Combinatoire, Physique Quantique et Complexité
extrême robustesse et calcul exact

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LIPN - Laboratoire d'Informatique de l'Université Paris Nord

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CIP

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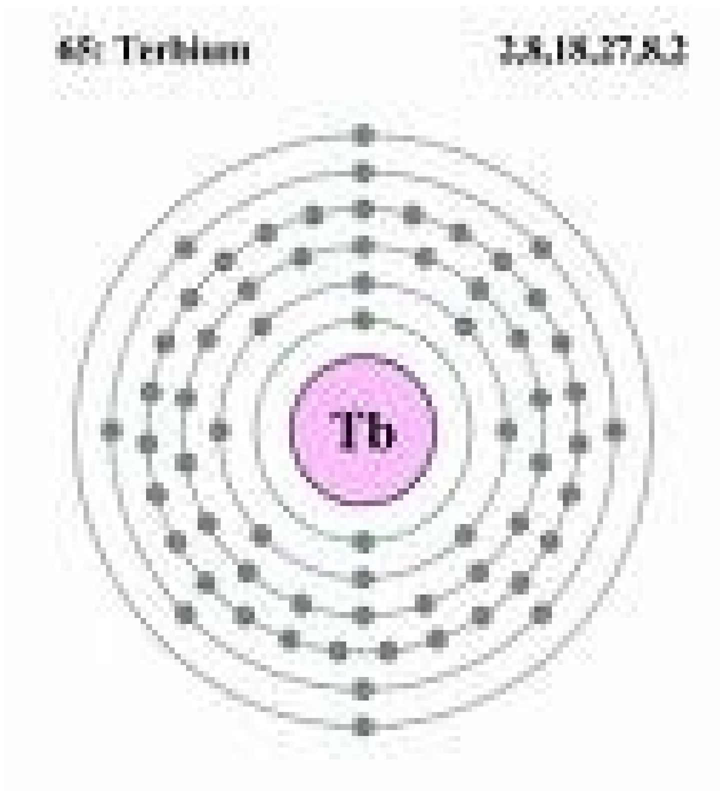


> l'Atome : un système complexe dont certains paramètres sont extrêmement robustes :

→ phénomènes collectifs quantiques

→ paramètres individuels (rayonnement, absorpsion, ...)

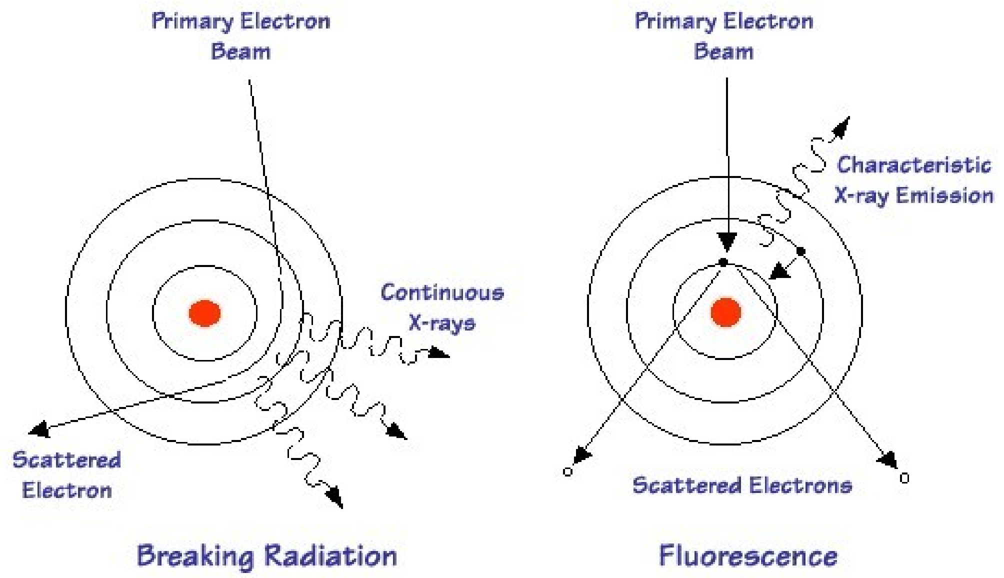
> nécessité de calcul exact → ***LOI(S)***

> motivations :

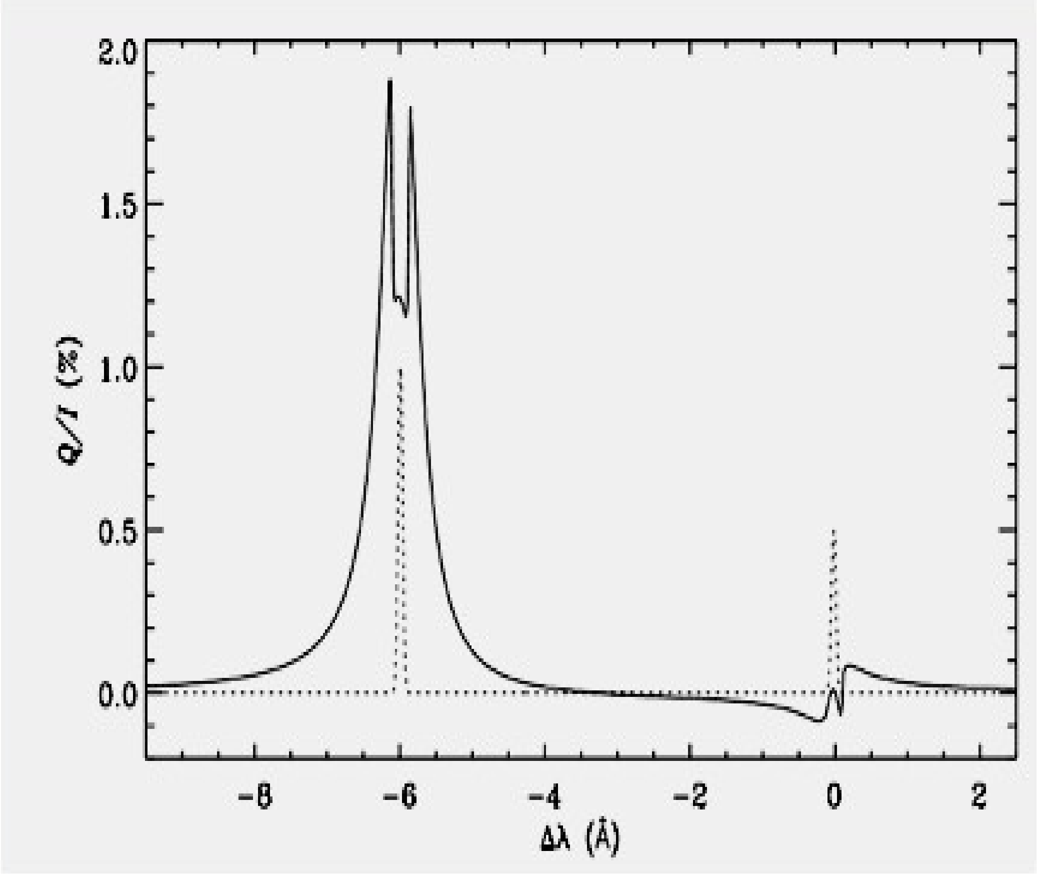
→ mécanique Statistique (modèles exactement résolus)

→ dénombrement (p. ex pour la somme sur toutes les histoires possibles)

→ traitement (exact) des opérateurs

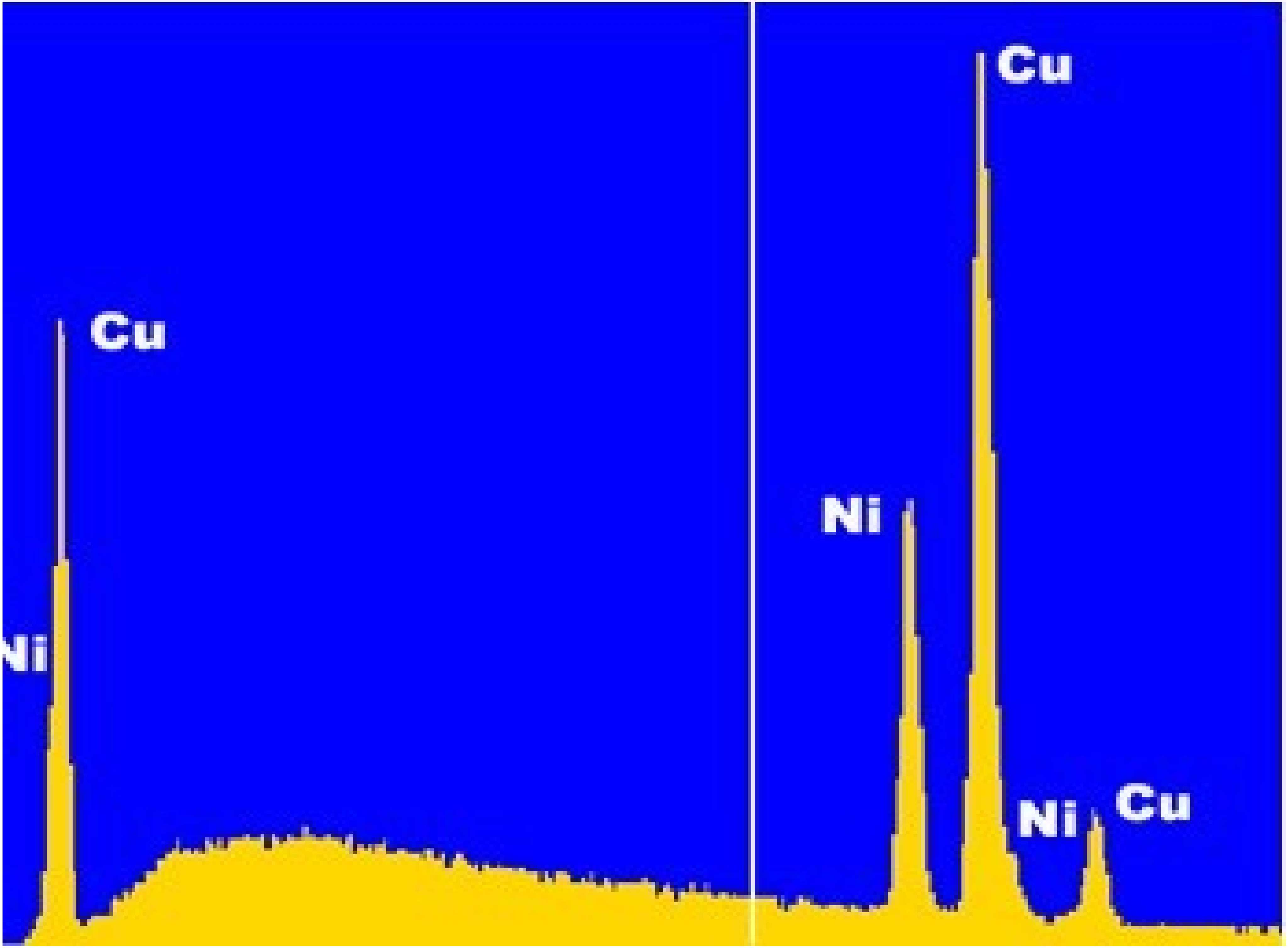


> le modèle standard pour les particules donne des résultats extrêmement précis (10 à 15 décimales)



Synthetic profiles of the fractional linear ploarization, *Q/I*, of the D_1 and D_2 lines of Na I observed off-limb (90° scattering) at a height of 0.01 Rsun, in the presence of a field of 45 G, inclinged of 90° from the local vertical and uniformly distributed in azimuth. The reference wavelength in this plot is that of the peak intensity of D_1, at λ 5896 Å (a normalized plot of the intensity profiles is superimposed for reference; see dotted line). We implemented the flat-spectum approximation using a Planckian illumination at 6000 K.

> raies spectrales typiques de l'élement sodium (Na)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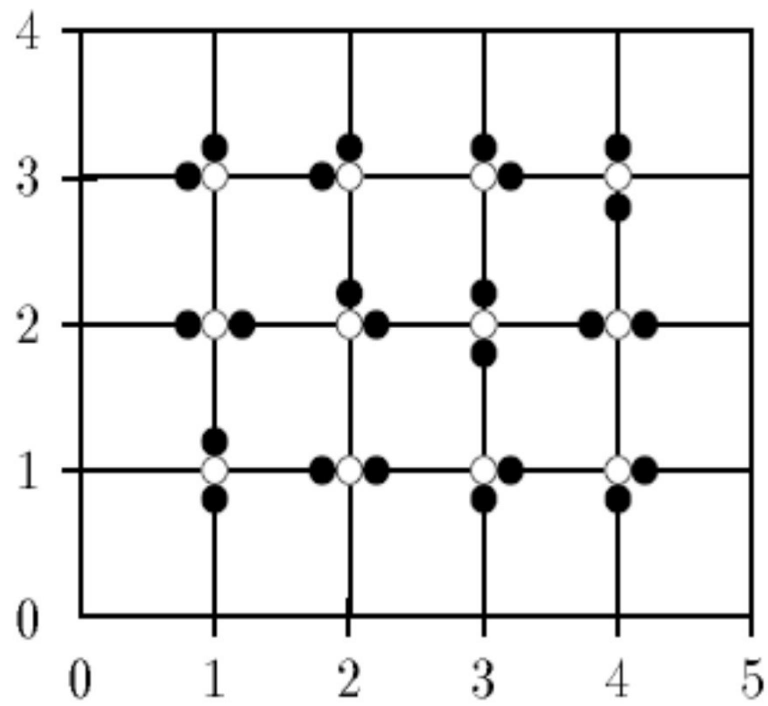


FIG. 2.1 - *Une configuration pour douze molécules d'eau.*

> le modèle de la glace dont s'inspire des configurations comme celles de Schelling.

OPERATEURS d'ECHELLE
LADDER (not SCALING !) OPERATORS

En Mécanique Quantique, les opérateurs vont souvent par paires :

$\longrightarrow$ un opérateur montant (on ajoute des particules ou des niveaux d'énergie)
$\longrightarrow$ un opérateur descendant (on enlève des particules ou des niveaux d'énergie)

$\longrightarrow$ une asymétrie (l'opérateur descendant finit par annuler tous les vecteurs)

Le commencement : les opérateurs de création/annihilation

$\rightarrow$ notés a^+ (opérateur montant, ajoute une particule ou un niveau d'énergie)
et a, opérateur descendant, enlève une particule ou un niveau d'énergie)

Ils doivent vérifier

$$[a\ ,\ a^+]=aa^+ - a^+a=1$$

Heisenberg : la mécanique des matrices (infinies). L'algèbre engendrée par ces opérateurs s'appelle “HW algebra”.

Remarque : On ne peut pas trouver deux matrices (non vides) A,B telles que

$$AB-BA=I_{nxn}$$

Un peu d'Informatique et de Combinatoire

o) On oriente l'égalité $aa^+ - a^+a = 1$ comme suit

$$aa^+ \longrightarrow a^+a + 1$$

l'application répétée de cette règle fournit une combinaison linéaire de monômes de la forme

$$(a^+)^i (a)^j$$

c'est une forme canonique (normal ordering : ordre normal).

$$aa^\dagger aaa^\dagger a \xrightarrow[{[a,a^\dagger]=1}]{\textit{normal ordering}} \underbrace{(a^\dagger)^2 a^4 + 4\, a^\dagger a^3 + 2\, a^2}$$

$a^\dagger$ - to the left $\quad a$ - to the right

Première façon de calculer : Théorème de Wick

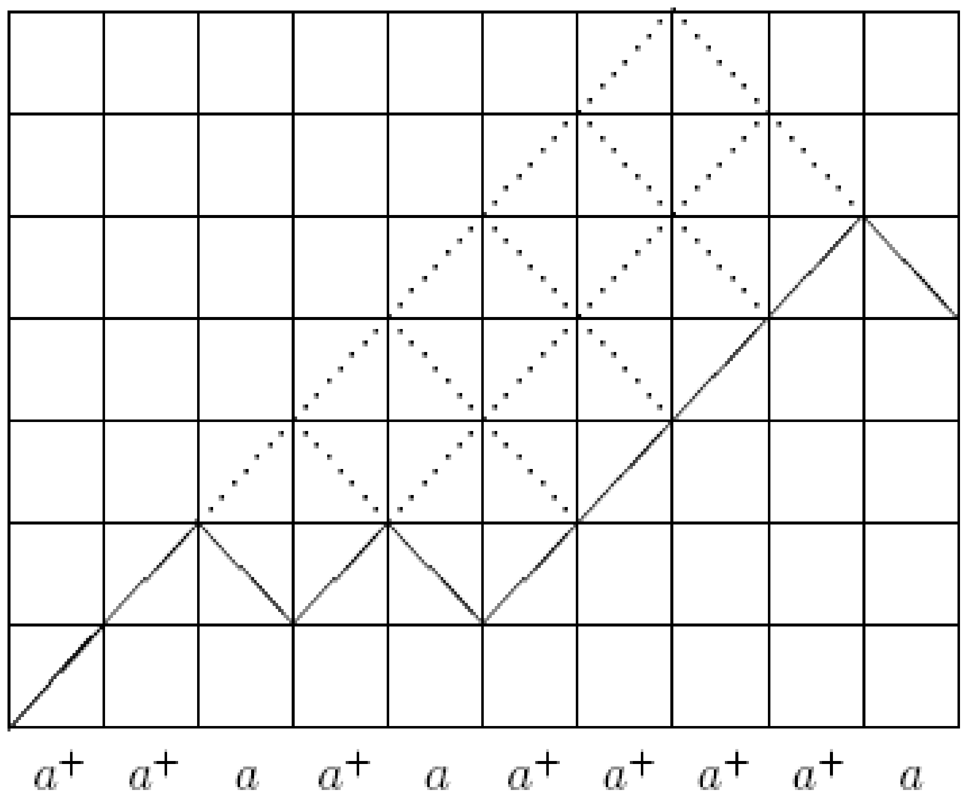
a precedes $a^\dagger$, called *contractions* in analogy to Quantum Field Theory,

$$a a^\dagger a a a^\dagger a = :\underbrace{a a^\dagger a a a^\dagger a}_{\text{no pair removed}}:$$

$$+ :\underbrace{\not a \not a^\dagger a a a^\dagger a + \not a a^\dagger a a \not a^\dagger a + a a^\dagger \not a a \not a^\dagger a + a a^\dagger a \not a \not a^\dagger a}_{\text{1 pair removed}}:$$

$$+ :\underbrace{\not a \not a^\dagger \not a a \not a^\dagger a + \not a \not a^\dagger a \not a \not a^\dagger a}_{\text{2 pairs removed}}: = (a^\dagger)^2 a^4 + 4\, a^\dagger a^3 + 2\, a^2.$$

Deuxième façon de calculer : Placement de tours



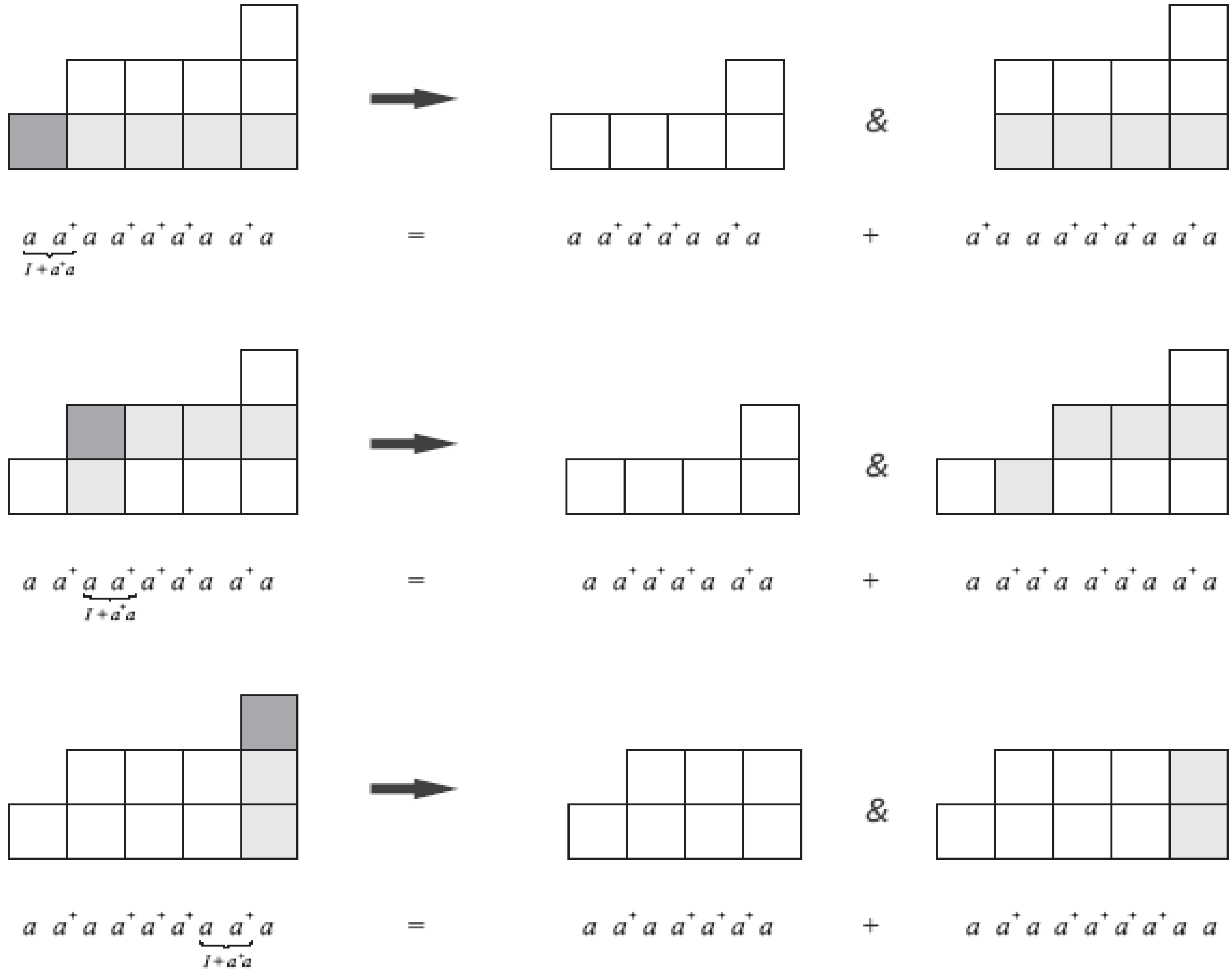


Figure 2. Three possible decompositions of the Ferrers board $\mathcal{B}_{\mathtt{w}}$ and the corresponding reduction of the word $\mathtt{w}$.

Représentations

La plus célèbre est la représentation de Bargmann-Fock

a ---> d/dx ; a+ ---> x

On a bien (d/dx)x-x(d/dx)[f]=f (f est un polynôme par exemple)

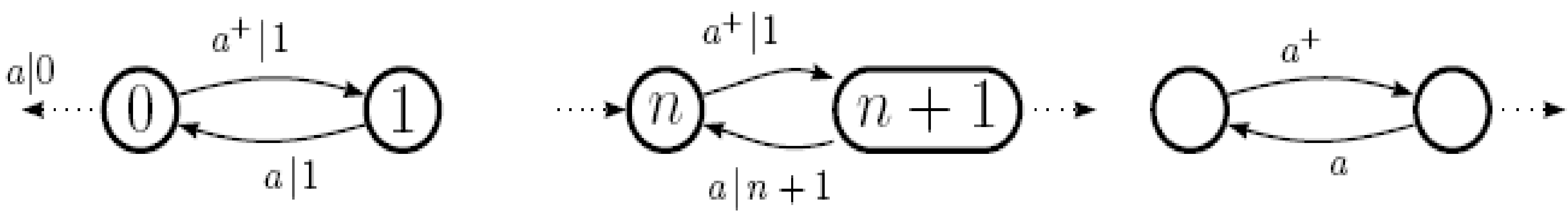


Figure 2: Bargman-Fock representation (flat).
L'état "n" est z^n et $a^+ \to z$; $a \to D = \frac{d}{dz}$

Représentations … suite

On peut aussi interpréter les états comme des urnes dans lesquelles il y a “n” boules distinctes mais indiscernables. Elles sont numérotées de 1 à n, mais on ne voit pas leur numéro, ceci est conforme à la représentation standard des particules.
On peut aussi changer les états.

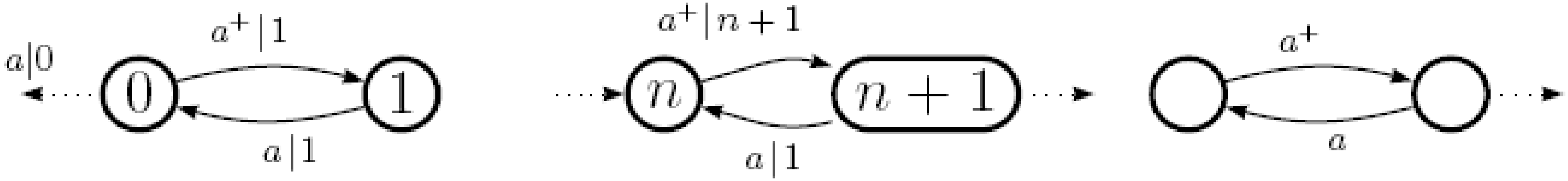


Figure 3: Représentation de Bargman-Fock avec états normalisés.
L'tat “n” is $\frac{z^n}{n!}$ and $a^+ \to z$; $a \to D = \frac{d}{dz}$

On peut aussi « équilibrer les charges » en prenant comme états

$$\frac{z^n}{n!}$$

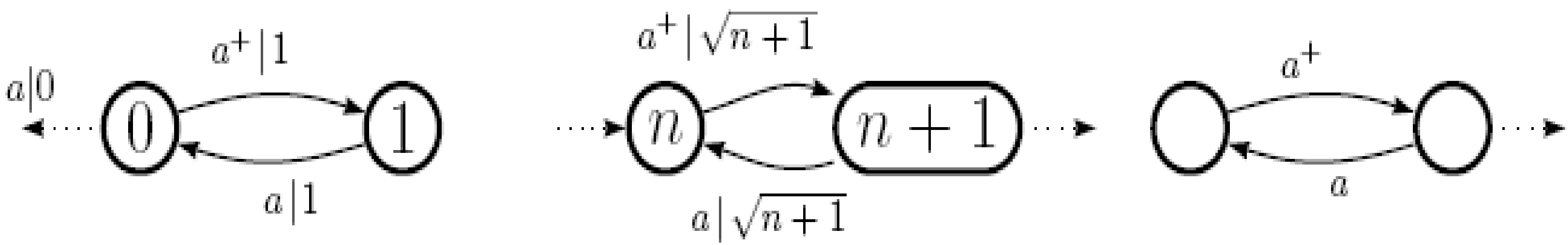


Figure 1: espace de Fock classique

Hence, a typical element in the HW algebra is of the form

$$\Omega = \sum_{k,l \geq 0} c(k,l)(a^{+})^k a^l$$

(normal form).

As can be seen from the Bargmann-Fock representation
Ω is homogeneous of degree e (excess) iff one has

$$\Omega = \sum_{\substack{k,l \geq 0 \\ k-l=e}} c(k,l)(a^{+})^k a^l$$

Due to the symmetry of the Weyl algebra, we can suppose, with no loss of generality that $e \geq 0$. For homogeneous operators one has generalized Stirling numbers defined by

$$\Omega^{n} = (a^{+})^{ne} \sum_{k \geq 0} S_{\Omega}(n,k)(a^{+})^{k} a^{k}$$

Example: $\Omega_1 = a^+a^+ a\, a^+ + a^+ a\, a^+a^+$ (e=2)

$\Omega_2 = a^+a^+ a + a^+a\, a^+$ (e=1)

If there is only one « a » in each monomial as in Ω_2, one can use the integration techniques of the Frascati(*) school (even for inhomogeneous) operators of the type

$$\Omega_2 = q(a^+)a + v(a^+)$$

(*) ***G. Dattoli, P.L. Ottaviani, A. Torre and L. Vàsquez, Evolution operator equations: integration with algebraic and finite difference methods, La Rivista del Nuovo Cimento 20 1 (1997).***

For $w = a^+a$, one gets the usual matrix of Stirling numbers of the second kind.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3}$$

For $w = a^+aa^+$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 18 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 96 & 72 & 16 & 1 & 0 & 0 & \cdots \\ 120 & 600 & 600 & 200 & 25 & 1 & 0 & \cdots \\ 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

For $w = a^+aaa^+a^+$, one gets

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

It can be proved that the matrices of coefficients for expressions with only a single « a » are matrices of special type : that of substitutions with prefunction factor.

2. The algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of sequence transformations

Let $\mathbf{C}^{\mathbf{N}}$ be the vector space of all complex sequences, endowed with the Frechet product topology [23]. It is easy to check that the algebra $\mathcal{L}(\mathbf{C}^{\mathbf{N}})$ of all continuous operators $\mathbf{C}^{\mathbf{N}} \to \mathbf{C}^{\mathbf{N}}$ is the space of *row-finite* matrices with complex coefficients. Such a matrix M is indexed by $\mathbf{N} \times \mathbf{N}$ and has the property that, for every fixed row index n, the sequence $(M(n,k))_{k \geq 0}$ has finite support. For a sequence $A = (a_n)_{n \geq 0}$, the transformed sequence $B = MA$ is given by $B = (b_n)_{n \geq 0}$ with

$$b_n = \sum_{k \geq 0} M(n,k) a_k \tag{6}$$

Remark that the combinatorial coefficients S_w defined above are indeed row-finite matrices.

2.1. *Substitutions with prefunctions*

Let $(d_n)_{n\geq 0}$ bet a fixed set of denominators. We consider, for a generating function f, the transformation

$$\Phi_{g,\phi}[f](x) = g(x)f(\phi(x)). \tag{9}$$

Where \phi(x)=\lambda x+higher terms and g(x)=1+higher terms.

The fact that, in the case of a single "a", the matrices of generalized Stirling numbers are matrices of substitutions with prefunctions is due to the fact that the one-parameter groups associated with the operators of type $\Omega=q(x)d/dx+v(x)$ are conjugate to vector fields on the line.

Use of exactly solved differential equations

Conjugacy trick :

Let $u_2=\exp(\int(v/q))$ and $u_1=q/u_2$ then

$u_1u_2=q$; $u_1u'_2=v$ and the operator $q(a^+)a+v(a^+)$

reads, via the Bargmann-Fock correspondence

$$(u_2u_1)d/dx+ u_1u'_2=u_1(u'_2 + u_2d/dx)= u_1d/dx\ u_2 =$$

$$1/u_2\,(u_1\ u_2\ d/dx\ )\ u_2$$

Which is conjugate to a vector field and integrates as a substitution with prefunction factor.

Example: The expression $\Omega_1 = a^+a^+ a a^+ + a^+ a a^+a^+$ above corresponds to the operator
(on the line below Ω_1 is in form q(x)d/dx+v(x))

$$\omega = x^2 \frac{d}{dx} x + x \frac{d}{dx} x^2 =$$

$$2x^3 \frac{d}{dx} + 3x^2 = x^{-3/2}(2x^3 \frac{d}{dx})x^{3/2} = x^{-3/2}(\phi)x^{3/2}$$

Now, φ_1 is a vector field and its one-parameter group acts by a one parameter group of substitutions. We can compute the action by another conjugacy trick which amounts to straightening Ω_1 to a constant field.

Thus set

$$\exp(\lambda\,\phi)[f(x)]=f(u\ \ (u(x)+\lambda))$$ for some u …

By differentiation w.r.t. λ at (λ=0) one gets

$$u'=1/(2x^3)\ ;\ u=-1/(4x^2)\ ;\ u^{-1}(y)=(-4y)^{-1/2}$$

Maple 9 - [fc_sem_algo.mws - [Server 1]]

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```
> expand(x^(-3/2)*2*x^3*diff(f(x)*x^(3/2),x));
```

$$2\,x^3\left(\frac{d}{dx}\,\mathrm{f}(x)\right)+3\,x^2\,\mathrm{f}(x)$$

The one-parameter group given by f(v(u(x)+λ); v being the (compositional) inverse of u, reads

```
> T1:=(lambda,x)->x*(1-4*lambda*x^2)^(-1/2);
```

$$T1 := (\lambda, x) \rightarrow \frac{x}{\sqrt{1-4\,\lambda\,x^2}}$$

Checking the tangent vector at the origin

```
> subs(lambda=0,diff(T1(lambda,x),lambda));
```

$$2\,x^3$$

... and the one-parameter group property

```
> simplify(T1(lambda1,T1(lambda2,x))^2-T1(lambda1+lambda2,x)^2);
```

$$0$$

In view of the conjugacy established previously we have that exp(λ ω)[f(x)] acts as

$$U_\lambda(f) = x^{-\frac{3}{2}} f(T(\lambda, x)).(T(\lambda, x))^{\frac{3}{2}}$$

$$= \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt{\frac{x^2}{1-4\lambda x^2}}\right)$$

which explains the prefactor. Again we can check by computation that the composition of (U_λ)s amounts to simple addition of parameters !!

Now suppose that exp(λ ω) is in normal form.

In view of what precedes, we must have

$$\exp(\lambda\omega) = \sum_{n\geq 0} \frac{\lambda^n \omega^n}{n!} = \sum_{n\geq 0} \frac{\lambda^n}{n!} x^{ne} \sum_{k=0}^{ne} S_\omega(n,k) x^k \left(\frac{d}{dx}\right)^k$$

Hence, introducing the eigenfunctions of the derivative (a method which is equivalent to the computation with coherent states) one can recover the mixed generating series of $S_\omega(n,k)$ from the knowledge of the one-parameter group of transformations.

$$\exp(\lambda\,\omega)\left[e^{yx}\right] = \Big(\sum_{n\geq 0}\frac{\lambda^n}{n!}x^{ne}\sum_{k=0}^{ne}S_\omega(n,k)x^k y^k\Big)e^{yx}$$

Thus, one can state

Proposition (*): With the definitions introduced, the following conditions are equivalent (where f → U_λ[f] is the one-parameter group exp(λω)).

1. $$\sum_{n,k\geq 0} S_\omega(n,\ k)\frac{x^n}{n!}\,y^k \;=\; g(x)e^{y\phi(x)}$$

2. $$U_\lambda[f](x) \;=\; g(\lambda x^e)f(x(1+\ \phi(\lambda x^e)))$$

Remark : Condition 1 is known as saying that S(n,k) is of « Sheffer » type.

G. Duchamp, A.I. Solomon, K.A. Penson, A. Horzela and P. Blasiak,
One-parameter groups and combinatorial physics,
World Scientific Publishing. arXiv: quant-ph/04011262

Example : With $\Omega = a^{+2}a\, a^{+} + a^{+}a\, a^{+2}$ (previous slide), we had e=2 and

$$U_{\lambda}[f](x) = \sqrt[4]{\frac{1}{(1-4\lambda x^2)^3}}\, f\left(\sqrt[2]{\frac{x^2}{1-4\lambda x^2}}\right)$$

Then, applying the preceding correspondence one gets

$$\sum_{n,k\geq 0} S_{\omega}(n,k)\frac{x^n}{n!}y^k = \sqrt[4]{\frac{1}{(1-4x)^3}}\, e^{y\left(\sqrt{\frac{1}{1-4x}}-1\right)} =$$

$$\sqrt[4]{\frac{1}{(1-4x)^3}}\, e^{y\left(\sum_{n\geq 1} c_n x^n\right)}$$

Where $c_n = \binom{2n}{n}$ re the central binomial coefficients.

```
> E1:=(1/((1-4*x)^3))^(1/4)*exp(y*(1/(1-4*x)^(1/2)-1));
```

$$E1 := \left(\frac{1}{(1-4\,x)^3}\right)^{(1/4)} \mathbf{e}^{\left(y\left(\frac{1}{\sqrt{1-4\,x}}-1\right)\right)}$$

```
> T1:=taylor(E1,x=0,6);
```

$$T1 := 1 + (2\,y+3)\,x + \left(12\,y + 2\,y^2 + \frac{21}{2}\right)x^2 + \left(59\,y + 18\,y^2 + \frac{4}{3}y^3 + \frac{77}{2}\right)x^3 + \left(270\,y + 115\,y^2 + 16\,y^3 + \frac{2}{3}y^4 + \frac{1155}{8}\right)x^4 + \left(\frac{4389}{8} + \frac{4767}{4}y + 637\,y^2 + 126\,y^3 + 10\,y^4 + \frac{4}{15}y^5\right)x^5 + \mathrm{O}(x^6)$$

```
> seq([sort(coeff(T1,x,n)*n!)],n=1..5);
```

$$[2\,y+3],\ [4\,y^2+24\,y+21],\ [8\,y^3+108\,y^2+354\,y+231],$$
$$[16\,y^4+384\,y^3+2760\,y^2+6480\,y+3465],$$
$$[32\,y^5+1200\,y^4+15120\,y^3+76440\,y^2+143010\,y+65835]$$

```
> M1:=matrix(5,5,(n,k)->coeff(coeff(T1,x,n)*n!,y,k));
```

$$M1 := \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 24 & 4 & 0 & 0 & 0 \\ 354 & 108 & 8 & 0 & 0 \\ 6480 & 2760 & 384 & 16 & 0 \\ 143010 & 76440 & 15120 & 1200 & 32 \end{bmatrix}$$

Proposition (*): With the definitions introduced, the following conditions are equivalent (where f → U_λ[f] is the one-parameter group exp(λω)).

$$1.\ \sum_{n,k\geq 0} S_\omega(n,k)\frac{x^n}{n!}y^k = g(x)e^{y\phi(x)}$$

$$2.\ U_\lambda[f](x) = g(\lambda x^e) f(x(1+\phi(\lambda x^e)))$$

Remark : Condition 1 is known as saying that S(n,k) is of « Sheffer » type.

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Remarks on the proof of the proposition :

2) → 1) Can be proved by direct computation.
1) → 2) Firstly the operator $\exp(\lambda\omega)$ is continuous for the Treves topology on the EGF. Secondly, the equality in (2) is linear and continous in f (both sides). Thirdly the set of $\exp(yx)$ for y complex is total in the spaces of EGF endowed with this topology and the equality is satisfied on this set.

Substitutions and gases of graphs

A great, powerful and celebrated result:
(For certain classes of graphs)

If *C(x)* is the EGF of CONNECTED graphs in a certain class, then *exp(C(x))* is the EGF of ALL graphs.
(Uhlenbeck, Mayer, Touchard,…)

This implies that the matrix

M(n,k)=number of graphs with n vertices and having k connected components

is the matrix of a substitution (like S□(n,k) previously but without prefactor).

Examples

1) Stirling numbers

M(n,k)=number of graphs complete by blocks (eq. relations of math) with n vertices and k blocks

2) Idempotent numbers

I(n,k)=number of graphs idempotent endofunctions on [1..n] and k fix points

One can prove, using a Zariski-like argument, that, if *M* is such a matrix (with identity diagonal) then, all its powers (positive, negative and fractional) are substitution matrices and form a one-parameter group of substitutions, thus coming from a vector field on the line which could (in theory) be computed.

But no nice combinatorial principle seems to emerge.

For example, to begin with the Stirling substitution $z \rightarrow e^z-1$. We know that there is a unique one-parameter group of substitutions $s_\lambda(z)$ such that, for λ integer, one has the good value (for ex. $s_2(z)$ corresponds to partition of partitions)

$$s_2(z) = e^{(e^z-1)} - 1;\ s_3(z) = e^{(e^{(e^z-1)}-1)} - 1;\ s_{-1}(z) = log(1+z)$$

But we have no nice description of this group nor of the vector field generating it.

Problème du Transporteur

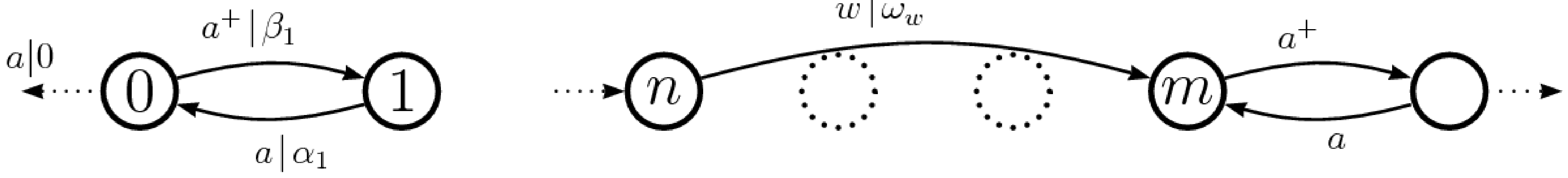
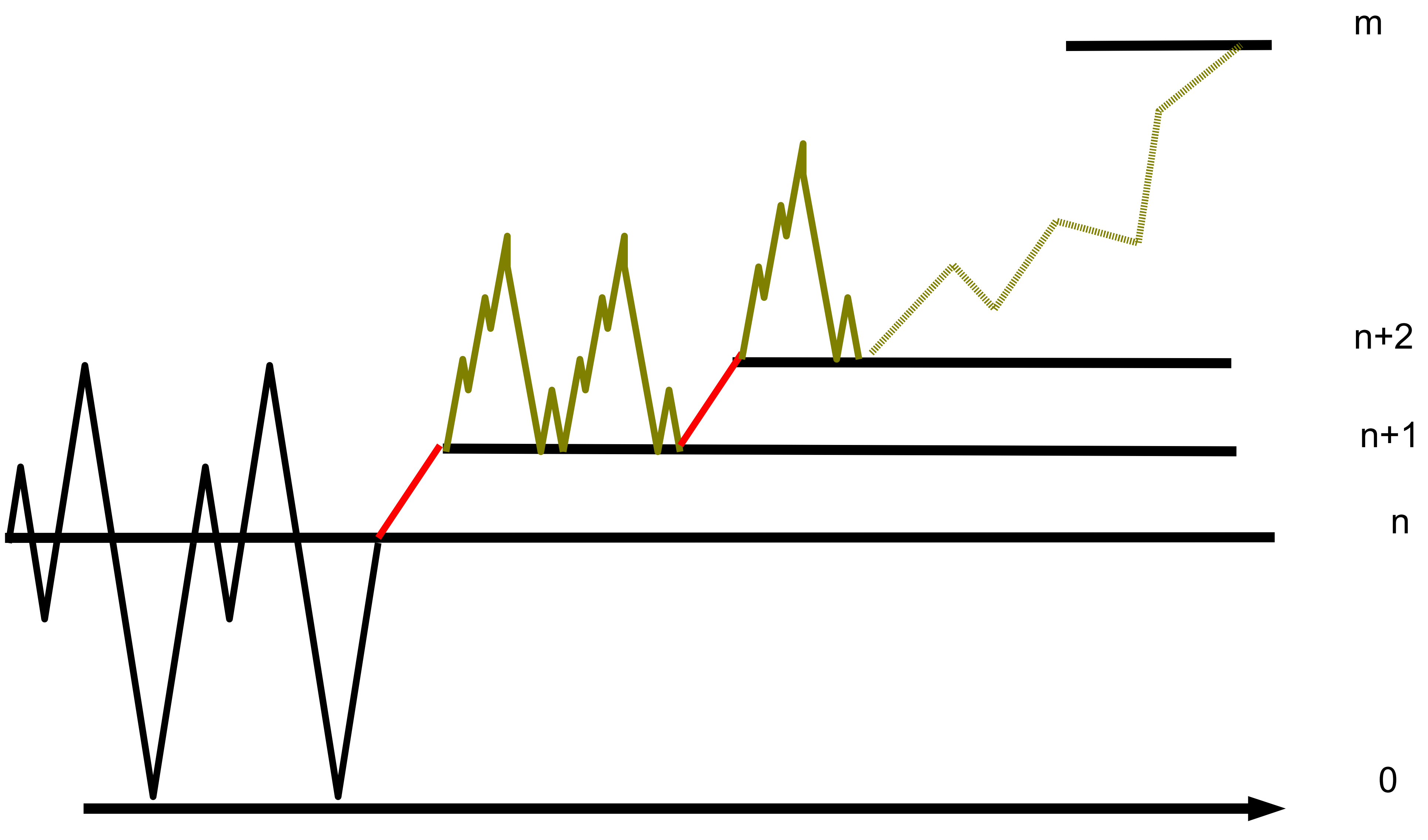


Figure 4: The transfer packet problem

Physicists need to know the sum of all weights created when one passes from level « n » to level « m ».

This problem has been called the « transfer packet problem » and is at once rephrased by combinatorists as the computation of a formal power series described by words.

Change of level



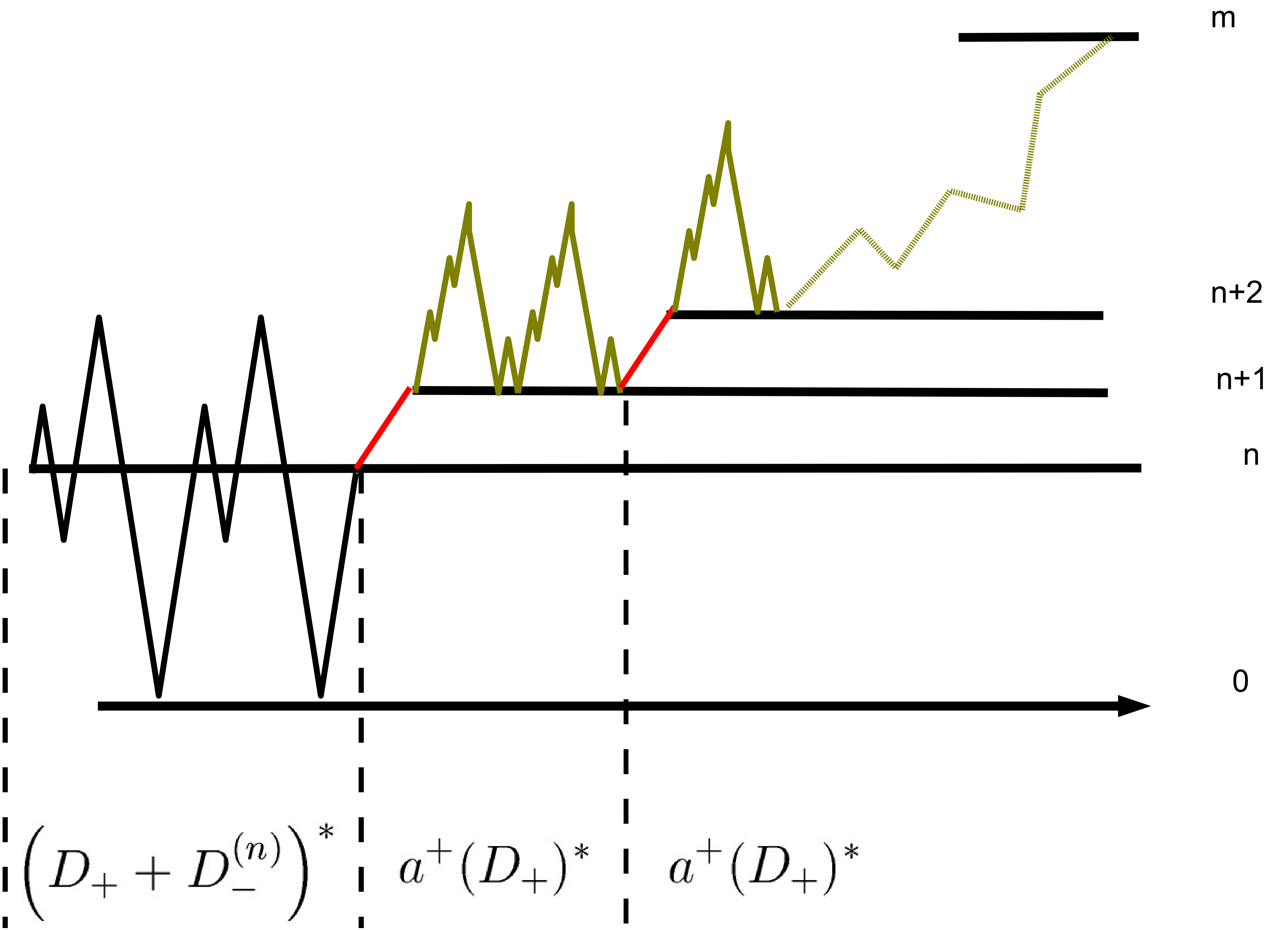
- The set of words which allow to pass from level « n »
- to level « m » in « i » steps clearly reads

$$W_k^{(i)} = \left\{ w \in \{a, a^+\}^* \middle| \pi_e(w) = k \text{ and } |w| = i \right\}$$

$$\text{with } k = m - n \text{ and } \pi_e(w) = |w|_{a^+} - |w|_a.$$

- The weight associated to this set is then

$$e_n.W_{m-n}^{(i)} = \omega_{n\to m}^{(i)}\ e_m\ ;\ T_{n\to n+k} := \sum_{i\geq 0} t^i \omega_{n\to n+k}^{(i)}$$


m
n+2
n+1
n
0
$\left(D_+ + D_-^{(n)}\right)^*$
$a^+(D_+)^*$
$a^+(D_+)^*$

Yap - [fig1]

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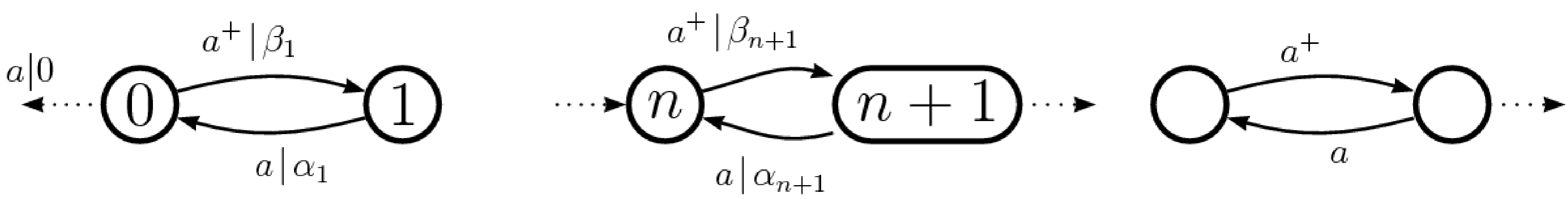

Figure 3: General setting: in order that the Fock space be bounded below, one must have $\alpha_0 = 0$.

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$$T_{n\to n}[t] = \cfrac{1}{1 - \cfrac{t^2\alpha_{n+1}\beta_{n+1}}{1-\cfrac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cdots}} - \cfrac{t^2\alpha_n\beta_n}{1-\cfrac{t^2\alpha_{n-1}\beta_{n-1}}{1-\cdots}}}$$

$$T^{+}_{n\to n}[t] = \cfrac{1}{1 - \cfrac{t^2\alpha_{n+1}\beta_{n+1}}{1-\cfrac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cdots}}}$$

Autres structures de transition

GRADED SCHELLING'S CITY SEFREGATION MODEL : DATA STRUCTURES AND IMPLEMENTATION NOTES.

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Definition 0.1 *Let X,Y be two sets $X \subset Y$. A* generalized derangement *from X to Y is an into (i. e. injective) mapping*[1] $\alpha : X \mapsto Y$ *such that*

$$(\forall x \in X)(\alpha(x) \neq x) \tag{1}$$

To describe Thomas Schelling's concurrent model, we start with a two-dimensional lattice board which is a rectangle of $n \times m$ (n lines and m columns) points (each point will be located by its coordinates (x, y) with $1 \leq x \leq m$; $1 \leq y \leq n$). A *state* of the board will be simply a mapping $s : [1..m] \times [1..n] \mapsto \{0, A, B\}^2$ indicating whether a point at (x, y) has a value corresponding to

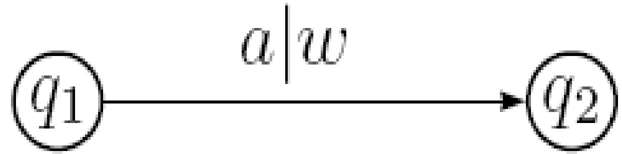
$$\begin{cases} \bullet \ \text{nothing } s(x,y) = 0 \\ \bullet \ \text{an element of type } A, \ s(x,y) = A \\ \bullet \ \text{an element of type } B, \ s(x,y) = B \end{cases} \tag{2}$$

the dynamics of the system will be described by a sequence of states $s_0, \ s_1, \cdots, \ s_n, \cdots$ generated by the following rules.

1. one fixes a threshold (in percent) $0 \leq t \leq 1$;
2. the original state is s_0 (i.e. a distribution of A, B and empty cells along the board);
3. at each step, for each (filled) cell of type X at (x, y), one counts the ratio $\rho(x, y)$ of neighbours of type X over the number of neighbours;
4. if $\rho \geq t$ the cell is marked r (remain), if not it is marked m (move);
5. let M be the set of cells marked "move" and E the set of empty cells;
6. choose randomly (uniform distribution) α among the generalized derangements $M \mapsto M \cup E$;
7. then $s_{n+1}(x, y) = s_n(x, y)$ if the cell was marked r and $s_{n+1}(x, y) = \alpha(s_n(x, y))$ otherwise.

2 Graphes et structures de transition

L'élément le plus complet de ces modèles est la flêche



i) $q_1, q_2 \in Q$ Q est un ensemble (fini ou infini d'états)

ii) a est le symbole qui agit

iii) $\alpha \in k$ est un coefficient. L'ensemble (k) des coefficients est muni de deux lois $+, \times$, associatives, $+$ est commutative et $\times$ soit distributive (à droite et à gauche) sur $+$.

Nous verrons plus bas que ces propriétés sont nécessaires au bon fonctionnement du modèle. Si f est la flêche ci-dessus nous noterons

- $\alpha(f) = q_1$, source ou origine de f
- $\beta(f) = q_2$, but ou extrémité de f
- $\lambda(f) = a$, étiquette de f
- $weight(f)$, poids, coût ou coefficient de transition de f

On a donc que les poids se multiplient en série et s'aditionnent en parallèle. Les diagrammes suivants montrent la nécessité des propriétés exigées pour $+$ et $\times$.

Diagramme	Identité	Nom
$\overset{a\|\alpha}{\longrightarrow}$ $p \overset{a\|\beta}{\longrightarrow} q$ $\overset{a\|\gamma}{\longrightarrow}$	$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$	Associativité de $+$
$p \underset{a\|\beta}{\overset{a\|\alpha}{\rightrightarrows}} q$	$\alpha + \beta = \beta + \alpha$	Commutativité de $+$
$p \overset{a\|\alpha}{\longrightarrow} q \overset{b\|\beta}{\longrightarrow} r \overset{c\|\gamma}{\longrightarrow} s$	$\alpha(\beta\gamma) = (\alpha\beta)\gamma$	Associativité de $\times$
$p \underset{a\|\beta}{\overset{a\|\alpha}{\rightrightarrows}} q \overset{b\|\gamma}{\longrightarrow} r$	$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$	Distributivité (droite) de $\times$ sur $+$
$p \overset{a\|\alpha}{\longrightarrow} q \underset{b\|\gamma}{\overset{b\|\beta}{\rightrightarrows}} r$	$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$	Distributivité (gauche) de $\times$ sur $+$

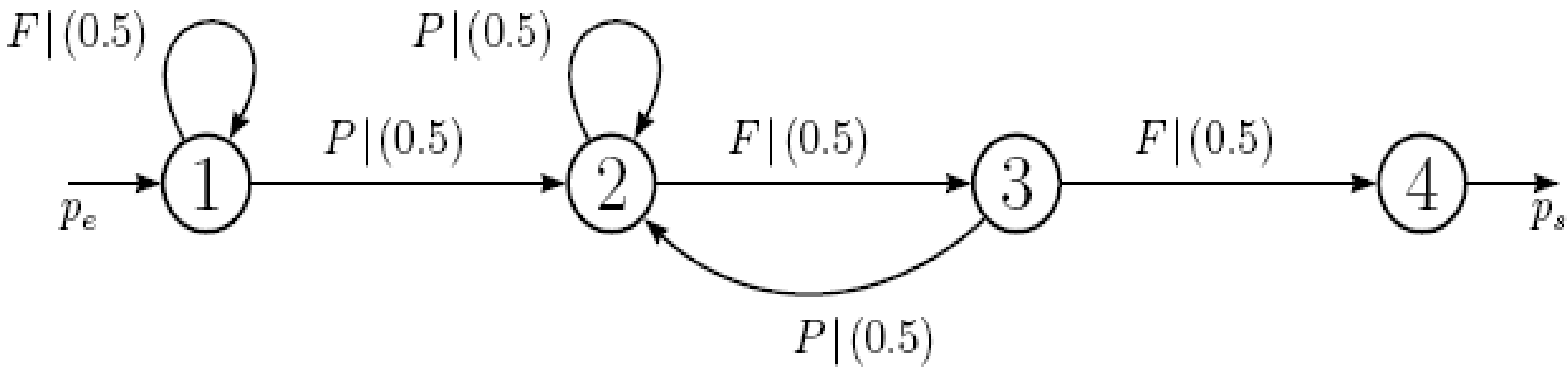


FIG. 2 – *Un automate proposé par les étudiants*

```
> M:=matadd(Mp,Mf);
```

$$M := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> I4:=matrix(4,4,(i,j)->if i=j then 1 else 0 fi);
```

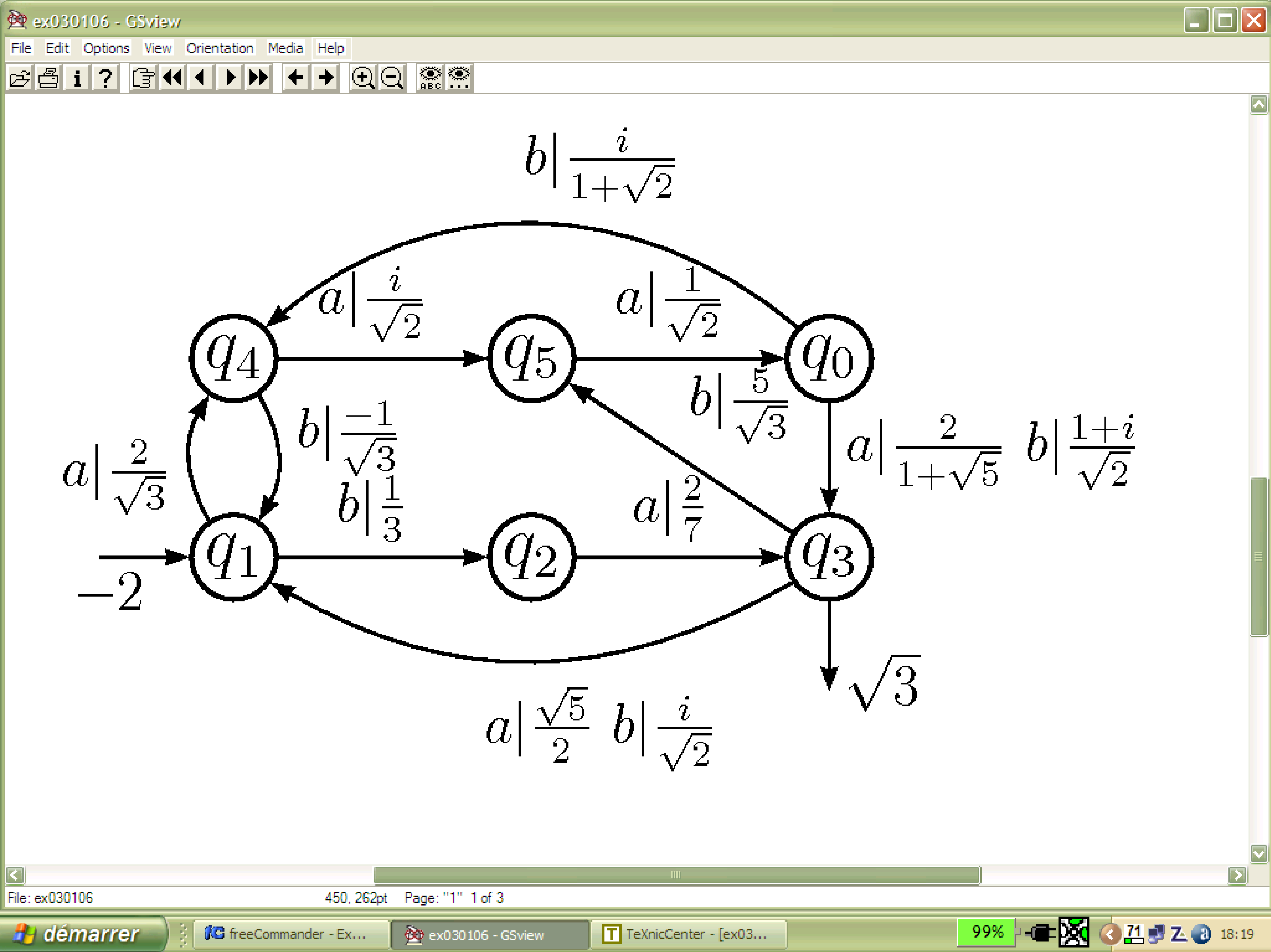
$$I4 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> M2:=matadd(I4,M,1,-z);
```

$$M2 := \begin{bmatrix} 1 - \frac{z}{2} & -\frac{z}{2} & 0 & 0 \\ 0 & 1 - \frac{z}{2} & -\frac{z}{2} & 0 \\ 0 & -\frac{z}{2} & 1 & -\frac{z}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> inverse(M2)[1,4];
```

$$\frac{z^3}{(-2+z)(-4+2z+z^2)}$$



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$b|\frac{-1}{\sqrt{3}}$
$a|\frac{2}{1+\sqrt{5}}$
$b|\frac{1+i}{\sqrt{2}}$
$a|\frac{2}{\sqrt{3}}$
$b|\frac{1}{3}$
$a|\frac{2}{7}$
q_1
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q_3
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$a|\frac{\sqrt{5}}{2}$
$b|\frac{i}{\sqrt{2}}$
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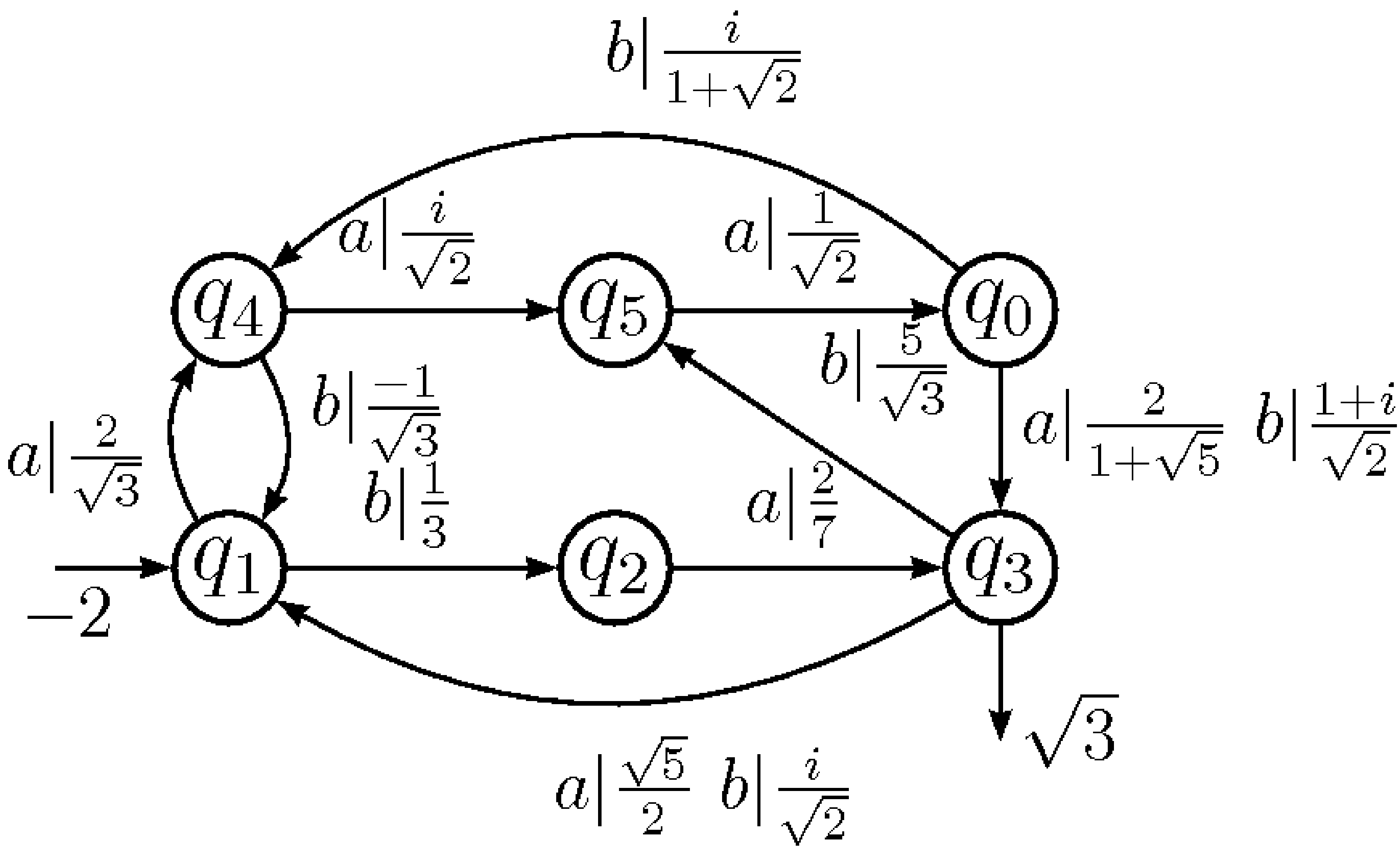
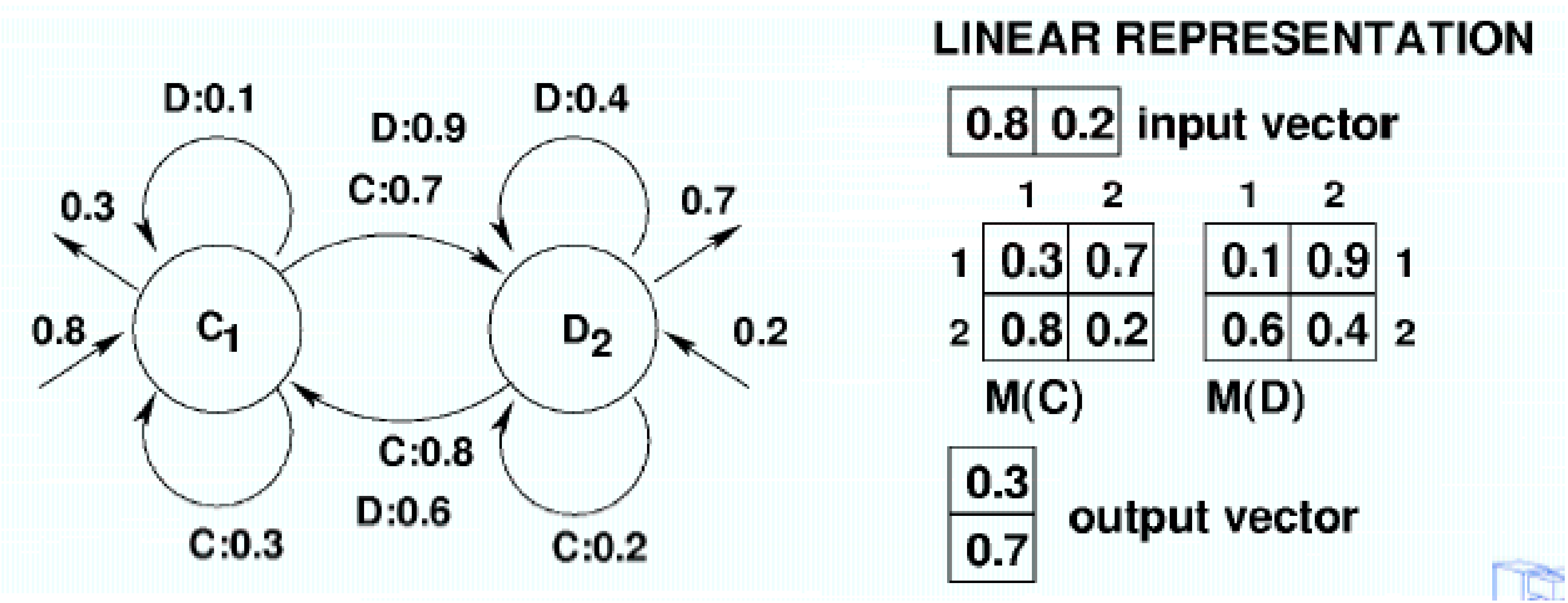
Example : An automaton generated by arbitrary transition coefficients.



Example of Probabilistic Automaton



Behaviour of an Automaton and how to compute it effectively

An automaton is a machine which takes a string (sequence of letters) and returns a value.

This value is computed as follows :

1) The weight of a path is the product of the weights (or coefficients) of its edges

2) The label of a path is the product (concatenation) of the labels of its edges

Behaviour ... (cont'd)

3) The behaviour between two states « r,s » w.r.t. A word « w » is the product of

3a) the ingoing coefficient of the first state (here « r ») by

3b) the sum of the weights of the paths going from « r » to « s » with label « w » by

3c) the outgoing coefficient of the second state (here « s »)

Behaviour ... (cont'd)

4) The behaviour of the automaton under consideration w.r.t. a word « w » is then the sum of all the behaviours of the automaton between two states « r,s » for all possible pairs of states.

Behaviour ... (cont'd)

There is a simple formula using the linear representation. The behaviour of an automaton with linear representation (I,M,T) is the product

$$IM(w)T$$

where M(w) is the canonical exention of M to the strings.

$$M(a_1 a_2 \ldots a_n) = M(a_1)M(a_2)\ldots M(a_n)$$

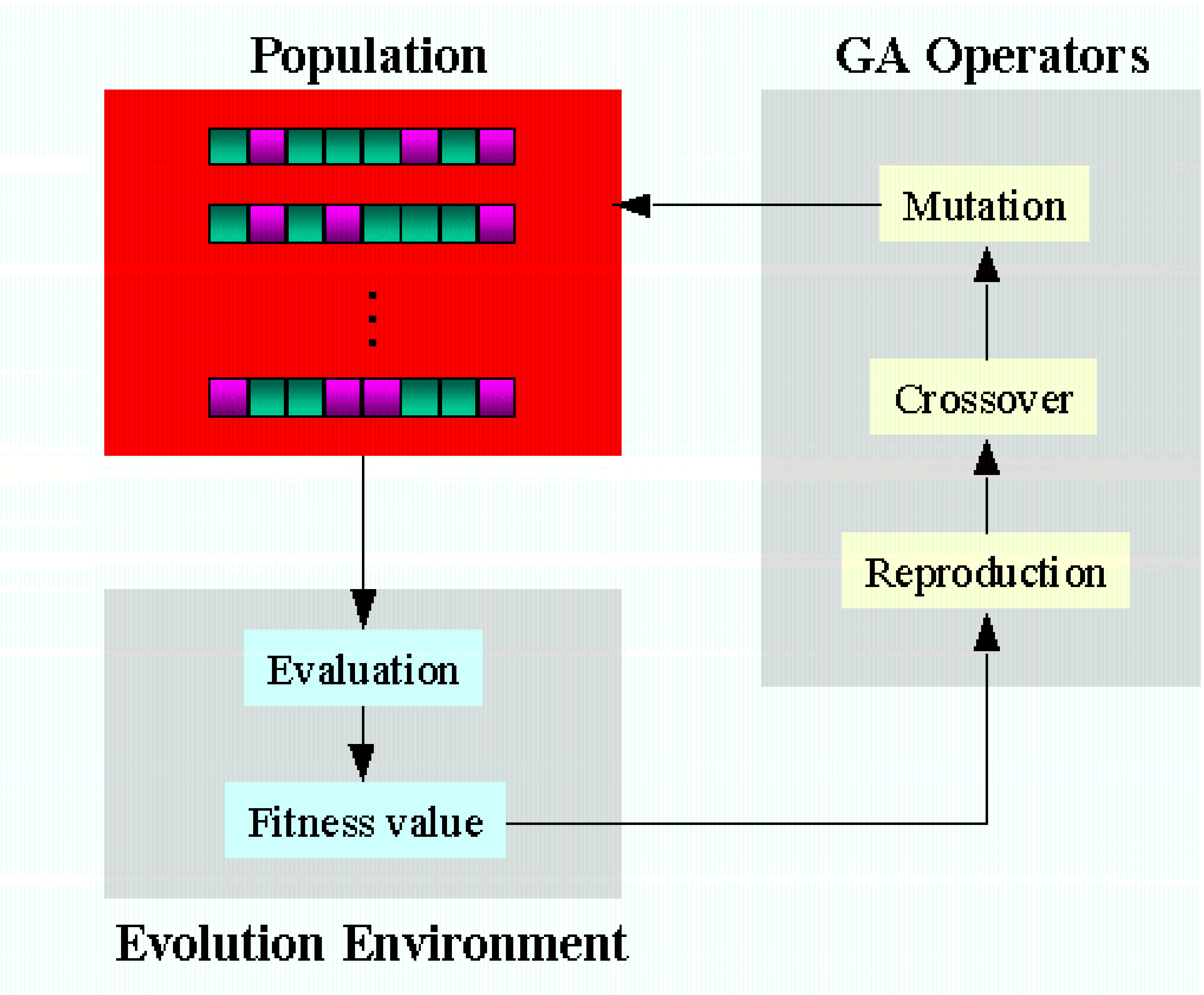
Behaviour ... (end)

The behaviour, as a function on words belongs to the rational class. If time permits, we will return to its complete calculation as a rational expression

distances between automata

Example -> use of genetic algorithms to control indirect (set of) parameters : the spectrum of a matrix.

Genetic algorithms : general pattern



Genetic Algorithm Evolution Flow

Genetic algorithms : implementation

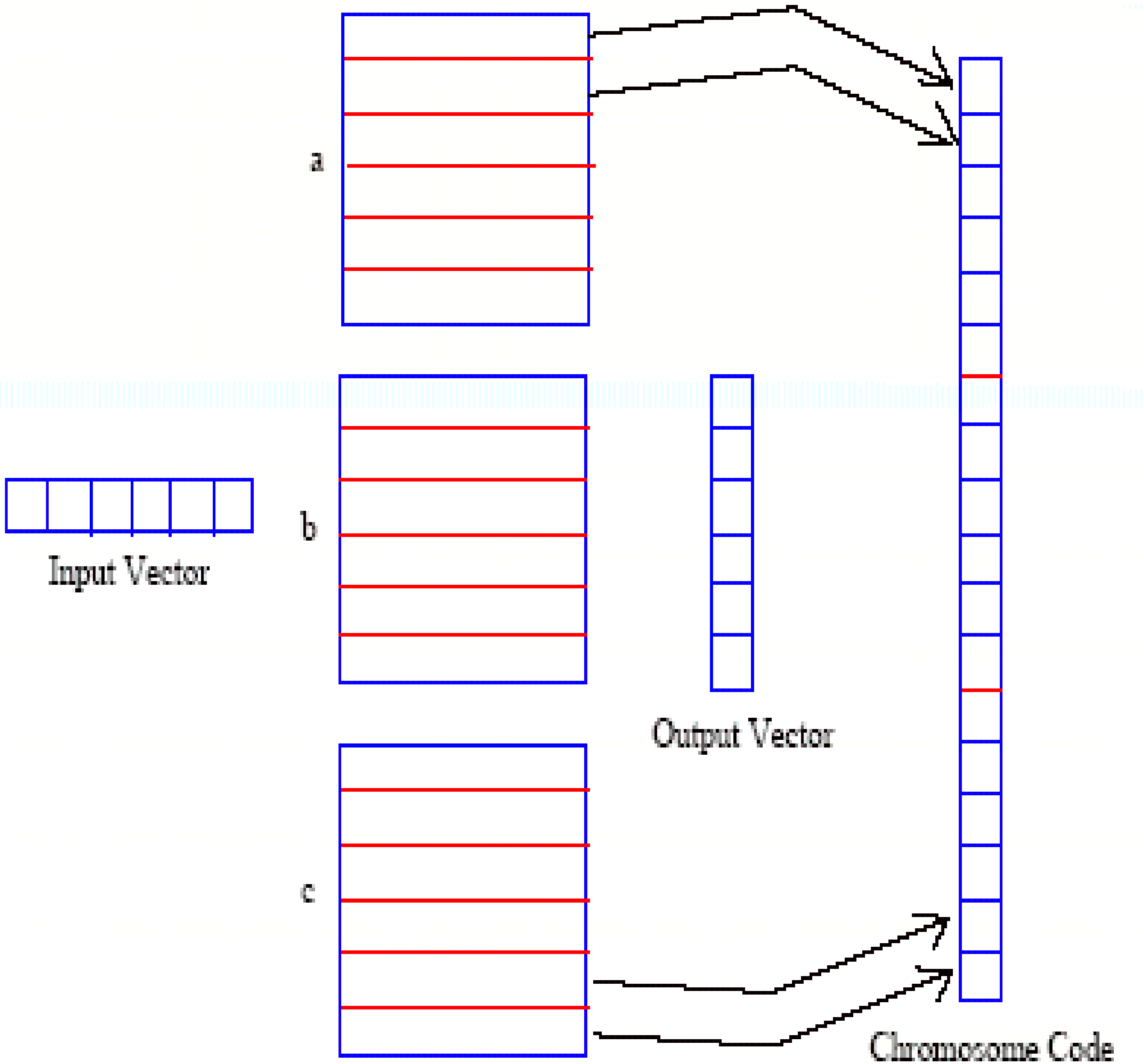


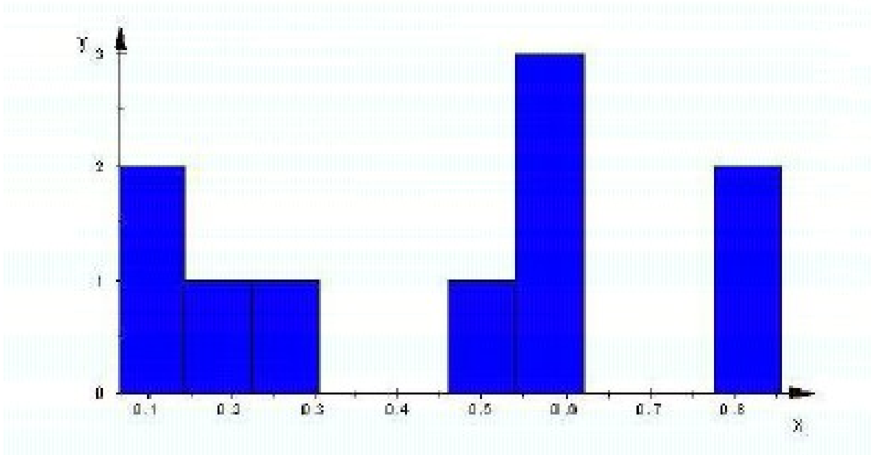
Figure 4.13: Chromosome code

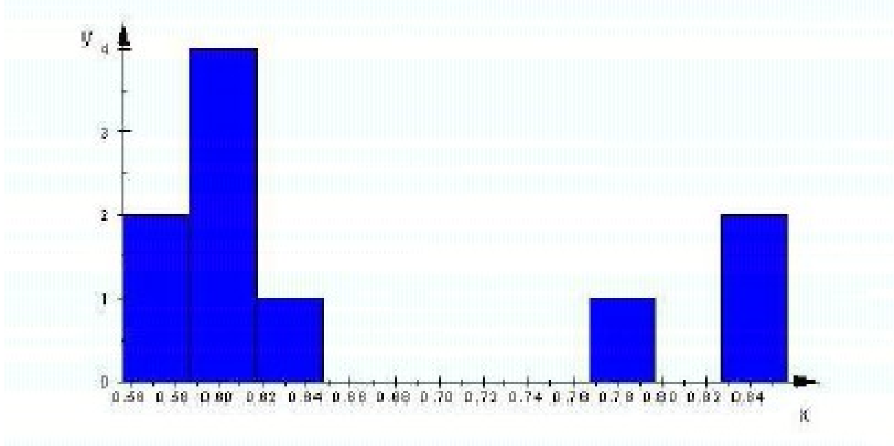
Genetic algorithms : implementation

Below, the results of an experiment aiming to control the second greatest eigenvalue of the transfer matrix of a population of probabilistic automata.

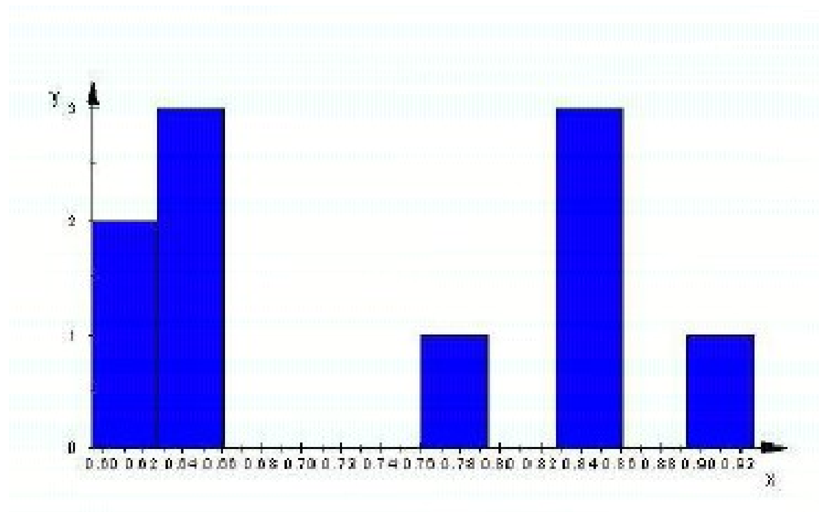
The fitness function of each automaton corresponds to the second greatest eigenvalue (in module).
The first being, of course, of value 1.

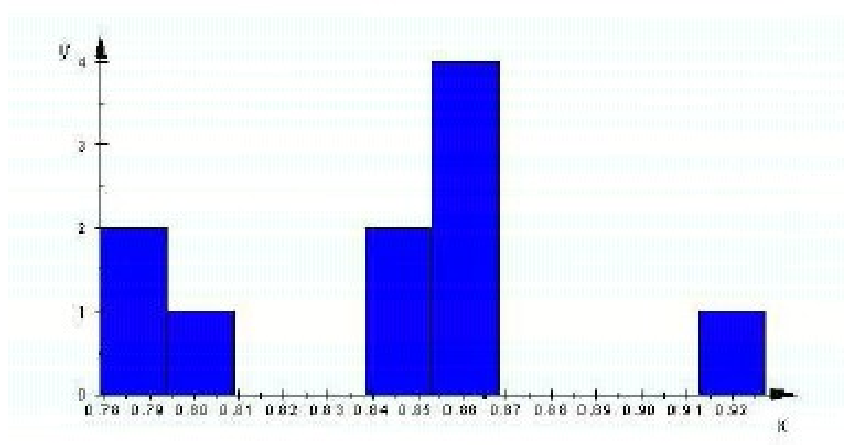
Genetic algorithms ; results



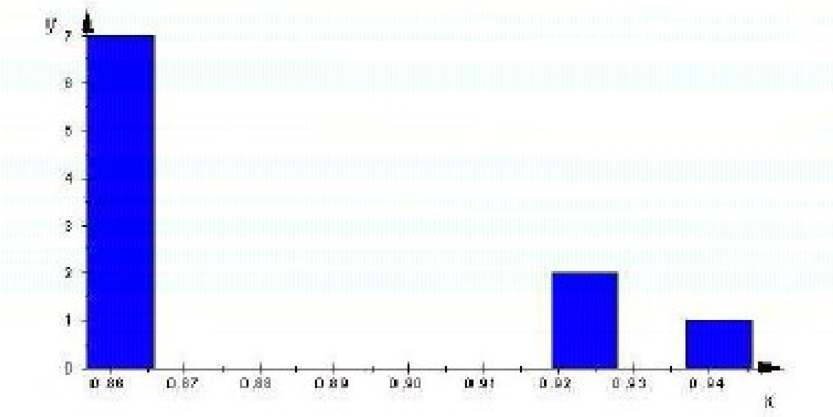


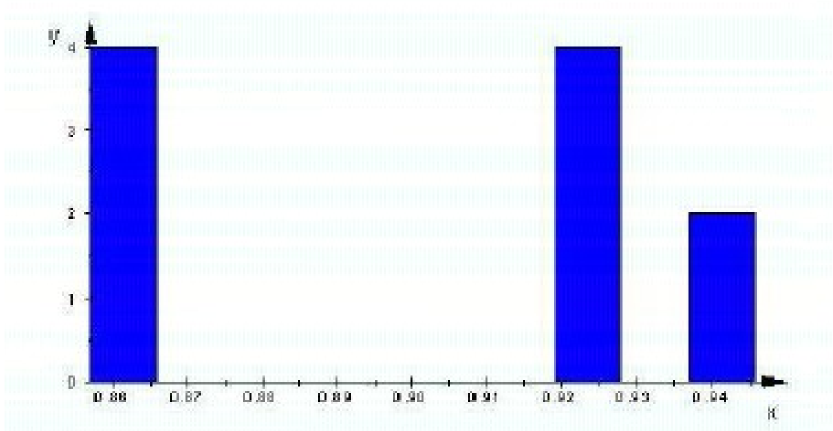
Genetic algorithms ; results



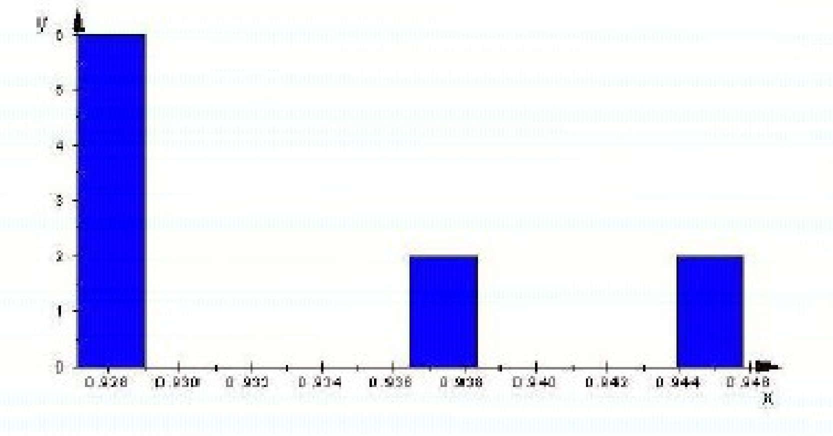


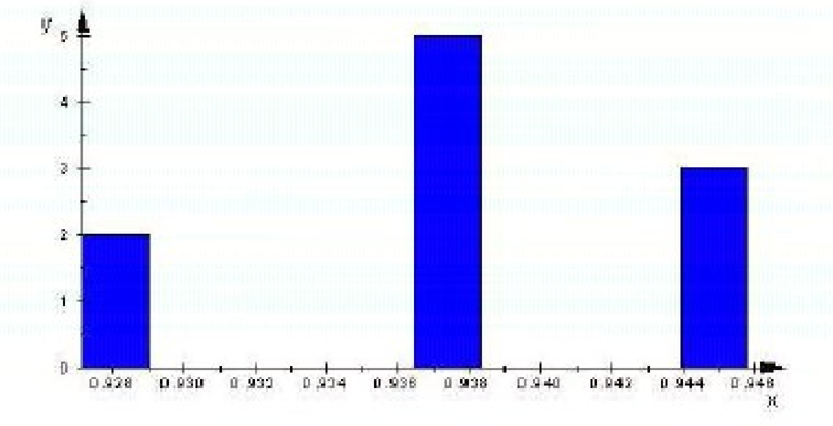
Genetic algorithms ; results



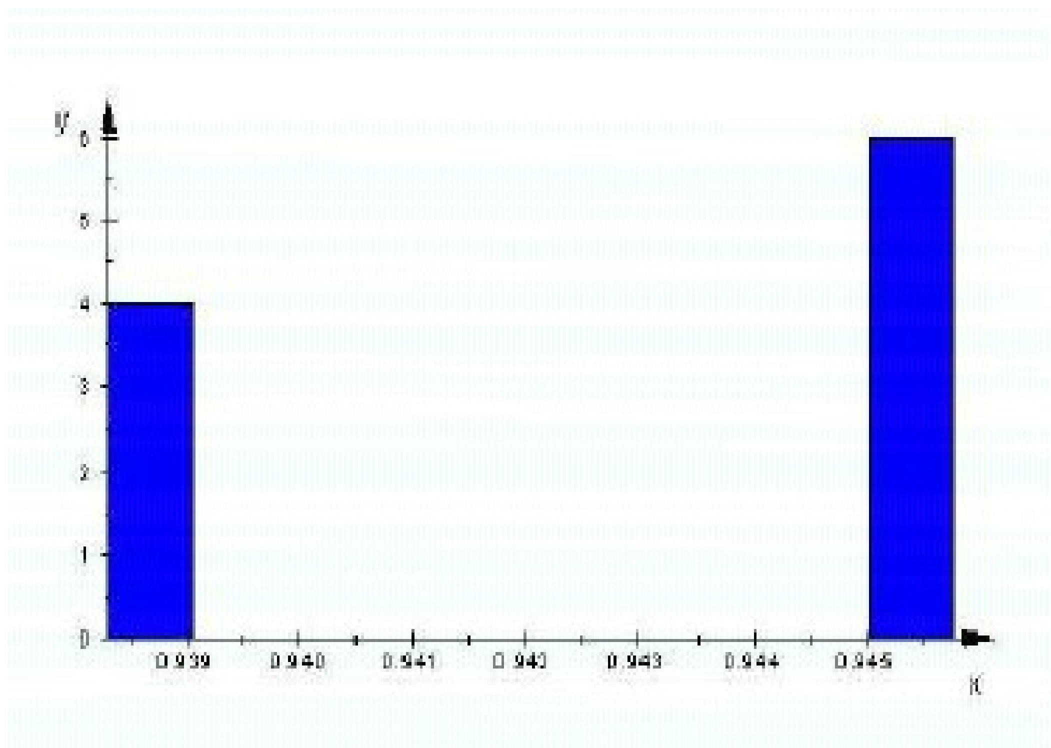


Genetic algorithms ; results





Genetic algorithms ; results



Final result : the population is rendered homogeneous

Behaviour ... (recall)

3) The behaviour between two states « r,s » w.r.t. A word « w » is the product of

3a) the ingoing coefficient of the first state (here « r ») by

3b) the sum of the weights of the paths going from « r » to « s » with label « w » by

3c) the outgoing coefficient of the second state (here « s »)

Question : Can we characterize (mathematically) the "behaviour" functions ?

Theorem A: For a function f on the free monoid, TFAE (the notations being as above)

i) There are functions f_i, g_i $i=1,2..n$ such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in A^* (the free monoid of alphabet A).

ii) There is a morphism of monoids $\mu: A^* \to k^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iii) (Schützenberger) (If A is finite) c lies in the rational closure of A within the algebra k<<A>>.

Schützenberger's theorem (known as the theorem of Kleene-Schützenberger) could be rephrased in saying that rational functions are exactly behaviours of finite (state and alphabet) automata.

In our case, we are obliged to allow infinitely many edges.

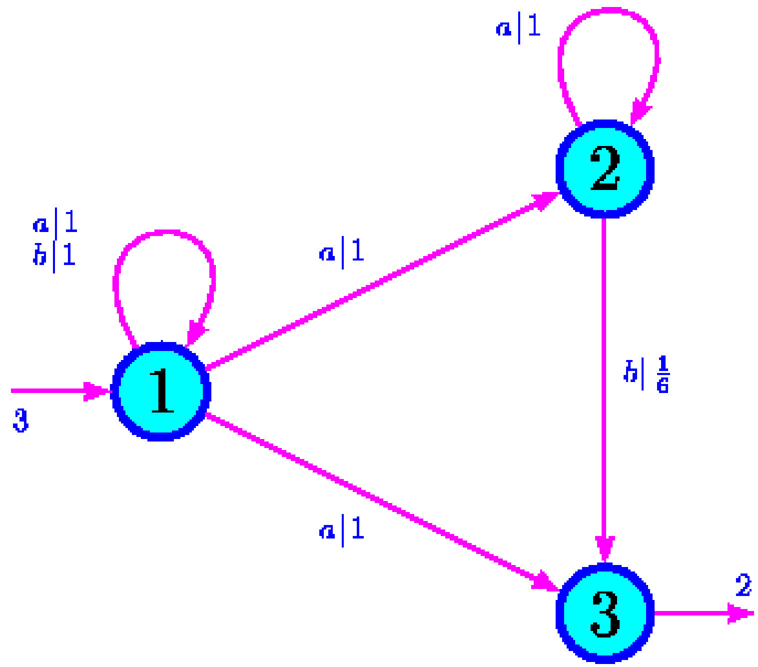
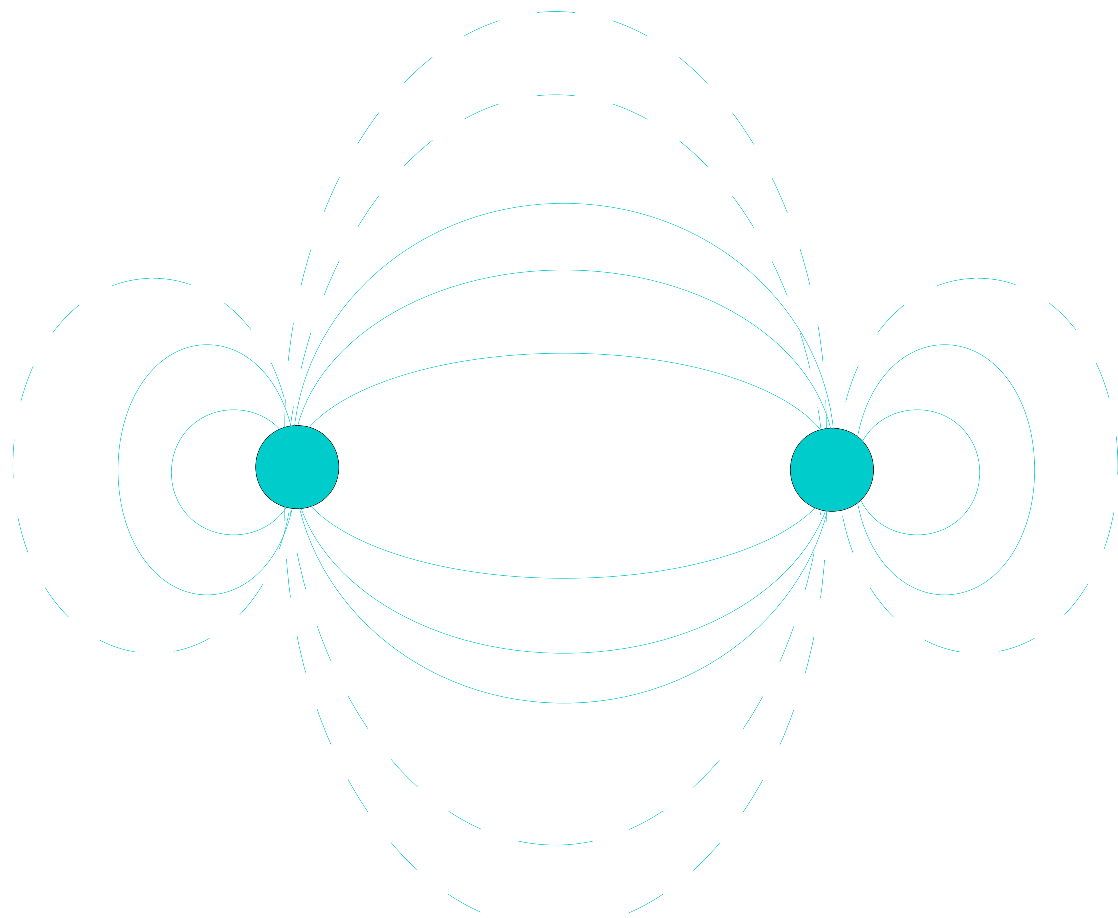


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6+a^*b).$$



We can safely apply the first three conditions of **<u>Theorem A</u>** to *Ldiag.* The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more "basic" functions. The modification reads

iii') (A is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in A} p(a)\, a$$

within the algebra k<<A>>.

Computations in $K<A>^0$, the space of behaviours

Summability : We say that a family $(f_i)_{i\in I}$ (I finite or not, f_i in $K<<A>>$) is summable if, for each $w \in A^*$, the family $(<f_i|w>)_{i\in I}$ is finitely supported and we set

$$(\sum_{i\in I} f_i) : w \to (\sum_{i\in I} <f_i|w>)$$

Identifying each word with the Dirac linear form located at the word, one has then, for each $f \in K<<A>>$

$$f = \sum_{w\in A^*} f(w)w$$

If $f \in K^{rat}\langle\langle A \rangle\rangle$, it exists a morphism of monoids $\mu: A^* \to K^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*, $f(w)=\lambda\mu(w)\xi$. Then

$$f= \sum_{w\in A^*} f(w)w=\sum_{w\in A^*} \lambda\mu(w)\xi\, w=\lambda\left(\sum_{w\in A^*} \mu(w)w\right)\xi=$$

$$\lambda\left(\sum_{w\in A^*} \mu(w)w\right)\xi=\lambda\left(\sum_{m\geq 0} \sum_{|w|=m} \mu(w)w\right)\xi$$

But, as words and scalars commute (it is so by construction of the convolution algebra $K^{n \times n}\langle\langle A \rangle\rangle$), one has

$$\sum_{m\geq 0} \sum_{|w|=m} \mu(w)w=\sum_{m\geq 0} \left(\sum_{a\in A} \mu(a)a\right)^m=\left(\sum_{a\in A} \mu(a)a\right)^*$$

hence

$$f=\lambda\left(\sum_{a\in A} \mu(a)a\right)^*\xi$$

where the “star” stands for the sum of the geometric series.

If Q is a finite set, the space $k^{Q\times Q}$ of square matrices with indices in Q and coefficients in k has a natural semiring structure with the usual operations (sum and product). A (right) star of $M \in k^{Q\times Q}$ (when it exists) is a solution of the equation $MY + 1_{Q\times Q} = Y$ (where $1_{Q\times Q}$ is the identity matrix). Let $M \in k^{Q\times Q}$ be given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} \in k^{Q_1\times Q_1}$, $a_{12} \in k^{Q_1\times Q_2}$, $a_{21} \in k^{Q_2\times Q_1}$ and $a_{22} \in k^{Q_2\times Q_2}$ such that $Q_1 + Q_2 = Q$. Let $N \in k^{Q\times Q}$ given by

$$N = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = (a_{11} + a_{12}a_{22}{}^{*}a_{21})^{*} \qquad (1)$$

$$A_{12} = a_{11}{}^{*}a_{12}A_{22} \qquad (2)$$

$$A_{21} = a_{22}{}^{*}a_{21}A_{11} \qquad (3)$$

$$A_{22} = (a_{22} + a_{21}a_{11}{}^{*}a_{12})^{*} \qquad (4)$$

$$A_{11} = (a_{11} + a_{12}a_{22}{}^{*}a_{21})^{*} \quad (1)$$

$$A_{12} = a_{11}{}^{*}a_{12}A_{22} \quad (2)$$

$$A_{21} = a_{22}{}^{*}a_{21}A_{11} \quad (3)$$

$$A_{22} = (a_{22} + a_{21}a_{11}{}^{*}a_{12})^{*} \quad (4)$$

Sketch of the proof

- Prove the equivalence

  equational star <=> iterative star

- Prove that the block-matrix A_{ij} solves the equational star equation

A (short) word on automata theory.

- The formulas (for the star* of a matrix) above are sufficiently "expressive" to be the crucial fact in the resolution of a conjecture in Noncommutative Geometry.

- For applications, automata theory had to cope with spaces of coefficients much more general than that of a field ... even the "minus" operation of the rings had to disappear to be able to cope with problems like shortest path or the Noncommutative problem or the shortest path with list of minimal arcs .

The emerging structure is that of a semiring. Think of a ring without the "minus" operation, nevertheless "transfer" matrix computations can be performed.

The input alphabet being set by the automaton under consideration, we will here rather focus on the definition of semirings providing transition coefficients. For convenience, we first begin with various laws on $\mathbb{R}_+ := [0, +\infty[$ including

1. $+$ (ordinary sum)
2. $\times$ (ordinary product)
3. min (if over $[0, 1]$, with neutral 1, otherwise must be extended to $[0, +\infty]$ and then, with neutral $+\infty$) or max
4. $+_a$ defined by $x +_a y := log_a(a^x + a^y)$ $(a > 0)$
5. $+_{[n]}$ (Hölder laws) defined by $x +_{[n]} y := \sqrt[n]{x^n + y^n}$
6. $+^s$ (shifted sum, $x +^c y := x + y - 1$, over whole $\mathbb{R}$, with neutral 1)
7. $\times^c$ (complemented product, $x + y - xy$, can be extended also to whole $\mathbb{R}$, stabilizes the range of probabilities or fuzzy $[0, 1]$ and is distributive over the shifted sum)

As (useful) examples, one has ([0,+∞], min, +), ([0,+∞[, max, +) or its (commutative or not) variants.

What remains for K<A> ? (free algebra)

K semiring :

- Universal properties (comprising – little known - tensor products)
- Complete semiring K<<A>>, summability is defined by pointwise convergence (see computation above).
- Rational closures and Kleene-Schützenberger Thm
- Rational expressions, Brzozowski theorem
- Automata theory, theory of codes
- Lazard's monoidal elimination

In fact, one call pull the operations on the functions back to the level of automata.

Proposition 2 *Let* $R : \mathcal{A}_r = (\lambda^r, \mu^r, \gamma^r)$ *(resp.* $S : \mathcal{A}_s = (\lambda^s, \mu^s, \gamma^s)$*) of rank* n *(resp.* m*). The linear representations of the sum, the concatenation and the star are respectively*
$R + S$:

$$\mathcal{A}_r \boxplus \mathcal{A}_s = \left(\begin{pmatrix} \lambda^r & \lambda^s \end{pmatrix}, \left(\begin{array}{c|c} \mu^r(a) & 0_{n\times m} \\ \hline 0_{m\times n} & \mu^s(a) \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^r \\ \gamma^s \end{pmatrix} \right) , \tag{1}$$

$R.S$:

$$\mathcal{A}_r \boxdot \mathcal{A}_s = \left(\begin{pmatrix} \lambda^r & 0_{1\times m} \end{pmatrix}, \left(\begin{array}{c|c} \mu^r(a) & \gamma^r\lambda^s\mu^s(a) \\ \hline 0_{m\times n} & \mu^s(a) \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^r\lambda^s\gamma^s \\ \gamma^s \end{pmatrix} \right) , \tag{2}$$

If $\lambda^s\gamma^s = 0$, S^* :

$$\mathcal{A}_s{}^{\circledast} = \left(\begin{pmatrix} 0_{1\times m} & 1 \end{pmatrix}, \left(\begin{array}{c|c} \mu^s(a) + \gamma^s\lambda^s\mu^s(a) & 0_{m\times 1} \\ \hline \lambda^s\mu^s(a) & 0 \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^s \\ 1 \end{pmatrix} \right) . \tag{3}$$

A provisional conclusion

This weighting structure is very flexible because it allows as well the set-theoretical old computation (logic, computer science) as more sophisticated (stochastic, non-commutative memories, operator valued etc …) weightings.

As was witnessed in other domains, function spaces say more on the data structures than the data themselves … but we still await our Grothendieck, or Einstein … !