# ONE-PARAMETER GROUPS AND COMBINATORIAL PHYSICS

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In this communication, we consider the *normal ordering* of operators of the type

$$\Omega = \sum_{\alpha+\beta=r} c_{\alpha,\beta}(a^+)^\alpha a(a^+)^\beta, \quad \alpha,\beta,r \text{ integers}$$

where $a$ (*resp.* $a^+$) is a boson annihilation (*resp.* creation) operator; these satisfy $[a,a^+] \equiv aa^+ - a^+a = 1$, and for the purposes of this note may be thought of as $a \equiv d/dx$ and $a^+ \equiv x$. We discuss the integration of the one-parameter groups $e^{\lambda\Omega}$ and their combinatorial by-products. In particular we show how these groups can be realized as groups of substitutions with prefactor functions.

## 1. Introduction

This text is the continuation of a series of works on the combinatorial and analytic aspects of normal forms of boson strings[1,2,3,4,5,11,13,14,15,20].
Let $w \in \{a,a^+\}^*$ be a word in the letters $\{a,a^+\}$ (i.e. a boson string), and define (as in Blasiak, Penson and Solomon[1,2,3,4]) by $r$, $s$, respectively $|w|_{a^+}$ (the number of creation operators), $|w|_a$ (the number of annihilation operators) and $m = min(r,s)$ then the normal form of $w^n$ is

$$\mathcal{N}(w^n) = \sum_{k=0}^{nm} S_w(n, nm-k)(a^+)^{nr-k}a^{ns-k} \tag{1}$$

which, according to whether the excess $e = r-s$ is positive or negative, reads

$$(a^+)^{ne}\left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k\right) \text{ or } \left(\sum_{k=0}^{\infty} S_w(n,k)(a^+)^k a^k\right)(a)^{n|e|} \tag{2}$$

It has been observed that the numbers $S_w$ are *rook numbers* [12,24] .

Consider, as examples, the upper-left corner of the following (doubly infinite) matrices.

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \tag{3}$$

For $w = a^+aa^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\,0\cdots \\ 1 & 1 & 0 & 0 & 0 & 0\,0\cdots \\ 2 & 4 & 1 & 0 & 0 & 0\,0\cdots \\ 6 & 18 & 9 & 1 & 0 & 0\,0\cdots \\ 24 & 96 & 72 & 16 & 1 & 0\,0\cdots \\ 120 & 600 & 600 & 200 & 25 & 1\,0\cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36\,1\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\,\vdots\,\ddots \end{bmatrix} \tag{4}$$

For $w = a^+aaa^+a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,0\cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0\,0\cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0\,0\cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0\,0\cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52\,1\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\,\vdots\,\ddots \end{bmatrix} \tag{5}$$

**Remark 1.1.** In each case, the matrix $S_w$ is of staircase form and the "step" depends on the number of $a$'s in the word $w$. More precisely, due to equation (1) one can prove that each row ends with a 'one' in the cell $(n, nr)$, where $r = |w|_a$ and we number the entries from $(0, 0)$. Thus all the matrices are row-finite and unitriangular iff $r = 1$, which case will be of special interest in the following. Moreover, the first column is $(1, 0, \cdots, 0, \cdots, 0, \cdots)$ iff $w$ ends with an $a$ (this means that $\mathcal{N}(w^n)$ has no constant term for all $n > 0$).

In this communication, we concentrate on boson strings and more generally (homogeneous) boson operators involving only one "$a$". We will see that this case is closely related to one-parameter substitution groups and their conjugates.

The structure of the paper is the following.

In section 2 we define the framework for our transformation matrices (spaces, topology and decomposition), then we concentrate on the *Riordan subgroup* (i.e. transformations which are substitutions with prefactor functions) and adapt the classical theory (*Sheffer condition*) to the present context. In section 3 we analyse *unipotent* transformations (Lie group structure and combinatorial examples). The divisibility property of the group of unipotent transformations tells us that every transformation is embedded

in a one-paramater group. This will be analysed in section 4 from the formal and analytic points of view. Section 5 is devoted to some concluding remarks and further interesting possibilities.

## 2. The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of sequence transformations

Let $\mathbf{C}^{\mathbf{N}}$ be the vector space of all complex sequences, endowed with the Frechet product topology [23]. It is easy to check that the algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of all continuous operators $\mathbf{C}^{\mathbf{N}} \to \mathbf{C}^{\mathbf{N}}$ is the space of *row-finite* matrices with complex coefficients. Such a matrix $M$ is indexed by $\mathbf{N} \times \mathbf{N}$ and has the property that, for every fixed row index $n$, the sequence $(M(n,k))_{k\geq 0}$ has finite support. For a sequence $A = (a_n)_{n\geq 0}$, the transformed sequence $B = MA$ is given by $B = (b_n)_{n\geq 0}$ with

$$b_n = \sum_{k\geq 0} M(n,k) a_k \tag{6}$$

Remark that the combinatorial coefficients $S_w$ defined above are indeed row-finite matrices.

We may associate a univariate series with a given sequence $(a_n)_{n\in\mathbf{N}}$, using a sequence of prescribed (non-zero) denominators $(d_n)_{n\in\mathbf{N}}$ , as follows:

$$\sum_{n\geq 0} a_n \frac{z^n}{d_n}. \tag{7}$$

For example, with $d_n = 1$, we get the ordinary generating functions (OGF), with $d_n = n!$, we get the exponential generating functions (EGF) and with $d_n = (n!)^2$, the doubly exponential generating functions (DEGF) and so on. Thus, once the denominators have been chosen, to every (linear continuous) transformation of generating functions, one can associate a corresponding matrix.

The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ possesses many interesting subalgebras and groups, such as the algebra of lower triangular transformations $\mathcal{T}(\mathbf{N},\mathbf{C})$, the group $\mathcal{T}_{inv}(\mathbf{N},\mathbf{C})$ of invertible elements of the latter (which is the set of infinite lower triangular matrices with non-zero elements on the diagonal), the subgroup of unipotent transformations $\mathcal{UT}(\mathbf{N},\mathbf{C})$ (i.e. the set of infinite lower triangular matrices with elements on the diagonal all equal to 1) and its Lie algebra $\mathcal{NT}(\mathbf{N},\mathbf{C})$, the algebra of locally nilpotent transformations (with zeroes on the diagonal). One has the inclusions (with $\mathcal{D}_{inv}(\mathbf{N},\mathbf{C})$, the set of invertible diagonal matrices)

$$\mathcal{UT}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}(\mathbf{N},\mathbf{C}) \subset \mathcal{L}(\mathbf{C}^{\mathbf{N}})$$

$$\mathcal{D}_{inv}(\mathbf{N},\mathbf{C}) \subset \mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) \text{ and } \mathcal{NT}(\mathbf{N},\mathbf{C}) \subset \mathcal{L}(\mathbf{C}^{\mathbf{N}}). \tag{8}$$

We remark that $\mathcal{T}_{inv}(\mathbf{N},\mathbf{C}) = \mathcal{D}_{inv}(\mathbf{N},\mathbf{C}) \bowtie \mathcal{UT}(\mathbf{N},\mathbf{C})$ because $\mathcal{UT}$ is normalized by $\mathcal{D}_{inv}$ and $\mathcal{T}_{inv} = \mathcal{D}_{inv}.\mathcal{UT}$ (every invertible transformation is the product of its diagonal by a unipotent transformation).

We now examine an important class of transformations of $\mathcal{T}$ as well as their diagonals: the substitutions with prefactor functions which, from now on, will be called substitutions with prefunctions.

### 2.1. *Substitutions with prefunctions*

Let $(d_n)_{n\geq 0}$ bet a fixed set of denominators. We consider, for a generating function $f$, the transformation

$$\Phi_{g,\phi}[f](x) = g(x)f(\phi(x)). \tag{9}$$

The matrix of this transformation $M_{g,\phi}$ is given by the transforms of the monomials $\frac{x^k}{d_k}$ hence

$$\sum_{n\geq 0} M_{g,\phi}(n,k)\frac{x^n}{d_n} = \Phi_{g,\phi}\left[\frac{x^k}{d_k}\right] = g(x)\frac{\phi(x)^k}{d_k}. \tag{10}$$

If $g, \phi \neq 0$ (otherwise the transformation is trivial), we can write

$$g(x) = a_l\frac{x^l}{d_l} + \sum_{r>l} a_r\frac{x^r}{d_r},\ \phi(x) = \alpha_m\frac{x^m}{d_m} + \sum_{s>m}\alpha_s\frac{x^s}{d_s} \tag{11}$$

with $a_l, \alpha_m \neq 0$ and then, by (9,10,11),

$$\Phi_{g,\phi}\left[\frac{x^k}{d_k}\right] = a_l(\alpha_m)^k\frac{x^{l+mk}}{d_l d_m^k d_k} + \sum_{t>l+mk} b_t\frac{x^t}{d_t}. \tag{12}$$

One then has

$$M_{g,\phi} \text{ is row} - \text{finite} \Longleftrightarrow \phi \text{ has no constant term} \tag{13}$$

and in this case it is always lower triangular.

The converse is true in the following sense. Let $T \in \mathcal{L}(\mathbf{C}^{\mathbf{N}})$ be a matrix with non-zero two first columns and suppose that the first index $n$ such that $T(n,k) \neq 0$ is less for $k = 0$ than $k = 1$ (which is, from (10) the case when $T = M_{g,\phi}$). Set

$$g(x) := d_0\sum_{n\geq 0} T(n,0)\frac{x^n}{d_n};\ \phi(x) := \frac{d_1}{g(x)}\sum_{n\geq 0} T(n,1)\frac{x^n}{d_n} \tag{14}$$

then $T = M_{g,\phi}$ iff, for all $k$,

$$\sum_{n\geq 0} T(n,k)\frac{x^n}{d_n} = g(x)\frac{\phi(x)^k}{d_k}. \tag{15}$$

**Remark 2.1.** Eq. (10) is called the *Sheffer condition* (see [16,20,21,25]) and, for EGF ($d_n = n!$) it amounts to stating that

$$\sum_{n,k\geq 0} T(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)}. \tag{16}$$

From now on, we will suppose that $\phi$ has no constant term ($\alpha_0 = 0$). Moreover $M_{g,\phi} \in \mathcal{T}_{inv}$ if and only if $a_0, \alpha_1 \neq 0$ and then the diagonal term with address $(n,n)$ is $\frac{a_0}{d_0}\left(\frac{\alpha_1}{d_1}\right)^n$. We get

$$M_{g,\phi} \in \mathcal{UT} \Longleftrightarrow \frac{a_0}{d_0} = \frac{\alpha_1}{d_1} = 1. \tag{17}$$

In particular for the EGF and the OGF, we have the condition that

$$g(x) = 1 + \text{higher terms } \textit{and } \phi(x) = x + \text{higher terms.} \tag{18}$$

**Note 2.1.** In classical combinatorics (for OGF and EGF), the matrices $M_{g,\phi}(n,k)$ are known as *Riordan matrices* (see [16,17] for example).

## 3. Unipotent transformations

### 3.1. *Lie group structure*

We first remark that $n \times n$ truncations (i.e. taking the $[0..n] \times [0..n]$ submatrix of a matrix) are algebra morphisms

$$\tau_n : \mathcal{T}(\mathbf{N},\mathbf{C}) \to \mathcal{M}([0..n]\times[0..n],\mathbf{C}). \tag{19}$$

We can endow $\mathcal{T}(\mathbf{N},\mathbf{C})$ with the Frechet topology given by these morphisms. We will not develop this point in detail here, but this topology is metrisable, complete and given by the following convergence criterion:

*a sequence $(M_k)$ of matrices in $\mathcal{T}(\mathbf{N},\mathbf{C})$ converges iff*
*for all fixed $n \in \mathbf{N}$*
*the sequence of truncated matrices $(\tau_n(M_k))$ converges.*

This topology is compatible with the $\mathbf{C}$-algebra structure of $\mathcal{T}(\mathbf{N},\mathbf{C})$.

The two maps $\exp : \mathcal{NT}(\mathbf{N},\mathbf{C}) \to \mathcal{UT}(\mathbf{N},\mathbf{C})$ and $\log : \mathcal{UT}(\mathbf{N},\mathbf{C}) \to \mathcal{NT}(\mathbf{N},\mathbf{C})$ are continous and mutually inverse.

### 3.2. *Examples*

3.2.1. *Provided by the exponential formula*

The "classical exponential formula" [7,9,21] tells us the following: consider a class [a] of finite labelled graphs $\mathcal{C}$ and denote by $\mathcal{C}^c$ the subclass of connected graphs in $\mathcal{C}$. Then the exponential generating series of $\mathcal{C}$ and $\mathcal{C}^c$ are related as follows:

$$EGF(\mathcal{C}) = e^{EGF(\mathcal{C}^c)}. \tag{20}$$

The following examples give us some insight into why combinatorial matrices of the type:

$T(n,k) =$ *Number of graphs of* $\mathcal{C}$ *on* $n$ *vertices having* $k$ *connected components*

give rise to substitution transformations.

**Example 3.1. Stirling numbers.**

From now on, let $v(\Gamma),\ c(\Gamma)$ denote parameters $n,\ k$ above, namely the number of vertices of $\Gamma$ and the number of connected components of $\Gamma$. Consider first the class of graphs $\Gamma$ of equivalence relations. Then using the statistics $x^{v(\Gamma)}y^{c(\Gamma)}$ we get

$$\sum_{n,k\geq 0} S(n,k)\frac{x^n}{n!}y^k = \sum_{\substack{\text{all equivalence}\\ \text{graphs } \Gamma}} \frac{x^{v(\Gamma)}}{v(\Gamma)!}y^{c(\Gamma)} =$$

$$\exp\left(\sum_{\Gamma\ connected} \frac{x^{v(\Gamma)}}{v(\Gamma)!}y^{c(\Gamma)}\right) = \exp\left(\sum_{n\geq 1} y\frac{x^n}{n!}\right) = e^{y(e^x-1)} \tag{21}$$

and we will see that the transformation associated with the matrix $S(n,k)$ is $f(x) \to f(e^x-1)$.

**Example 3.2. Idempotent numbers.**
We consider here the class of graphs of endofunctions (i.e. functions from a finite set into itself). Then, using the statistics $x^{v(\Gamma)}y^{c(\Gamma)}$ and denoting by $I(n,k)$ the number of graphs of idempotent endofunctions (i.e. being equal to their compositional square) of a given set with $n$ elements and having $k$ connected components, we get

[a] Closed under relabelling (of the vertices), disjoint union, and taking connected components.

$$\sum_{n,k\geq 0} I(n,k)\frac{x^n}{n!}y^k = \sum_{\substack{\text{all graphs } \Gamma \text{ of} \\ \text{endofunctions in the class } (1,2)}} \frac{x^{v(\Gamma)}}{v(\Gamma)!}y^{c(\Gamma)} =$$

$$\exp\left(\sum_{\Gamma\ connected} \frac{x^{v(\Gamma)}}{v(\Gamma)!}y^{c(\Gamma)}\right) = \exp\left(\sum_{n\geq 1} y\frac{nx^n}{n!}\right) = e^{yxe^x} \quad . \quad (22)$$

Corresponding to these numbers we get the (doubly) infinite matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 24 & 12 & 1 & 0 & 0 & \cdots \\ 0 & 5 & 80 & 90 & 20 & 1 & 0 & \cdots \\ 0 & 6 & 240 & 540 & 240 & 30 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (23)$$

and we will see that the transformation associated with this matrix is $f(x) \to f(xe^x)$

**Remark 3.1.** i) More generally, call the *Burnside class* with parameters $a < b$ ($a, b$ integers), the class of graphs of (finite) endofunctions such that $f^a = f^b$ (compositional power) and $Burn_{a,b}(n,k)$, the corresponding numbers as in section (3.2.1). Then, exponential formula shows us that the matrices $Burn_{a,b}$ are all matrices of substitutions.
ii) For the special cases $a = 1,\ b = l+1$, the mixed series (EGF in $x$ and OGF in $y$) is explicit and reads

$$\sum_{n,k\geq 0} Burn_{1,l+1}(n,k)\frac{x^n}{n!}y^k = e^{y\left(\sum_{h|l} \frac{x^h}{h} e^{hx}\right)} \quad (24)$$

3.2.2. *Normal ordering powers of boson strings*

To get unipotent matrices, one has to consider boson strings with only one annihilation operator. In the introduction, we have given examples with $a^+a, a^+aa^+$ (the matrix of the third string, $a^+aaa^+a^+$, with two annihilators, is not unipotent). Such a string is then $w = (a^+)^{r-p}a(a^+)^p$ and we will see shortly that

- if $p = 0$, $S_w(n,k)$ is the matrix of a unipotent substitution

- if $p > 0$, $S_w(n,k)$ is the matrix of a unipotent substitution with prefunction

To cope with the matrices coming from the normal ordering of powers of boson strings we have to make a small detour to analysis and formal groups.

## 4. One-parameter subgroups of $UT(\mathbf{N},\mathbf{C})$

### 4.1. *Exponential of elements of* $NT(\mathbf{N},\mathbf{C})$

Let $M = I + N \in UT(\mathbf{N},\mathbf{C})$ ($I = I_{\mathbf{N}}$ is the indentity matrix). One has

$$M^t = \sum_{k\geq 0} \binom{t}{k} N^k \tag{25}$$

where $\binom{t}{k}$ is the generalized binomial coefficient defined by

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}. \tag{26}$$

One can see that, for $k \leq n$, due to the local nilpotency of $N$, the matrix coefficient $M^t(n,k)$ is well defined and, in fact, a polynomial of degree $n-k$ in $t$ (for $k > n$, this coefficient is 0). We have the additive property $M^{t_1+t_2} = M^{t_1}M^{t_2}$ and the correspondence $t \to M^t$ is continuous. Conversely, let $t \to M_t$ be a continous local one-parameter group in $UT(\mathbf{N},\mathbf{C})$; that is, for some real $\epsilon > 0$

$$|t_1| \text{ and } |t_2| < \epsilon \Longrightarrow M_{t_1}M_{t_2} = M_{t_1+t_2} \tag{27}$$

then there exists a unique matrix $H \in NT(\mathbf{N},\mathbf{C})$ such that $M_t = exp(tH)$. (This may be proved using the projections $\tau_n$ and the classical theorem about continuous one-parameter subgroups of Lie groups, see [10], for example).
When $M_t = M^t$ is defined by formula (25) we have
$H = log(I+N) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} N^k$.

The mapping $t \to M^t$ will be called a *one parameter group* of $UT(\mathbf{N},\mathbf{C})$.

**Proposition 4.1.** *Let $M$ be the matrix of a substitution with prefunction; then so is $M^t$ for all $t \in \mathbf{C}$.*

The proof will be given in a forthcoming paper and uses the fact that "to be the matrix of a substitution with prefunction" is a property of polynomial type. But, using composition, it is straightforward to see that $M^t$ is the matrix of a substitution with prefunction for all $t \in \mathbf{N}$. Thus, using a "Zariski-type" argument, we get the result that the property is true for all $t \in \mathbf{C}$.

### 4.2. *Link with local Lie groups : Straightening vector fields on the line*

We first treat the case $p = 0$ of subsection (3.2.2). The string $(a^+)^r a$ corrresponds, in the Bargmann-Fock representation, to the vector field $x^r \frac{d}{dx}$ defined on the whole line.
Now, we can try (at least locally) to straighten this vector field by a diffeomorphism $u$ to get the constant vector field (this procedure has been introduced by G. Goldin in the context of current algebras[8]). As the one-parameter group generated by a constant field is a shift, the one-parameter (local) group of transformations will be, on a suitable domain

$$U_\lambda[f](x) = f\left(u^{-1}\left(u(x) + \lambda\right)\right). \tag{28}$$

Now, we know from section (4.1) that, if two one-parameter groups have the same tangent vector at the origin, then they coincide *(tangent paradigm).* Direct computation gives this tangent vector :

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} f\left(u^{-1}\left(u(x) + \lambda\right)\right) = \frac{1}{u'(x)} f'(x) \tag{29}$$

and so the local one-parameter group $U_\lambda$ has $\frac{1}{u'(x)}\frac{d}{dx}$ as tangent vector field. Here, we have to solve $\frac{1}{u'(x)} = x^r$ in order to get the diffeomorphism $u$. In the case $r \neq 1$, we have (with $\mathcal{D} = ]0, +\infty[$ as domain)

$$u(x) = \frac{x^{1-r}}{1-r} = y;\ u^{-1}(y) = ((1-r)y)^{\frac{1}{1-r}} \tag{30}$$

and

$$e^{\lambda x^r \frac{d}{dx}}[f](x) = f\left(\frac{x}{(1 - \lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}\right) \tag{31}$$

The substitution factor $s_\lambda(x) = \frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}}$ has been already obtained by other means in [1]. The computation is similar for the case

when $r = 1$ and, for this case, we get

$$e^{\lambda x \frac{d}{dx}}[f](x) = f\left(e^{\lambda} x\right) \tag{32}$$

with $s_\lambda(x) = e^\lambda x$ as substitution factor.
These first examples are summarized in the following table

| $r$ | $s_\lambda(x)$ | Name |
|---|---|---|
| 0 | $x + \lambda$ | Shift |
| 1 | $e^\lambda x$ | Dilation |
| 2 | $\frac{x}{1-\lambda x}$ | Homography |
| 3 | $\frac{x}{\sqrt{1-2\lambda x^2}}$ | - |

**Comment 4.1.** If one uses classical analysis (i.e. convergent Taylor series), one must be careful about the domain where the substitutions are defined and the one-parameter groups are defined only locally.
For each of these examples, one can check by hand that for suitable (and small) values of $\lambda, \mu$, one has $s_\lambda(s_\mu(x)) = s_{\lambda+\mu}(x)$ (one-parameter group property).
It is possible to avoid discussion of the domains by considering $\lambda$, $\mu$ as new variables and applying the "substitution principle"; namely by claiming that it is possible to substitute a series without constant term in a series (in the algebra $\mathbf{C}[[x, \lambda, \mu]]$).

Using the same method, one can start with more complicated operators. Examples and substitution factors are given below

| Operator | Substitution Factor | Description |
|---|---|---|
| $\left(1 + (a^+)^2\right) a$ | $s_\lambda(x) = \dfrac{xcos(\lambda) + sin(\lambda)}{cos(\lambda) - xsin(\lambda)}$ | One-parameter group of homographies |
| $\dfrac{\sqrt{1 + (a^+)^2}}{a^+}$ | $s_\lambda(x) = \sqrt{x^2 + 2\lambda\sqrt{1 + x^2} + \lambda^2}$ | Composition of quadratic direct and inverse functions |

### 4.3. *Case $p > 0$: another conjugacy trick and a surprising formula*

Regarding vector fields as infinitesimal generators of one-parameter groups leads to conjugacy since, if $U_\lambda$ is a one-parameter group of transformation, so too is $VU_\lambda V^{-1}$ ($V$ being a continuous invertible operator). We could formally consider $(a^+)^{r-p}a(a^+)^p;\ p > 0$ as conjugate to $\left((a^+)^r a\right)$ in this context. More generally, supposing all the terms well-defined, if

$$\Omega = u_1(x)\frac{d}{dx}u_2(x) = \frac{1}{u_2(x)}\left(u_1(x)u_2(x)\frac{d}{dx}\right)u_2(x)$$

then

$$e^{\lambda\Omega} = \frac{1}{u_2(x)}\left(e^{\lambda u_1(x)u_2(x)\frac{d}{dx}}\right)u_2(x) \tag{33}$$

This rather surprising formula (33) may be understood as an operator equality.

Now, the tangent paradigm (see section 4.2) tells us that, if we adjust this tangent vector to coincide with $x^{r-p}\frac{d}{dx}x^p$ (recall that the original problem was the integration of the operator $\Omega = (a^+)^{r-p}a(a^+)^p;\ p > 0$), then we get the right one-parameter group. Using this "conjugacy trick" we get

$$e^{\lambda\Omega}[f](x) = \left(\frac{s_\lambda(x)}{x}\right)f(s_\lambda(x)) \text{ with } s_\lambda(x) = \frac{x}{(1-\lambda(r-1)x^{r-1})^{\frac{1}{r-1}}} \tag{34}$$

**Remark 4.1.** (i) It can be checked that, if $s_\lambda(x)$ is a substitution factor (i.e. at least locally $s_\lambda(s_\mu(x)) = s_{\lambda+\mu}(x)$) such that $s_\lambda(0) = 0$ for every $\lambda$ (which is the case in most of our examples) then the transformations defined by $U_\lambda[f](x) = \left(\frac{s_\lambda(x)}{x}\right)f(s_\lambda(x))$ form a one-parameter (possibly local) group.

(ii) It is also possible to use the "ad" operator (Lie adjoint) instead of "Ad" (conjugacy) to obtain integration formulas (see Dattoli [6]).

### 4.4. *Characteristic series ↔ one parameter group correspondence*

The preceding allows us to extend integration processes to linear combinations of boson strings in the following sense. The algebra $W_{1,\infty}$ generated by $a^+, (a^+)^{-1}, a$ is graded by

$$\text{weight}(a^+) = 1,\ \text{weight}\left((a^+)^{-1}\right) = \text{weight}(a) = -1 \tag{35}$$

and every homogeneous operator of this algebra which is of the form

$$\Omega = \sum_{|w|_a=1,\ \mathrm{weight}(w)=e} \alpha_w w \tag{36}$$

(there is only one annihilation operator in each monomial) can be integrated as above. So one would like to reconstruct the characteristic series

$$\sum_{n,k} S_\Omega(n,k)\frac{x^n}{n!}y^k \tag{37}$$

from knowledge of the one-parameter subgroup $e^{\lambda\Omega}$.
This is the aim of the following paragraph.

For every homogeneous operator as above with $e \geq 0$, one defines the coefficients $S_\Omega(n,k)$ as in the Introduction by

$$\mathcal{N}(\Omega^n) = (a^+)^{ne}\sum_{k=0}^{\infty} S_\Omega(n,k)(a^+)^k a^k \tag{38}$$

One has the following proposition

**Proposition 4.2.** *With the definitions introduced, the following conditions are equivalent:*

$$\sum_{n,k\geq 0} S_\Omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)} \tag{39}$$

$$U_\lambda[f](x) = g(\lambda x^e) f\left(x\left(1+\phi(\lambda x^e)\right)\right) \tag{40}$$

Which solves the problem.

## 5. Conclusion and remaining problems

We have considered a class of elements of $W_{1,\infty}$ (see section 4.4 for a definition) which describe some rational vector fields on the line. For these operators, we have established a correspondence

One-parameter group (=integration of the field) $\leftrightarrow$
Characteristic series (=coefficients of the normal ordering)

We have then seen that families of combinatorial matrices give rise through the exponential formula to substitutions.
Further work which remains is to study the vector fields associated with these combinatorial matrices. Also it would be desirable to adapt this machinery to other algebras ($q$uons, several boson modes).

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