Statistics on Graphs, Exponential Formula and Combinatorial Physics

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The Exponential formula can be traced back to works by Touchard ("Sur les cycles des substitutions", 1939) and by Ridell & Uhlenbeck ("On the theory of the virial development of the equation of state of monoatomic gases", 1953).

The title of the talk has been chosen in reference to the work of the latter physicists.

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Exponential Formula : Informal Version

Informally speaking, the exponential formula means that "*the*

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EGF(S; z) = e^{EGF(S_c; z)}.
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 (1)

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Informally speaking, the exponential formula means that "*the exponential generating function* EGF(*S*; z) *of a class S of (combinatorial) structures is equal to the exponential* $e^{EGF(S_c;z)}$ *of those of the connected substructures Sc*", *i.e.*,

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Objective

The exponential formula occurs quite naturally in many physical **contexts.** Nevertheless, applying the exponential paradigm one can

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The exponential formula occurs quite naturally in many physical contexts. Nevertheless, applying the exponential paradigm one can feel sometimes incomfortable wondering whether "one has the right" to do so.

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Objective

The exponential formula occurs quite naturally in many physical contexts. Nevertheless, applying the exponential paradigm one can feel sometimes incomfortable wondering whether "one has the right" to do so.

The objective of this talk is to present a general and formal framework in which the exponential formula holds.

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In the informal version of the exponential paradigm, there are (at least) two indefinite notions :

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- **1** The notion of connected substructures :
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In the informal version of the exponential paradigm, there are (at least) two indefinite notions :

- **1** The notion of connected substructures :
- ² Classes of structures admiting an exponential generating function.

 $\mathbf{E} = \mathbf{A} \oplus \mathbf{A} + \mathbf{A} \oplus \mathbf{A} + \mathbf{A} \oplus \mathbf{A} + \mathbf{A} \oplus \mathbf{A}$

Connected structures

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Connected structures

For a given class of structures *S*, it is often possible to define a subclass S_c of connected structures. They can be seen as the

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 $\mathcal{A} \equiv \mathcal{F} \times \{ \mathcal{B} \} \times \{ \mathcal{B} \times \{ \mathcal{B} \} \}$

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For a given class of structures *S*, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some "bigger" structures.

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For a given class of structures *S*, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some "bigger" structures. Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

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Examples

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- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every
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- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of distinct prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

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Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.

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Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.

Regarding the previous examples, we deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.

Convention : Since we will deal with a partially defined function, we adopt the following convention. If *f* is a partial function, then " $f(x) = f(y)$ " means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

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- 3 There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of

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If $D = M \times M$, that is, \oplus is totally defined, then M is a (total) usual monoid.

 $A \equiv 1 + \left(\sqrt{p} \right) \times \left(\frac{p}{p} \right)$

Notations

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Notations

A sum
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x_1 \oplus x_2 \oplus \cdots \oplus x_n
$$
 is written as $\bigoplus_{i=1}^n x_i$, and, $n.x := x \oplus \cdots \oplus x$
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for an integer *n*.

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Let (M, \oplus) be a partial commutative monoid. An indecomposable

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Let (M, \oplus) be a partial commutative monoid. An indecomposable element of *M* is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously,

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A decomposition of $x \in M$ is a mapping f from $I(M)$ to N, with only **finitely many non-zero values,** such that $x = \bigoplus f(p) \cdot p$. If *x* has a

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unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that *M* has the unique decomposition property.

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1 Cancellation:

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- **1** Cancellation:
- ² Well-founded divisibility relation ;
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- ² Well-founded divisibility relation ;
- ³ Indecomposable elements are "primes" with respect to divisibility.

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Cancellation

A partial commutative monoid *M* is cancellative if $x \oplus y = x \oplus z$ **implies that** $y = z$ **.** The partial monoid of graphs with the direct sum is

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Divisibility relation

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Divisibility relation

Let (M, \oplus) be a partial commutative monoid, and $x, y \in M$. We say that *y* divides *x*, denoted by *y*|*x*, if there is $y' \in M$, such that $x = y \oplus y'$.

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 $\mathcal{A} \equiv \mathcal{F} \times \{ \mathcal{B} \} \times \{ \mathcal{B} \times \{ \mathcal{B} \} \}$

Characterization of the unique decomposition property

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Characterization of the unique decomposition property

A partial commutative monoid *M* has the unique decomposition property iff

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A partial commutative monoid *M* has the unique decomposition property iff

- **1** *M* is cancellative ;
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A partial commutative monoid *M* has the unique decomposition property iff

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Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

 $\mathcal{A} \ \Box \ \rightarrow \ \mathcal{A} \ \overline{\partial} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \quad \ \overline{\mathcal{B}}$

A partial commutative monoid *M* with the unique decomposition **property is called square-free** if for every $x \in M$, and every $p \in I(P)$,

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 $\mathcal{A} \equiv \mathcal{F} \rightarrow \mathcal{A} \equiv \mathcal{F} \rightarrow \mathcal{A} \equiv \mathcal{F} \rightarrow \mathcal{A}$

A partial commutative monoid *M* with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, **then** ∂ **_x**(*p*) ∈ {**0**, **1**}. The intuitive meaning is that no indecomposable

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 $\mathcal{A} \equiv \mathcal{F} \times \{ \mathcal{B} \} \times \{ \mathcal{B} \times \{ \mathcal{B} \} \}$

A partial commutative monoid *M* with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, then $\partial_r(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The

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A partial commutative monoid *M* with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, then $\partial_r(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The graphs and the square-free integers are examples of square-free partial commutative monoids.

 $\mathcal{A} \ \Box \ \rightarrow \ \mathcal{A} \ \overline{\partial} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \quad \ \overline{\mathcal{B}}$

Set-theoretical support (1/2)

Let (M, \oplus) a square-free partial commutative monoid with *D* for the domain of ⊕. Now *M* is considered as a class of structures, *i.e.*, there

$$
\begin{array}{rcl}\n\sigma(x) & = & \emptyset \\
D & = & \{(x, y) \in M^2 : \sigma(x) \cap \sigma(y) = \emptyset\}, \\
\sigma(x \oplus y) & = & \sigma(x) \cup \sigma(y).\n\end{array}
$$
\niff

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Set-theoretical support (1/2)

Let (M, \oplus) a square-free partial commutative monoid with *D* for the domain of ⊕. Now *M* is considered as a class of structures, *i.e.*, there exists a set *X* and a set-theoretical mapping $\sigma : M \to \mathcal{P}_{\text{fin}}(X)$, called support mapping, such that

$$
\begin{array}{rcl}\n\sigma(x) & = & \emptyset \\
D & = & \{(x, y) \in M^2 : \sigma(x) \cap \sigma(y) = \emptyset\}, \\
\sigma(x \oplus y) & = & \sigma(x) \cup \sigma(y)\n\end{array} \quad \text{iff} \quad x = 0 \tag{2}
$$

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 $\mathcal{A} \ \Box \ \rightarrow \ \mathcal{A} \ \overline{\partial} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \quad \ \overline{\mathcal{B}}$

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Set-theoretical support (2/2)

For instance, let us consider the square-free partial commutative monoid $\mathcal{G}(\mathbb{N})$ of graphs with integer numbers as vertices. Then, the

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 $\mathcal{A} \equiv \mathcal{V} \rightarrow \mathcal{A} \oplus \mathcal{V} \rightarrow \mathcal{A} \oplus \mathcal{V} \rightarrow \mathcal{A} \oplus \mathcal{V} \quad .$

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Set-theoretical support (2/2)

For instance, let us consider the square-free partial commutative monoid $\mathcal{G}(\mathbb{N})$ of graphs with integer numbers as vertices. Then, the mapping $V : \mathcal{G}(\mathbb{N}) \to \mathcal{P}_{\text{fin}}(\mathbb{N})$ which maps a graph G to its set of vertices $V(G)$ is a support mapping.

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A 3-tuple (M, X, σ) defined as in the previous slide is called a square-free partial commutative monoid with support in (the finite subsets of) *X*.

Let (M, X, σ) be a square-free partial commutative monoid with **support in** *X*. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$
N_Y := \{ x \in N : \sigma(x) = Y \} . \tag{3}
$$

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N^Y is the set of all elements of *M* with support equals to *Y*. We say that (M, X, σ) is locally finite if for every finite subset *Y* of *X*, *M^Y* is also finite, *i.e.*, there is only finitely many elements supported

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\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]). \tag{4}
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From a combinatorial point of view, the elements of *M* should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid *M* is a mapping from *M* to a (unitary) ring *R* of characteristic zero such that

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Proposition : Equivariance property on *M^Y*

Let *M* be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let *Y*, *Y'* be two finite

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\mu(M_Y) = \mu(M_{Y'}) := \mu(M[n]) . \tag{5}
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 $\mathcal{A} \ \Box \ \rightarrow \ \mathcal{A} \ \overline{\partial} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \ \mathcal{A} \ \ \overline{\mathcal{B}} \ \rightarrow \quad \ \overline{\mathcal{B}}$

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Let *M* be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let $N \in \{M, I(M)\}$. We

$$
EGF(N; z) := \sum_{n=0}^{\infty} \mu(N[n]) \frac{z^n}{n!} . \tag{6}
$$

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(Recall that $\mu(N[n])$ is the common value of $\mu(N_Y)$ for every finite subset *Y* of *X* of cardinality *n*.)

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Exponential formula for *M*

We have

$$
EGF(M; z) = \mu(0) - 1 + e^{EGF(I(M); z)}.
$$
 (7)

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 (8)

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Example

Let $\mathfrak E$ be the set of all equivalence relations on finite subsets of $\mathbb N$:

$$
\mu(\mathfrak{E}[n]) := \sum_{k \ge 0} S_2(n,k) \mathbf{x}^k . \tag{9}
$$

$$
EGF(I(\mathfrak{E}); z) = x(e^z - 1) \tag{10}
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Let $\mathfrak E$ be the set of all equivalence relations on finite subsets of $\mathbb N$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}, X$ finite, such that *E* is an **equivalence relation on** *X*. Every element of \mathfrak{E} may be identified with

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Let $\mathfrak E$ be the set of all equivalence relations on finite subsets of $\mathbb N$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}, X$ finite, such that *E* is an equivalence relation on *X*. Every element of $\mathfrak E$ may be identified with its graph. Let $S_2(n, k)$ be the Stirling numbers of second kind, *i.e.*, the

$$
\mu(\mathfrak{E}[n]) := \sum_{k \ge 0} S_2(n,k) \mathbf{x}^k . \tag{9}
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$$
 (11)

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Example

Let $\mathfrak E$ be the set of all equivalence relations on finite subsets of $\mathbb N$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}, X$ finite, such that *E* is an equivalence relation on *X*. Every element of $\mathfrak E$ may be identified with its graph. Let $S_2(n, k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality *n* with exactly *k* connected components. We choose as statistic

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\mu(\mathfrak{E}[n]) := \sum_{k \ge 0} S_2(n,k) \mathbf{x}^k . \tag{9}
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and therefore,

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