Laguerre-type Derivatives: Dobiński relations and combinatorial identities

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Abstract.
We consider properties of the operators $D(r,M) = a^r(a^\dagger a)^M$ (which we call generalized Laguerre-type derivatives), with $r = 1, 2, \ldots$, $M = 0, 1, \ldots$, where a and $a^\dagger$ are boson annihilation and creation operators respectively, satisfying $[a, a^\dagger] = 1$. We obtain explicit formulas for the normally ordered form of arbitrary Taylor-expandable functions of $D(r,M)$ with the help of an operator relation which generalizes the Dobiński formula. Coherent state expectation values of certain operator functions of $D(r,M)$ turn out to be generating functions of combinatorial numbers. In many cases the corresponding combinatorial structures can be explicitly identified.

1. Introduction

Among many ways of generalizing the ordinary derivative $\frac{d}{dx}$, the notion of the so-called Laguerre derivative [1] seems to be particularly fruitful. The idea is to extend the operator $\frac{d}{dx}$ to a simple homogeneous counterpart D_x, which we define as in [2],[3] (note that here we omit the factor (-1) present in these references):

$$D_x = \frac{d}{dx}x\frac{d}{dx}. \tag{1}$$

In Refs. [1],[2],[3] many important consequences of the replacement $\frac{d}{dx} \longrightarrow D_x$ in the integral transform methods and in the operational calculus were investigated. The link between D_x and the Laguerre polynomials becomes clear if one notices the operational relation (see Eq.(5) of Ref.[2])

$$e^{yD_x}x^n = n!\; y^n L_n(-\frac{x}{y}), \tag{2}$$

where $L_n(z)$ are Laguerre polynomials, which *a posteriori* justifies the name Laguerre derivative for D_x. Eq. (2) permits one to obtain the result of action of $e^{\lambda D_x}$ on various functions, using different generating function of Laguerre polynomials listed in the Section 5.11 of [4]. In particular, using the well known ordinary generating function of $L_n(x)$ (formula 5.11.2.1 for $\alpha = 0$ of [4]) one obtains [5]

$$e^{\lambda D_x}e^{-bx} = \frac{1}{1+b\lambda}\exp\left(-\frac{bx}{1+b\lambda}\right) \tag{3}$$

valid for $|b\lambda| < 1$ [5]. Analogously, using the formula 5.11.2.6 of [4], one gets

$$e^{\lambda D_x}_1 F_1\,[b],[1],x) = \frac{1}{\sqrt{1-\lambda^{b]}}{}_1F_1\left[b],[1],\frac{x}{1-\lambda}\right)}, \tag{4}$$

with ${}_1F_1$ the hypergeometric function which for many values of b specializes to elementary or known special functions. Note that for both these examples the action of $e^{\lambda D_x}$ results in a substitution and a prefactor which is a reminiscent of the so-called Sheffer-type operators [6], [7].

We now employ the operational equivalence

$$[\frac{d}{dx}, x] = 1 \longleftrightarrow [a, a^\dagger] = 1, \tag{5}$$

where a, $a^\dagger$ are boson annihilation and creation operators respectively and rewrite $D_x \longleftrightarrow D$ as

$$D = aa^\dagger a. \tag{6}$$

By going one step further we extend Eq.(6) into the definition of the generalized Laguerre derivative $D(r, M)$ as

$$D(r,M) = a^r(a^\dagger a)^M \quad \sim \quad D_x(r,M) = (\tfrac{d}{dx})^r(x\tfrac{d}{dx})^M, \qquad r = 1,2,\ldots, \qquad M = 0,1,\ldots\,. \tag{7}$$

These operators are the object of our present study. Although the equivalence in Eq.(7) between $D(r, M)$ and $D_x(r, M)$ is formal since the domains of a, $a^\dagger$ and $\frac{d}{dx}$, x are different, we shall show that it provides one with an effective calculational tool.

Since $(a^\dagger a)^M$ conserve the number of bosons, operators $D(r, M)$ act as monomials in boson operators which annihilate r bosons. Recent experiments in quantum optics have shown how one may produce quantum states with specified numbers of photons. This in turn raises the interesting possibility of producing exotic coherent states; that is states other than the standard ones which satisfy $a|z\rangle = z|z\rangle$ [?]. The current work introduces operators whose eigenstates may be used to model new coherent states which, while having many of the features of the standard ones, still permit explicit analytic description. The explicit forms of these new generalized coherent states can be used to evaluate relevant physical parameters, such as the photon distribution and the Mandel parameter, squeezing factors and signal-to-noise ratio, etc.

Much theoretical work has been devoted to the description of nonstandard coherent states; for example, the so-called non-linear coherent states [8], multiphoton coherent states [9] and q-deformed coherent states [10]. The structure embodied in definition Eq.(7) is a special case of the extension of boson operators put forward in the construction of non-linear coherent states [8]. In these references one defines the generalized boson annihilator b by

$$b = af(a^\dagger a) \tag{8}$$

choosing the $f(x)$ that most suits the problem in question. Evidently for this identification $r = 1$ and $f(x) = x^M$. In this case the commutator is equal to

$$[D(1, M), D^\dagger(1, M)] = (a^\dagger a + 1)^{2M+1} - (a^\dagger a)^{2M+1}. \tag{9}$$

This emphasizes the fact that although $D(1, M)$ and $D^\dagger(1, M)$ annihilate and create one boson, respectively, they are *not* canonical boson operators (unless $M = 0$). Eq.(9) is a special case of

$$\begin{aligned}[D(r, M), D^\dagger(r, M)] = (a^\dagger a + r)^{2M} &\left(\sum_{k=1}^{r+1} |\sigma(r+1, k)|(a^\dagger a)^{k-1}\right) \\ &- (a^\dagger a)^{2M} \left(\sum_{k=1}^{r} |\sigma(r, k)|(a^\dagger a)^{k}\right),\end{aligned} \tag{10}$$

where the $\sigma(r, k)$ are Stirling numbers of the first kind [11].

Eq.(10) was obtained by using the following two equations

$$a^r (a)^{\dagger\ r} = \prod_{p=1}^{r} (a^\dagger a + p) \tag{11}$$

$$= \sum_{k=1}^{r+1} |\sigma(r+1, k)|(a^\dagger a)^{k-1} \tag{12}$$

Eq.(11) is readily proved by induction. To prove Eq.(12) we use the generating function for $|\sigma(r + 1, k)|$ in the form [?]

$$\textstyle\prod_{k=1}^{r+1} |\sigma(r+1, k)| x^{r+1-k} \quad = \quad \displaystyle\prod_{p=1}^{r} (1 + px) \tag{13}$$

from which, by substituting $x = 1/n$ and using Eq.(11), Eq.(12) follows.

The basic objective of this work is the investigation of arbitrary powers of $D(r, M)$ which in turn will allow one to evaluate Taylor-expandable functions of $D(r, M)$. We achieve our goal following recently developed methods of construction of normally ordered products [14], [15], [16]. As we shall show, results derived in this way have a combinatorial flavour and lend themselves to a combinatorial interpretation.

The paper is organized as follows. In Section 2 we introduce generalizations of the Stirling and Bell numbers well known from classical combinatorics and relate them to the normally ordered powers of operators $D(r, M)$. These numbers, as shown in Section 3, may be explicitly found using generalized Dobiński relations. In Section 4 we compare calculations of purely analytical origin with those based on methods of graph theory and give a combinatorial interpretation of our results. Examples of various applications of our approach are presented in Section 5 while Section 7 summarizes the paper.

2. Normal ordering: Generalized Stirling and Bell Numbers

The *normally ordered* form of $F(a, a^\dagger)$, denoted by $F_{\mathcal{N}}(a, a^\dagger)$ [17] is obtained by moving all annihilators to the right using the canonical commutation relation of Eq.(5). It satisfies $F_{\mathcal{N}}(a, a^\dagger) = F(a, a^\dagger)$. On the other hand the *double dot* operation $: G(a, a^\dagger) :$ means that we are applying the same ordering procedure but without taking account of the commutation relation. Conventionally the solution to the normal ordering problem is obtained if a function $G(a, a^\dagger)$ is found satisfying

$$F_{\mathcal{N}}(a, a^\dagger) = \, : G(a, a^\dagger) : \quad . \tag{14}$$

A large body of research has been recently devoted to finding the solution of Eq.(14) [18]. A general approach which facilitates a combinatorial interpretation of quantum mechanical quantities is to use the coherent state representation. Standard coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \tag{15}$$

with the number states $|n\rangle$ satisfying $a^\dagger a|n\rangle = n|n\rangle$, $\langle n|n'\rangle = \delta_{n,n'}$ and z complex, are eigenstates of the annihilation operator, i.e. $a|z\rangle = z|z\rangle$. The latter eigenstate property shows that having solved the normal ordering problem Eq.(14) for an operator $F(a, a^\dagger)$ we immediately find

$$\langle z|F_{\mathcal{N}}(a, a^\dagger)|z\rangle = \; G(z, z^*) \; . \tag{16}$$

An early observation on how to extract combinatorial content from normally ordered forms [20] was based on the formula $e^{\lambda a^\dagger a} = \, : e^{a^\dagger a(e^\lambda - 1)} :$, attributed to Schwinger. It led to the identification

$$\langle z|(a^\dagger a)^n|z\rangle \stackrel{z=1}{=} B(n), \tag{17}$$

where the $B(n)$ are conventional Bell numbers described in [11]. Eq.(17) may be taken as a *definition* of the Bell numbers. For Stirling numbers of the second kind [29] we have [11],

$$(a^\dagger a)^n = \sum_{k=1}^{n} S(n,k)(a^\dagger)^k a^k \tag{18}$$

(which may also be used as a practical definition) in terms of which one defines the Bell polynomials by

$$B(n,x) = \sum_{k=1}^{n} S(n,k)x^k. \tag{19}$$

We have extended and developed the coherent state method methodology for operators other than $a^\dagger a$ in [13], [14] and [15].

After the seminal observation by Katriel [20], combinatorial methods found widespread application in this context [22],[14], [15], [16], [7]. We apply these methods to $F(a,a^\dagger) = [D(r,M)]^n$, $n = 1, 2, \ldots$.

Formally, we write $[D(r,M)]^n$ in normally ordered form as

$$[D(r,M)]^n = \left[\sum_{k=M}^{Mn} S_r^{(M)}(n,k)(a^\dagger)^k a^k\right] a^{rn}. \tag{20}$$

Clearly, from Eq.(18) the integers $S_r^{(M)}(n,k)$ are generalizations of the conventional Stirling numbers of the second kind (which are recovered for $r = 0, M = 1$). Analogously to Eq.(19) the numbers $S_r^{(M)}(n,k)$ serve to define the generalized Bell polynomials

$$B_r^{(M)}(n,x) = \sum_{k=M}^{Mn} S_r^{(M)}(n,k)x^k. \tag{21}$$

Finding the explicit form of these generalized Stirling numbers will give the normally ordered form of $[D(r,M)]^n$. We proceed to do this in the next section by use of a generalization of the famous Dobiński formula.

3. Generalized Dobiński formula

We first write Eq.(20) in derivative form as

$$[D_x(r,M)]^n = \left[\sum_{k=M}^{Mn} S_r^{(M)}(n,k)x^k \left(\frac{d}{dx}\right)^k\right] \left(\frac{d}{dx}\right)^{rn}. \tag{22}$$

Acting with the r.h.s. of Eq.(22) on e^x one obtains $B_r^{(M)}(n,x)e^x$. The action of the l.h.s. of Eq.(22) on e^x is obtained by acting with generalized Laguerre derivatives on monomials x^n

$$D_x(r,M)\ x^n = n^{\underline{r}} n^M x^{n-r}, \tag{23}$$

where $n^{\underline{r}} = n(n-1)\ldots(n-r+1)$ is the falling factorial, then extending it to the p-th power

$$[D_x(r,M)]^p\, x^n = \left[\prod_{j=0}^{p-1} (n-rj)^{\underline{r}}(n-rj)^M\right] x^{n-rp}, \tag{24}$$

and next summing up contributions for $x^n/n!$. This leads to the Dobiński-type representation of generalized Bell polynomials [30],[31],[16]:

$$B_r^{(M)}(n,x) = e^{-x}\sum_{l=0}^{\infty}\left[\prod_{i=1}^{n}(l+(i-1)r)\right]^M \frac{x^l}{l!}. \tag{25}$$

The classic Dobiński formula [11] corresponds to $r=0, M=1$

$$B(n,x) = e^{-x}\sum_{l=0}^{\infty}\frac{l^n x^l}{l!}. \tag{26}$$

From Eq.(25) the generalized Stirling numbers are obtained by standard Cauchy multiplication of series

$$S_r^{(M)}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left[\prod_{i=1}^{n}(j+(i-1)r)\right]^M. \tag{27}$$

We point out that Eqs.(25) and (27) are the central results for further calculations. For practical applications it is useful to note that the generalized Stirling and Bell numbers, as well as generalized Bell polynomials, can be expressed through generalized hypergeometric functions ${}_pF_q$. ‡

Below we quote some examples of such relations.

$$S_1^{(M)}(n,k) = \frac{(-1)^k (n!)^M}{k!}\cdot {}_{M+1}F_M([-k,\underbrace{n+1,...,n+1}_{M\ times}],[\underbrace{1,...,1}_{M\ times}],1) \tag{28}$$

$$B_1^{(M)}(n,x) = e^{-x}(n!)^M\cdot {}_MF_M([\underbrace{n+1,...,n+1}_{M\ times}],[\underbrace{1,...,1}_{M\ times}],x) \tag{29}$$

The numbers $B_1^M(n) = B_1^M(n,1)$ can be shown to be related to the numbers $B_{p,p}(n)$ characterizing the normal order of $[(a^(\dagger)^p a^p]$ introduced in Ref.[13] by the formula

$$B_1^M(n) = B_{n,n}(M+1)\ , \tag{30}$$

aseen by comparing Eq.[] in Ref.[13] with the Eq.(25) in the present work.

$$\begin{aligned} B_2^{(M)}(n,x) = 2^{Mn-1}e^{-x}x\Bigg(& x(n!)^M\cdot {}_MF_{M+1}([\underbrace{n+1,...,n+1}_{M\ times}],[\ \underbrace{1,...,1}_{M-1\ times}\ ,3/2,2],x^2/4) \\ & +2\left(\frac{\Gamma(n+1/2)}{\sqrt{\pi}}\right)^M\cdot {}_MF_{M+1}([\underbrace{n+1/2,...,n+1/2}_{M\ times}],[\underbrace{1/2,...,1/2}_{M\ times},3/2],x^2/4)\Bigg) \end{aligned} \tag{31}$$

‡ We use a convenient and self-explanatory notation for the hypergeometric functions of type ${}_qF_p$: ${}_qF_p$([List of p upper parameters],[List of q lower parameters],x).

$$B_3^{(M)}(n) = e^{-1} 3^{Mn} \Bigg(\frac{1}{6}(n!)^M \cdot {}_M F_{M+2}([\underbrace{n+1, ..., n+1}_{M\ times}], [\underbrace{1, ..., 1}_{M-1\ times}, 4/3, 5/3, 2], 1/27)$$

$$+ \left(\frac{\sqrt{3}}{2\pi} \Gamma(2/3)\Gamma(n+1/3) \right)^M \cdot {}_M F_{M+2}([\underbrace{n+1/3, ..., n+1/3}_{M\ times}], [\underbrace{1/3, ..., 1/3}_{M-1\ times}, 1/3, 2/3, 4/3], 1/27)$$

$$+ \frac{1}{2} \left(\frac{\Gamma(n+2/3)}{\Gamma(2/3)} \right)^M \cdot {}_M F_{M+2}([\underbrace{n+2/3, ..., n+2/3}_{M\ times}], [\underbrace{2/3, ..., 2/3}_{M-1\ times}, 2/3, 4/3, 5/3], 1/27) \Bigg) \quad (32)$$

We conjecture that in general $B_r^{(M)}(n)$ is a combination of r hypergeometric functions of typa ${}_M F_{Mr}$ of argument $\frac{1}{r^r}$.

Examples of numbers resulting from Eqs.(29)-(32) for $n = 0, \ldots, 6$ are

$$M = 1 \quad B_1^{(1)}(n) = 1, 2, 7, 34, 209, 1546, 13227,$$

$$M = 2 \quad B_1^{(2)}(n) = 1, 5, 87, 2971, 163121, 12962661, \qquad (33)$$

$$M = 3 \quad B_1^{(3)}(n) = 1, 15, 1657, 513559, 326922081, 363303011071,$$

which are positive integers and as such admit combinatorial interpretation. Indeed, the first two sequences above may be identified as A002720 (which enumerates matching numbers of a perfect graph $K(n,n)$) and A069948 respectively in Ref. [25].

We note in passing that the numbers $B_r^{(M)}(n)$ are solutions to the Stieltjes moment problem, *i.e.* are the n-th moments of positive weight functions on the positive half axis. This can be deduced from their Dobiński-type relations Eq.(25), whose form allows one to obtain the weight functions for any r and M. For the first two sequences in Eq.(33) the Stieltjes weights are given in [25] under their entries.

As a second illustration of our approach we shall apply it to $D(r,1)$. Note that

$$\left[a^r (a^\dagger a)^M \right]^n = :B_r^{(M)}(n, a^\dagger a)\, a^{rn}: , \qquad (34)$$

which upon using the Dobiński relation Eq.(25) for $M = 1$ leads to

$$e^{\lambda D(r,1)} = : \frac{1}{1 - \lambda r a^r} \exp\left(\frac{a^\dagger a}{(1 - \lambda r a^r)^{1/r}} - a^\dagger a \right) : \ . \qquad (35)$$

The operator $D(r,1)$ is of Sheffer-type viewed through hermitean conjugation (see refs [7]) and Eq.(35) can also be obtained through the methods developed in these references (see Appendix). Consequently,

$$\langle z | e^{\lambda D(r,1)} | z \rangle \stackrel{z=1}{=} \frac{1}{1 - r\lambda} \exp\left(\frac{1}{(1 - r\lambda)^{1/r}} - 1 \right) \qquad (36)$$

$$\equiv \sum_{n=0}^{\infty} B_r^{(1)}(n) \frac{\lambda^n}{n!}, \qquad (37)$$

where [25]

$$\langle z|\left(a^r a^\dagger a\right)^n|z\rangle \overset{z=1}{=} B_r^{(1)}(n) = \sum_{p=1}^{n+1} |\sigma(n+1,p)| r^{n-p+1} B(p-1)\ . \qquad (38)$$

In Eq.(38) $\sigma(n,k)$ are the Stirling numbers of the first kind and $B(n)$ are conventional Bell numbers.

Using Eqs.(20) and (29) we obtain for $r=1$ the following formula in a compact notation

$$[D(1,M)]^n = \ (n!)^M : e^{-a^\dagger a} \cdot {}_M F_M([\underbrace{n+1,...,n+1}_{M\ times}],[\underbrace{1,...,1}_{M\ times}],a^\dagger a)a^n : (39)$$

The last formula can be used to normally order $H(\lambda D(1,M))$ for any Taylor-expandable $H(x)$.

4. Combinatorics of normally ordered Laguerre derivatives

In previous Sections we considered the normal ordering of Laguerre derivatives for which the results heavily exploited combinatorial identities stemming from the underlying iterative character of the problem. Indeed, the reordering of the operators a and $a^\dagger$ is a purely combinatorial task which can be interpreted in terms of graphs [26],citearXiv, [28]. Briefly, to each operator in the normally ordered form $H = \sum_{r,s} \alpha_{r,s}\, a^{\dagger\ r} a^s$ one associates a set of one-vertex graphs such that each vertex $\bullet$ carries weight $\alpha_{r,s}$ and has r outgoing and s incoming lines whose free ends are marked with white $\circ$ and gray $\circledcirc$ spots respectively. Multi-vertex graphs are built in a step-by-step manner by adding one vertex at each consecutive step and joining some of its incoming lines with some the free outgoing lines of the graph constructed in the previous step. Additionally, one keeps track of the history by labeling each vertex by the number of steps in which it was introduced. As a result, one obtains a set of increasingly labeled multi-vertex graphs with some free incoming and outgoing lines. It can be shown that normal ordering of powers of operator H can be obtained by enumeration of such structures. Namely, the coefficient of $a^{\dagger\ k} a^l$ in the normally ordered form of the operator H^n is obtained by counting all possible graphs with n vertices $\bullet$ and having k white $\circ$ and l gray $\circledcirc$ spots respectively. For illustration, we give two examples of Laguerre derivatives $D(1,1) = aa^\dagger a = a^\dagger a^2 + a$ and $D(2,1) = a^2 a^\dagger a = a^\dagger a^3 + 2\,a^2$ and their graph representation leading to the solution of the normal ordering problem by simple enumeration, see Fig. 1. One should compare these "graphical results" with the explicit formulas of Eqs.(27) and (25) or the expansion coefficients of the generating function in Eq.(36) for $r = 1,2$ and $M = 1$. Thus, using Eq.(**??**) the coefficients multiplying the operators in Fig.(1a) are the first two terms in $B_1^{(1)}(n) = 2,7,34,209$ for $n = 1,2,\ldots$ (A002720). Similar coefficients in Fig. (1b) are the first two terms in $B_2^{(1)}(n) = 3,16,121,1179$ for $n = 1,2,\ldots$ (A121629).

Figure 1. Building blocks (in the inset) and the associated graphs of order $n = 1, 2$ for Laguerre derivatives: (a) $D(1, 1)$ and (b) $D(2, 1)$.

5. Examples

1. For $r = M = 1$, i.e. for $D(1, 1) = aa^\dagger a$ one obtains [32]:

$$[D(1, 1)]^n = n! : L_n(-a^\dagger a) : a^n, \tag{40}$$

where $L_n(y)$ are Laguerre polynomials and Eq.(40) is derived from Eq.(39) and using the definition of $L_n(y)$ via the function ${}_1F_1$. Then

$$e^{\lambda D(1,1)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [D(1,1)]^n$$
$$= \ : \sum_{n=0}^{\infty} L_n(-a^\dagger a)(\lambda a)^n : = : \frac{1}{1-\lambda a} \exp\left(\frac{\lambda a^\dagger a^2}{1-\lambda a}\right) : , \tag{41}$$

(compare Eq. (3)), where in Eq.(41) we have used the ordinary generating function (o.g.f.) for the Laguerre polynomials [4].

Using other generating functions listed on p.704 of Ref.[4] one can derive further formulas of type Eq.(41). (In a), b) and c) below: $\lambda \neq 0,\ p = 1, 2, ...$).

a) Consider the formula 5.11.2.8 of [4]:

$$\sum_{n=0}^{\infty} \binom{n+p}{n} t^n L_{n+p}(x) = \frac{1}{(1-t)^{p+1}} e^{-\frac{tx}{1-t}} L_p\left(\tfrac{x}{1-t}\right) \tag{42}$$

Using Eq.(40) we obtain the normally ordered form of $[\lambda D(1,1)]^p \exp(\lambda D(1,1))/p!$:

$$\frac{1}{p!}\sum_{n=0}^{\infty} \frac{[\lambda D(1,1)]^{n+p}}{n!} = \sum_{n=0}^{\infty} \frac{(n+p)!}{p!n!} : L_{n+p}(-a^\dagger a)(\lambda a)^{n+p} : \tag{43}$$
$$= \ : \frac{1}{(1-\lambda a)^{p+1}} e^{\frac{\lambda a^\dagger a^2}{1-\lambda a}} L_p(-\tfrac{a^\dagger a}{1-\lambda a}) : (\lambda a)^p \ . \tag{44}$$

b) Formula 5.11.2.9 of [4]w for $\alpha = 0$:

$$\sum_{n=0}^{\infty} \frac{(p+1)_n}{(1)_n} t^n L_n(x) = \frac{1}{(1-t)^{p+1}} e^{-\frac{tx}{1-t}} L_p(\tfrac{tx}{1-t}), \tag{45}$$

gives the normally ordered form

$$e^{\lambda D(1,1)} L_p(-\lambda D(1,1)) = : \frac{1}{(1-\lambda a)^{p+1}} e^{-\frac{\lambda a^\dagger a^2}{1-\lambda a}} L_p(-\tfrac{\lambda a^\dagger a^2}{1-\lambda a}) : \ . \tag{46}$$

In Eq.(45) $(y)_k$ is the Pochhammer symbol.

c) Similarly, formula 5.11.2.6 of [4] for $\alpha = 0$ provides the normal ordering of

$${}_1F_1\,[b],[1],(1,1)) =: \frac{1}{(1-\lambda a)^b}\,{}_1F_1\left[b],[1],\frac{\lambda a^\dagger a^2}{1-\lambda a}\right) : \ . \tag{47}$$

which for b integer and half-integer can be written down in terms of known functions. Examples are:

$${}_1F_1\,[3],[1],(1,1)) =: \frac{1}{(1-\lambda a)^3} L_2\left(\frac{\lambda a^\dagger a^2}{1-\lambda a}\right) \exp\left(\frac{\lambda a^\dagger a^2}{1-\lambda a}\right) : \ . \tag{48}$$

$${}_1F_1\left[\frac{3}{2}],[1],(1,1)\right) =: \frac{1}{(1-\lambda a)^{\frac{3}{2}}} \exp\frac{1}{2}\Theta\right)\left(I_0(\frac{\Theta}{2})(1+\Theta) +_1 (\frac{\Theta}{2})\right) : \tag{49}$$

where $\Theta = \frac{\lambda a^\dagger a^2}{1-\lambda a}$.

2. The normal order of the modified Bessel function of the first kind $I_0\left(2(\lambda D(1,1))^{1/2}\right)$ may be derived:

$$I_0(2(\lambda D(1,1))^{1/2}) = :\sum_{n=0}^{\infty}\frac{L_n(-a^\dagger a)(\lambda a)^n}{n!}:$$
$$= : e^{\lambda a} J_0(2\sqrt{\lambda(-aa^\dagger a)}) := : e^{\lambda a} I_0(2\sqrt{\lambda a^\dagger a^2}) : , \qquad (50)$$

where in the last line we have used the exponential generating function (e.g.f.) of Laguerre polynomials [4]. Analogous formula for $J_0\left(2(\lambda D(1,1))^{1/2}\right)$ reads

$$J_0(2(\lambda D(1,1))^{1/2}) = :\sum_{n=0}^{\infty}\frac{L_n(-a^\dagger a)(-\lambda a)^n}{n!}:$$
$$= : e^{-\lambda a} I_0(2\sqrt{\lambda a^\dagger a^2}) : \;. \qquad (51)$$

3. We quote here the eigenfunctions of $D_x(r,M)$ with eigenvalue 1 satisfying $D_x(r,M)E(r,M;x) = E(r,M,x)$, with the following r boundary conditions:

$$E(r,M;0) = 1, \qquad \frac{\mathrm{d}}{\mathrm{dx}^r}E(r,M;x)|_{x=0} = 0, \quad p = 1,2,3,\ldots,(p-1) \qquad (52)$$

which are[**?**]

$$E(r,M;x) = {}_0F_{M+r-1}([\;],[\underbrace{1/r,2/r,...,(r-1)/r}_{r-1\ times},\underbrace{1,...,1}_{M\ times}],x^r/r^{r+M}) \qquad (53)$$

Useful normal ordering formulas can be obtained by applying the Dobiński relations to the eigenfunctions of $D_x(1,M)$ *with the argument taking operator values*, see Eq.(53), i.e. $E(r,M;D_x(1,M))$. We briefly show the calculation, in boson notation, for $E(1,2;\lambda D(1,2)) ={}_0F_2([\;],[1,1];\lambda D(1,2))$, see Eq.(36) :

$${}_0F_2([\;],[1,1];\lambda D(1,2)) = \sum_{n=0}^{\infty}\frac{\lambda^n}{(n!)^3}[a(a^\dagger a)^2]^n \qquad (54)$$
$$= \;: e^{-a^\dagger a}\sum_{l=0}^{\infty}\frac{(a^\dagger a)^l}{l!}\sum_{n=0}^{\infty}\frac{((n+l)!)^2}{(l!)^3}(\lambda a)^n : \qquad (55)$$
$$= \;: e^{-a^\dagger a}\sum_{l=0}^{\infty}\frac{(a^\dagger a)^l}{l!}\;{}_2F_2([1+l,1+l],[1,1],\lambda a) \qquad (56)$$

and similarly

$$E(1,M;\lambda D(1,M)) = {}_0F_M([\;],[\underbrace{1,1,...1}_{M\ times}],\lambda D(1,M)) = \qquad (57)$$
$$= : e^{-a^\dagger a}\sum_{l=0}^{\infty}\frac{(a^\dagger a)^l}{l!}\;{}_MF_M([\underbrace{1+l,1+l,...,1+l}_{M\ times}],[\underbrace{1,1,...,1}_{M\ times}],\lambda a) : , \qquad (58)$$

which indicates a pattern appearing in the course of this procedure.

Indeed, by evaluating the coherent state expectation value of Eq.(59) between $\langle z = 1|\ldots|z = 1\rangle$ in the spirit of Eq.(**??**) we furnish the hypergeometric generating functions of the numbers $B_1^{(M)}$ as then

$$e^{-1}\sum_{l=0}^{\infty}\frac{1}{l!}\ {}_MF_M([\underbrace{l+1,l+1,\ldots,l+1}_{M\ times}],[\underbrace{1,1,...1}_{M\ times}],\lambda)) = \tag{59}$$

$$= \sum_{l=0}^{\substack{(M)\\ 1](n)\frac{\lambda^l}{(l!)^{M+1}}\ .}} \tag{60}$$

In spite of its apparent complexity the l.h.s of the above equation can be straighforwardly handled by computer algebra systems.

6. Appendix

We derive Eq.(3) with the help of methods developed in Ref.[]. First, observe that $D(r,1) = a^\dagger a^{r+1} + ra^r$ from which it follows that $D^\dagger(r,1) = (a^\dagger)^{r+1}a + r(a^\dagger)^r$ is the operator of Sheffer-type: $D^\dagger(r,1) = v(a^\dagger) + q(a^\dagger)a$ with $q(x) = x^{r+1}$ and $v(x) = rx^r$. The normally ordered form of $\exp(\lambda D^\dagger(r,1))$ is obtained by solving the linear differential equations (Eqs.(2) and (3) of Ref.[]) for $T(\lambda,x)$ and $g(\lambda,x)$ yielding

$$T(\lambda,a^\dagger) = \frac{a^\dagger}{(1-\lambda r(a^\dagger)^r)^{1/r}}, \tag{61}$$

and

$$g(\lambda,a^\dagger) = \frac{1}{1-\lambda r(a^\dagger)^r}. \tag{62}$$

According to Eq.(29) of [] the normally ordered form of $e^{\lambda D(r,1)}$ is

$$e^{\lambda D(r,1)} = \left[e^{\lambda D^\dagger(r,1)}\right]^\dagger = :g(\lambda,a)e^{a^\dagger(T(\lambda,a)-a)}: \tag{63}$$

which gives Eq.(3).

7. Conclusions and outlook

We have used generalized Dobiński relations to investigate the properties of Laguerre-type differential operators. We provided a large number of operational formulas involving functions of Laguerre derivatives, which can alternatively be applied within the boson language. The framework developed above enables one to construct and analyze new coherent states relevant to nonlinear quantum optics, which is a subject of forthcoming research.

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