Markov chains (prob. automata when finite)

Example in Physics : annilhilation/creation operators



The (classical, for bosons) normal ordering problem goes as follows.

- Weyl (two-dimensional) algebra defined as

$$< a^+ , a \; ; \; [a , a^+] = 1 >$$

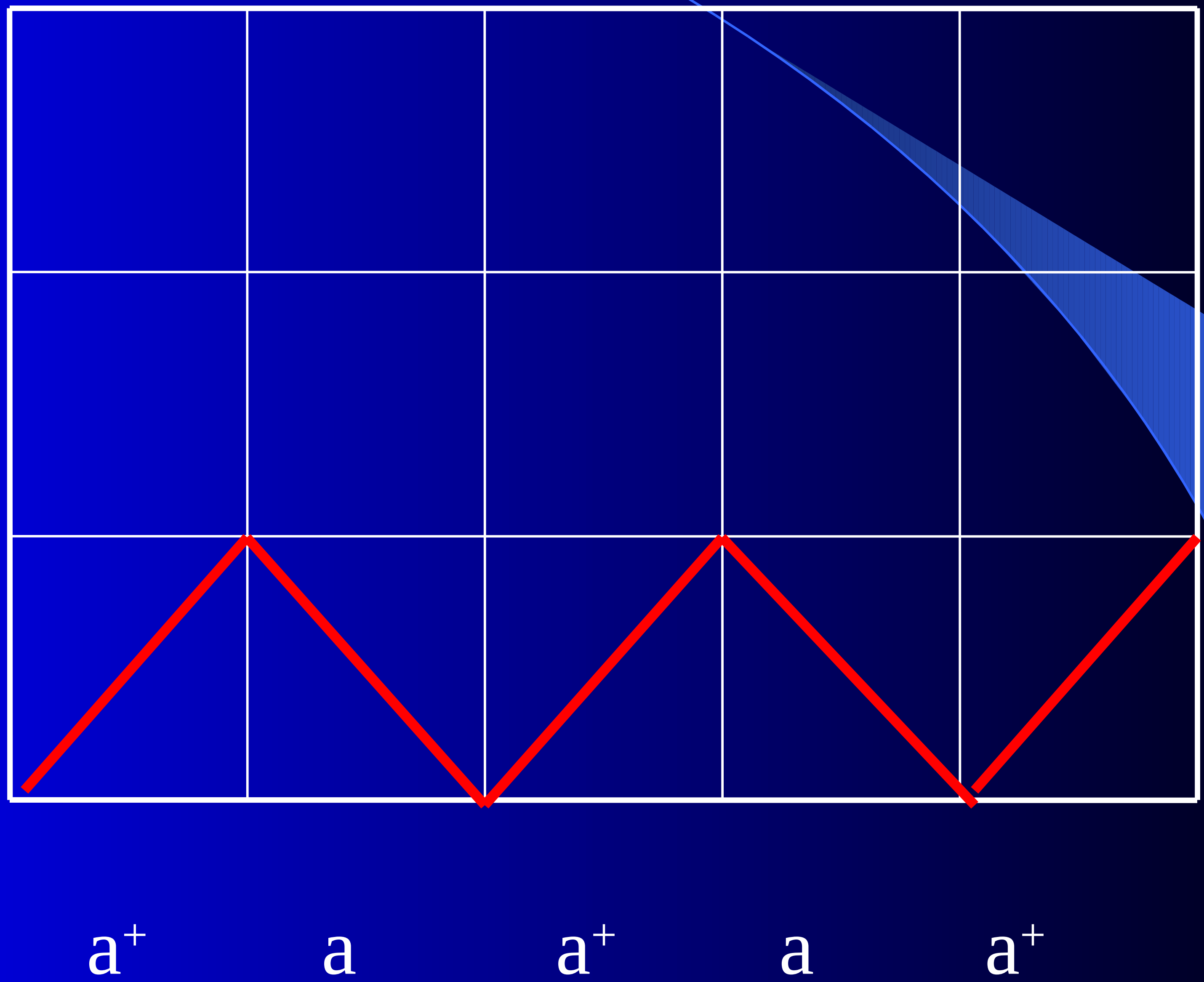
- Known to have no (faithful) representation by bounded operators in a Banach space.

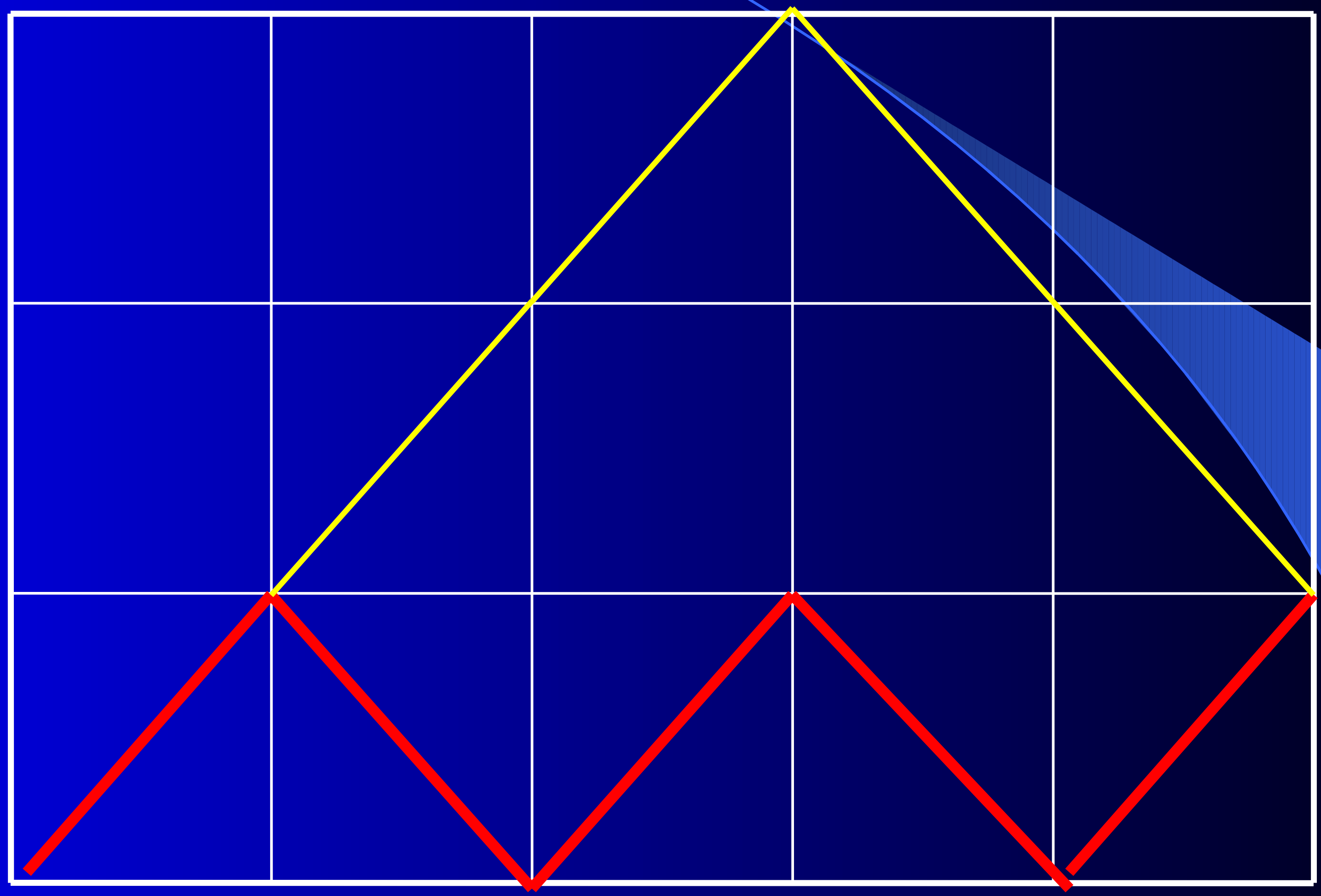
There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

$$a \to d/dx \; ; \; a^+ \to x$$

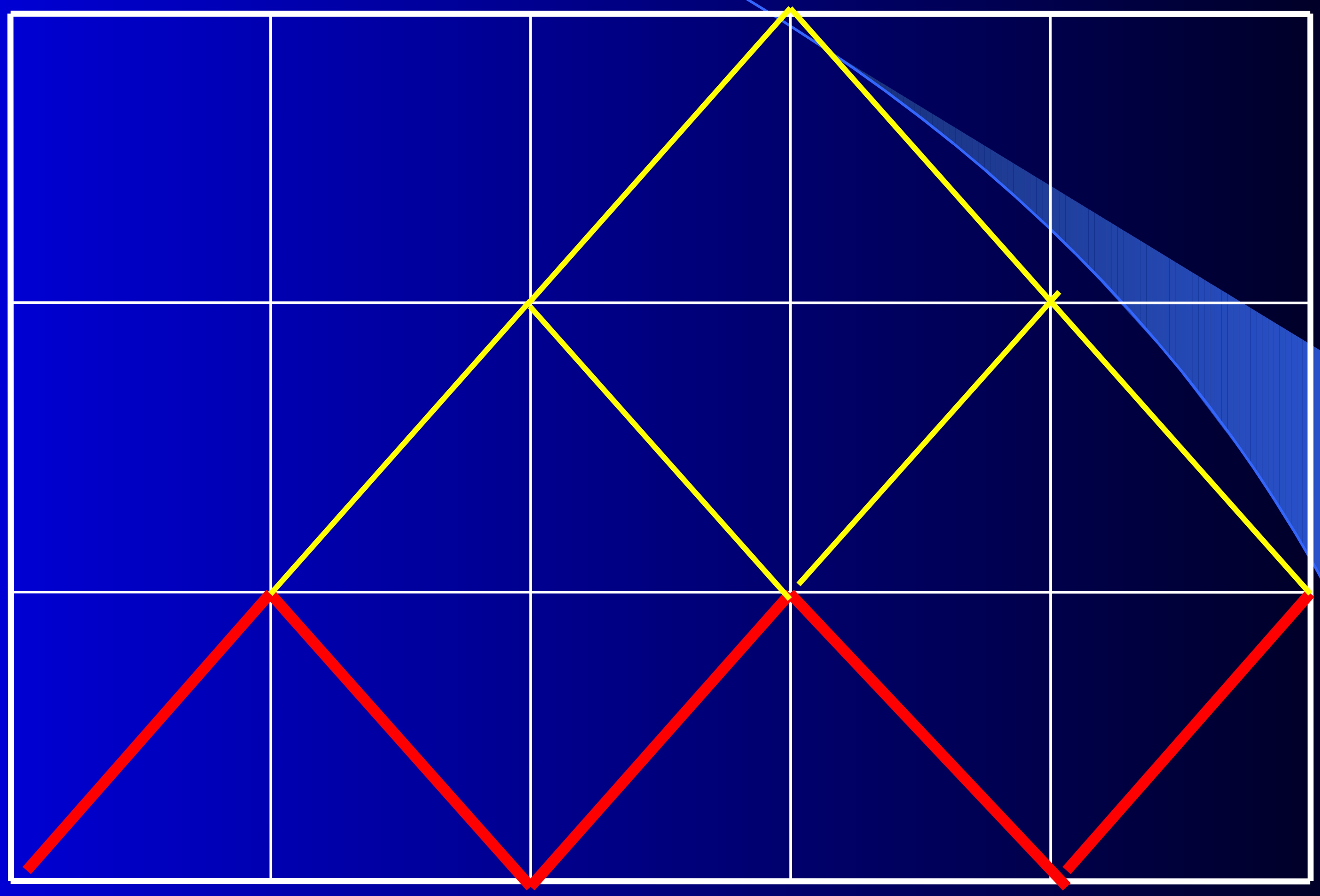
where a has degree -1 and a^+ has degree 1.

Example with $\Omega = a^+ a\ a^+ a\ a^+$

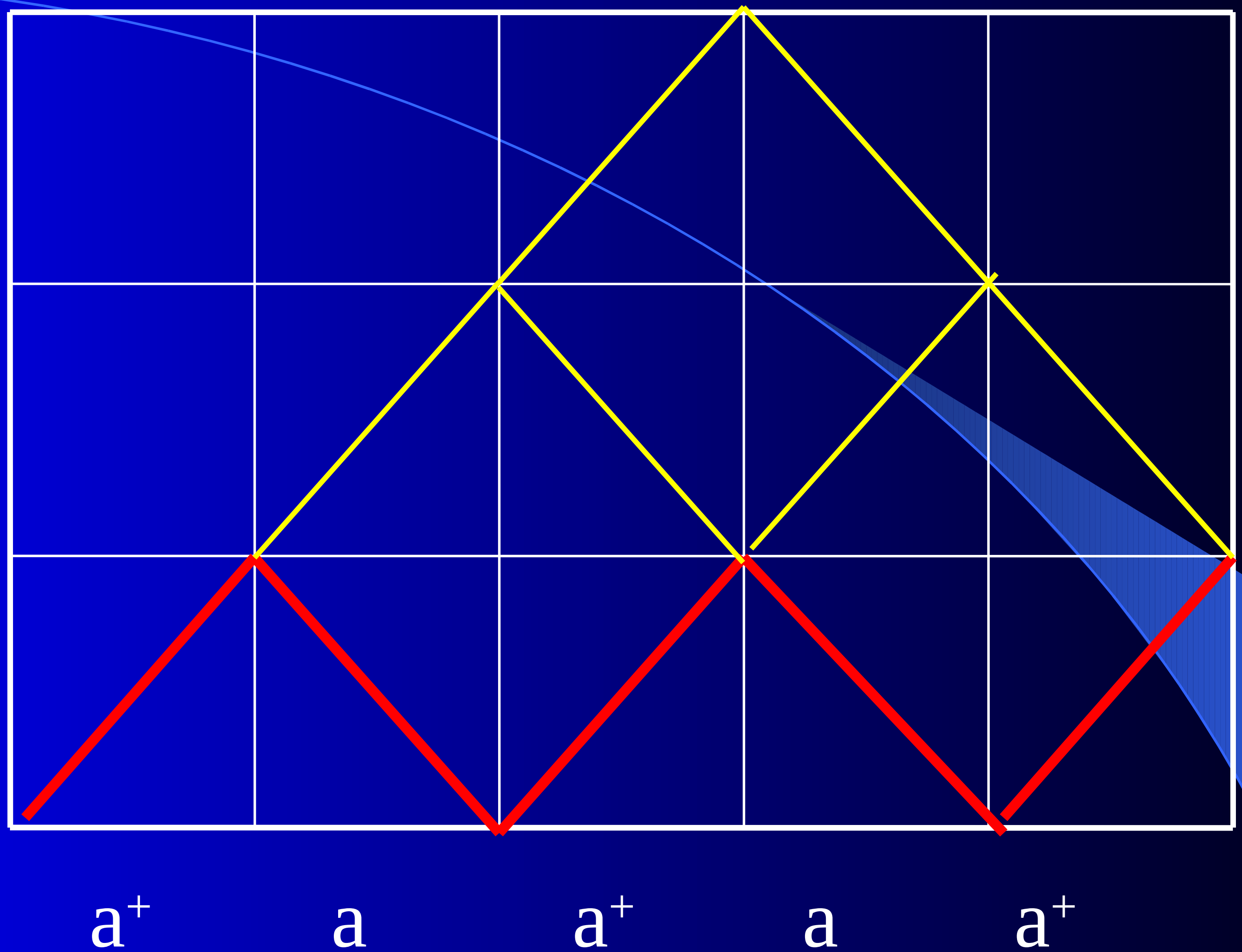




a^+ a a^+ a a^+



a^+ a a^+ a a^+



$$a^+aa^+aa^+ = 1\ a^+a^+a^+aa + 3\ a^+a^+a + 1\ a^+$$

Through Bargmann-Fock representation

$$a \to d/dx \; ; \; a^{+} \to x$$

Operators who have only one annihilation have exponentials who act as one-parameter groups of substitutions.

One can thus use computer algebra to determine their generating function.

For example, with

$$\Omega = a^{+2} a \, a^{+} + a^{+} a \, a^{+2}$$

the computation reads

Maple 9 - [Untitled (1) - [Server 1]]

File Edit View Insert Format Spreadsheet Window Help

One parameter group by f(v(u(x)+λ)); v is reciprocal of u

```
> T1(lambda,x):=(-4*(-1/(4*x^2)+lambda))^(-1/2);
```

$$T1(\lambda, x) := \frac{1}{\sqrt{\frac{1}{x^2} - 4\lambda}}$$

We suppose x>0

```
> T1:=(lambda,x) -> x/((1-4*lambda*x^2)^(1/2));
```

$$T1 := (\lambda, x) \rightarrow \frac{x}{\sqrt{1 - 4\lambda x^2}}$$

Checking the tangent vector

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))  -T1(lambda1+lambda2,x)  );
```

$$0$$

Time: 0.2s Bytes: 1.94M Available: 1.33G

démarrer | Microsoft PowerPoint ... | Maple 9 - [Untitled (1... | Le Robert & Collins | FR 14:25

And the action of exp(λ ω) on [f(x)] is

$$U_\lambda(f) = x^{-\frac{3}{2}} f(s_\lambda(x)).(s_\lambda(x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f(\sqrt{\frac{x^2}{1-4\lambda x^2}})$$

which explains the prefactor. Again we can check by computation that the composition of (U_λ)s amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of Eq1 (slide 15) we must have

$$\exp(\lambda\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k (\frac{d}{dx})^k$$

So, using this new technique, one can compute easily the coefficients of the matrix giving the normal forms.

Maple 9 - [calc1.mws - [Server 1]]

File Edit View Insert Format Spreadsheet Window Help

```
> g1:=exp(y*x*exp(x));d1:=taylor(g1,x=0,7);
```

$$g1 := e^{(y\,x\,e^{x})}$$

$$d1 := 1 + y\,x + \left(y + \frac{1}{2}y^2\right)x^2 + \left(\frac{1}{2}y + y^2 + \frac{1}{6}y^3\right)x^3 + \left(\frac{1}{6}y + y^2 + \frac{1}{2}y^3 + \frac{1}{24}y^4\right)x^4 + \left(\frac{1}{24}y + \frac{2}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{6}y^4 + \frac{1}{120}y^5\right)x^5 + \left(\frac{1}{120}y + \frac{1}{3}y^2 + \frac{3}{4}y^3 + \frac{1}{3}y^4 + \frac{1}{24}y^5 + \frac{1}{720}y^6\right)x^6 + O(x^7)$$

```
> matrix(7,7,(i,j)->(i-1)!*coeff(coeff(d1,x,i-1),y,j-1));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 \end{bmatrix}$$

For these one-parameter groups and conjugates of vector fields

G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,

One-parameter groups and combinatorial physics,

Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003. arXiv : quant-ph/0401126.

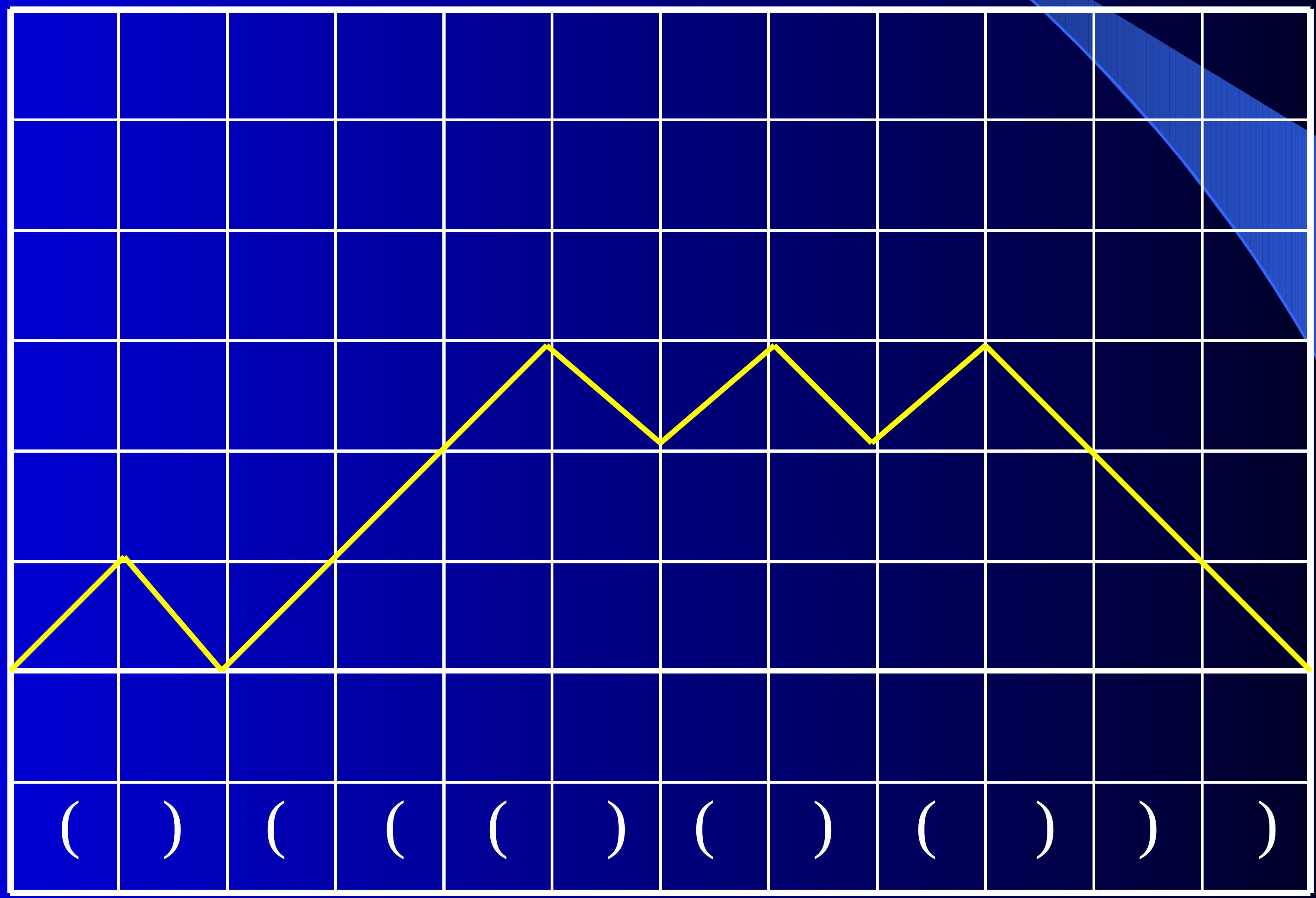
For the Sheffer-type sequences and coherent states

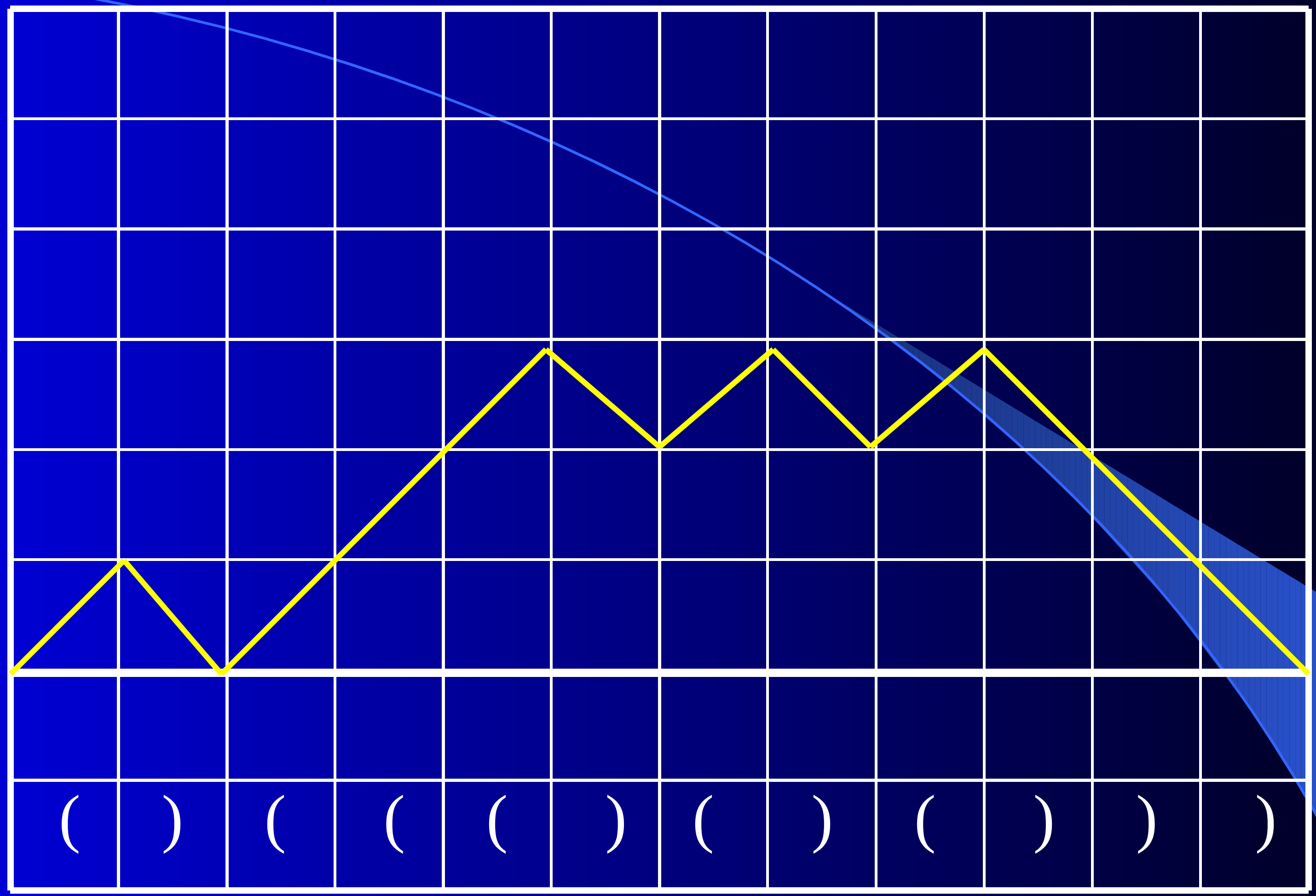
K A Penson, P Blasiak, G H E Duchamp, A Horzela and A I Solomon,

Hierarchical Dobinski-type relations via substitution and the moment problem,

J. Phys. A: Math. Gen. 37 3457 (2004) arXiv : quant-ph/0312202

A second application : Dyck paths
(systems of brackets, trees, physics, …)

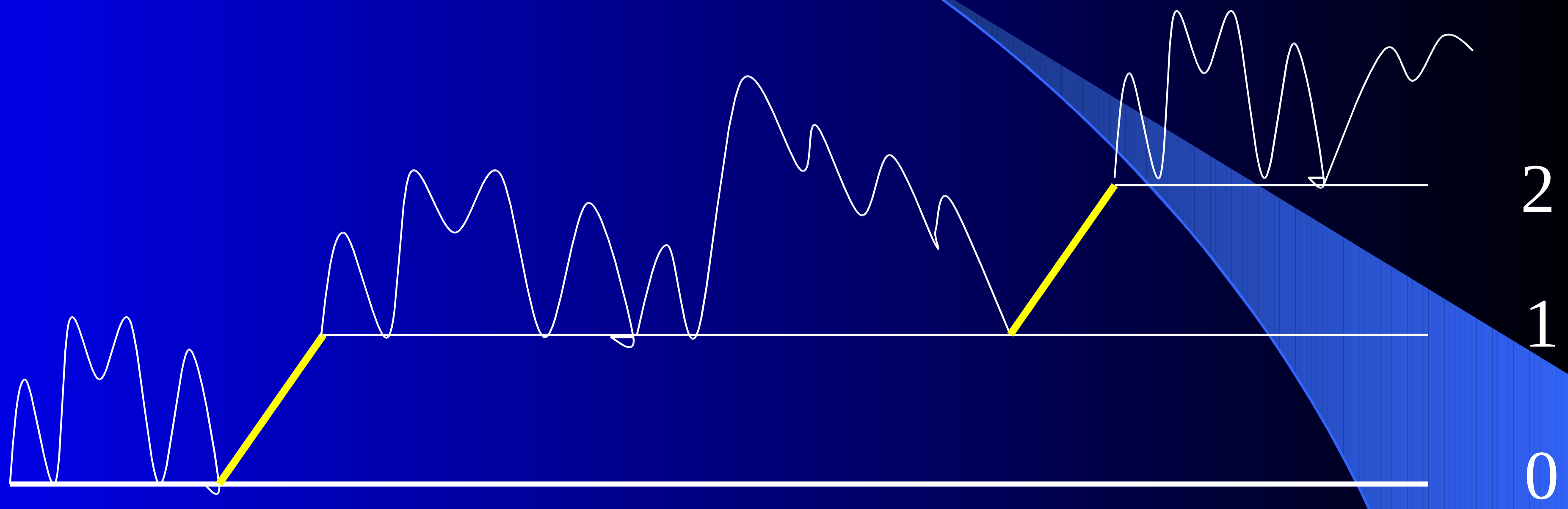




Equation : D = void + (D) D … one counts strings using an « x » by bracket and one finds $T(x)=x^0 + x^2 T^2(x)$ which can be solved by elementary methods …

$x^2 T^2 - T + 1 = 0$ Variable : T Parameter : x

Change of level (physics)



Positifs = $D(aD)^*$

$$Pos := \frac{Dyck}{1 - x\,Dyck}$$

```
> solve(x^2*T^2-T+1=0,T);
```

$$\frac{1+\sqrt{1-4\,x^2}}{2\,x^2},\ \frac{1-\sqrt{1-4\,x^2}}{2\,x^2}$$

```
> f:=1/(2*x^2)*(1-(1-4*x^2)^(1/2));
```

$$f := \frac{1-\sqrt{1-4\,x^2}}{2\,x^2}$$

```
> taylor(f,x=0,20);
```

$$1 + x^2 + 2\,x^4 + 5\,x^6 + 14\,x^8 + 42\,x^{10} + 132\,x^{12} + 429\,x^{14} + 1430\,x^{16} + \mathrm{O}(x^{18})$$

```
> seq(binomial(2*k,k)/(k+1),k=1..8);
```

$$1, 2, 5, 14, 42, 132, 429, 1430$$

```
>
```

```
> Pos:=simplify(Dyck/(1-x*Dyck));
```

$$Pos := -\frac{2}{-1-\sqrt{1-4\,x\,y+2\,x}}$$

```
> coeftayl(Pos,[x,y]=[0,0],[6,4]);
```

$$90$$

```
> S:=0:for l from 0 to 6 do for k from 0 to 6 do
  S:=S+coeftayl(Pos,[x,y]=[0,0],[k,l])*x^k*y^l od
  od:S;
```

$$1+x+x\,y+20\,x^6\,y^2+14\,x^5\,y^2+5\,x^3\,y^3+2\,x^2\,y^2+x^3+28\,x^5\,y^3+x^4+x^5$$
$$+x^6+x^2+132\,x^6\,y^5+2\,x^2\,y+5\,x^3\,y^2+90\,x^6\,y^4+42\,x^5\,y^5+3\,x^3\,y$$
$$+132\,x^6\,y^6+4\,x^4\,y+14\,x^4\,y^4+14\,x^4\,y^3+5\,x^5\,y+9\,x^4\,y^2+48\,x^6\,y^3$$
$$+42\,x^5\,y^4+6\,x^6\,y$$

```
>
```

Automata and rationality

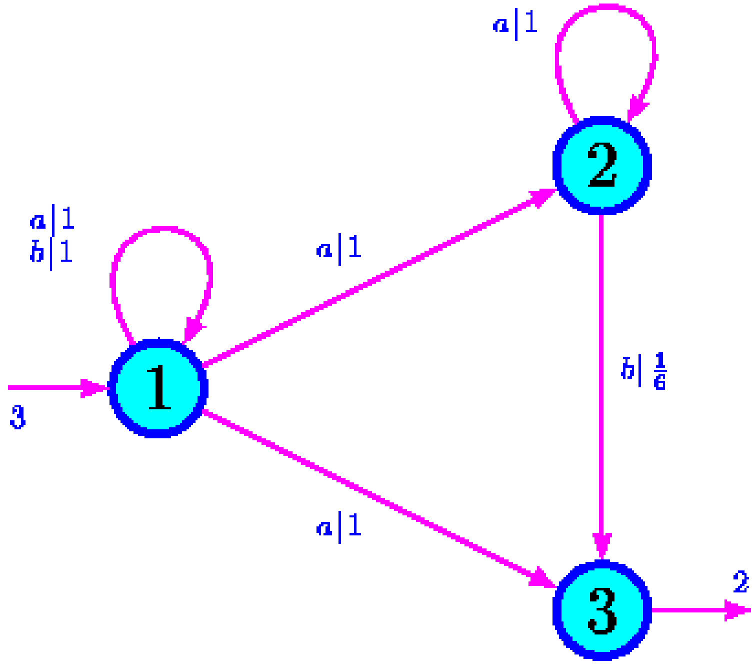


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6+a^*b).$$

Un type particulier d'automate à multiplicités est constitué des automates à multiplicités avec des ε-transitions.

Un k-ε-automate $\mathcal{A}_\varepsilon$ est un k-automate sur l'alphabet $A_\varepsilon = A \cup \{\tilde{\varepsilon}\}$.

Exemple :

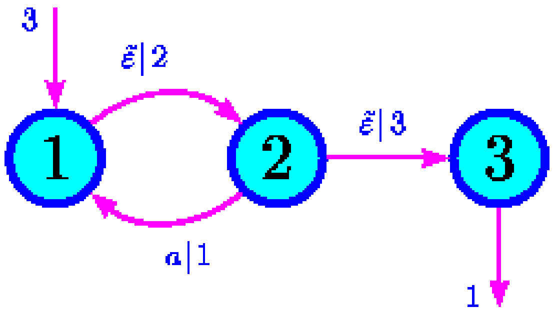
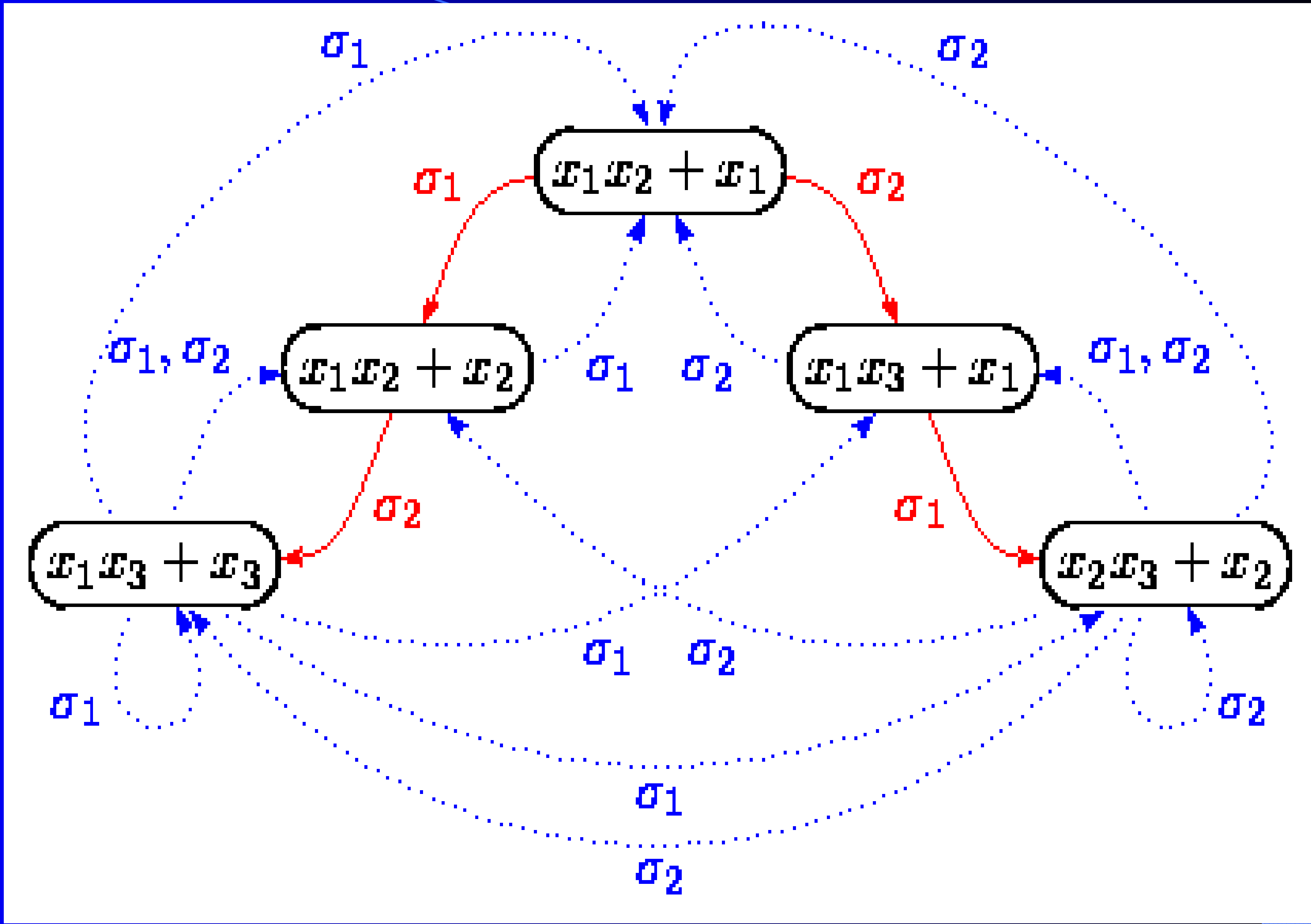


FIG. 2 – Un $\mathbb{N}$-ε-automate $\mathcal{A}_\varepsilon$

$$\text{comportement}(\mathcal{A}_\varepsilon) = 18\tilde{\varepsilon}\left(\sum_{i \in \mathbb{N}} 2^i (a\tilde{\varepsilon})^i\right)\tilde{\varepsilon}.$$



$x_1x_2 + x_1$
$x_1x_2 + x_2$
$x_1x_3 + x_1$
$x_1x_3 + x_3$
$x_2x_3 + x_2$
σ_1
σ_2
σ_1, σ_2

A correct implementation of Schelling's model

Problem : If one scans the board, addressing the inhabitants one after one, result is sensitive to the order of scanning.

Solution : Invent a (combinatorial) data structure which adapted to the parallel structure of the moving intentions of the inhabitants.

Problem : If one scans the board, addressing the inhabitants one after one, result is sensitive to the order of scanning.

Solution : Invent a (combinatorial) data structure which adapted to the parallel structure of the moving intentions of the inhabitants --> this must be a global model.

Combinatorics (mathematics)

Complex
Systems

Information
(comp. sci.)

Physics
(class. quant.)

Thank You