Statistics on Graphs, Exponential Formula and Combinatorial Physics

G.H.E. Duchamp, Laurent Poinsot, S. Goodenough and K.A. Penson

UMR 7030 - Université Paris 13 - Institut Galilée and UMR 7600 - Université Pierre et Marie Curie

ICCSA 2009 - CPC 2009 “Combinatorics, Physics and Complexity”

The Exponential formula can be traced back to works by Touchard ("Sur les cycles des substitutions", 1939) and by Ridell & Uhlenbeck ("On the theory of the virial development of the equation of state of monoatomic gases", 1953).

The title of the talk has been chosen in reference to the work of the latter physicists.

The Exponential formula can be traced back to works by Touchard (“Sur les cycles des substitutions”, 1939) and by Ridell & Uhlenbeck (“On the theory of the virial development of the equation of state of monoatomic gases”, 1953).

The title of the talk has been chosen in reference to the work of the latter physicists.

The Exponential formula can be traced back to works by Touchard ("Sur les cycles des substitutions", 1939) and by Ridell & Uhlenbeck ("On the theory of the virial development of the equation of state of monoatomic gases", 1953).

The title of the talk has been chosen in reference to the work of the latter physicists.

The Exponential formula can be traced back to works by Touchard ("Sur les cycles des substitutions", 1939) and by Ridell & Uhlenbeck ("On the theory of the virial development of the equation of state of monoatomic gases", 1953).

The title of the talk has been chosen in reference to the work of the latter physicists.

Exponential Formula : Informal Version

Informally speaking, the exponential formula means that *"the exponential generating function* $\mathsf{EGF}(S;\mathrm{z})$ *of a class S of (combinatorial) structures is equal to the exponential* $e^{\mathsf{EGF}(S_c;\mathrm{z})}$ *of those of the connected substructures* S_c", i.e.,

$$\mathsf{EGF}(S;\mathrm{z}) = e^{\mathsf{EGF}(S_c;\mathrm{z})} \,. \qquad (1)$$

Exponential Formula : Informal Version

Informally speaking, the exponential formula means that "*the exponential generating function* $\mathsf{EGF}(S; \mathtt{z})$ *of a class* S *of (combinatorial) structures is equal to the exponential* $e^{\mathsf{EGF}(S_c;\mathtt{z})}$ *of those of the connected substructures* S_c", *i.e.*,

$$\mathsf{EGF}(S; \mathtt{z}) = e^{\mathsf{EGF}(S_c;\mathtt{z})} . \tag{1}$$

Exponential Formula : Informal Version

Informally speaking, the exponential formula means that “*the exponential generating function* $\mathsf{EGF}(S; \mathtt{z})$ *of a class* S *of (combinatorial) structures is equal to the exponential* $e^{\mathsf{EGF}(S_c;\mathtt{z})}$ *of those of the connected substructures* S_c", *i.e.*,

$$\mathsf{EGF}(S; \mathtt{z}) = e^{\mathsf{EGF}(S_c;\mathtt{z})} \ . \qquad (1)$$

Objective

The exponential formula occurs quite naturally in many physical contexts. Nevertheless, applying the exponential paradigm one can feel sometimes incomfortable wondering whether "one has the right" to do so.

The objective of this talk is to present a general and formal framework in which the exponential formula holds.

Objective

The exponential formula occurs quite naturally in many physical contexts. Nevertheless, applying the exponential paradigm one can feel sometimes incomfortable wondering whether “one has the right” to do so.

The objective of this talk is to present a general and formal framework in which the exponential formula holds.

Objective

The exponential formula occurs quite naturally in many physical contexts. Nevertheless, applying the exponential paradigm one can feel sometimes incomfortable wondering whether “one has the right” to do so.

The objective of this talk is to present a general and formal framework in which the exponential formula holds.

In the informal version of the exponential paradigm, there are (at least) two indefinite notions :

1. The notion of connected substructures ;
2. Classes of structures admiting an exponential generating function.

In the informal version of the exponential paradigm, there are (at least) two indefinite notions :

1. The notion of connected substructures ;
2. Classes of structures admiting an exponential generating function.

In the informal version of the exponential paradigm, there are (at least) two indefinite notions :

1. The notion of connected substructures ;
2. Classes of structures admiting an exponential generating function.

Connected structures

For a given class of structures S, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some "bigger" structures. Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

Connected structures

For a given class of structures S, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some “bigger” structures. Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

Connected structures

For a given class of structures S, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some "bigger" structures.

Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

Connected structures

For a given class of structures S, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some “bigger” structures. Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

Connected structures

For a given class of structures S, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some "bigger" structures. Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a "disjoint" product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

Objective

Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.

Regarding the previous examples, we deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.

Convention : Since we will deal with a partially defined function, we adopt the following convention. If f is a partial function, then "$f(x) = f(y)$" means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

Objective

Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.

Regarding the previous examples, we deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.

Convention : Since we will deal with a partially defined function, we adopt the following convention. If f is a partial function, then "$f(x) = f(y)$" means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

Objective

Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.

Regarding the previous examples, we deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.

Convention : Since we will deal with a partially defined function, we adopt the following convention. If f is a partial function, then "$f(x) = f(y)$" means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

Objective

Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.
Regarding the previous examples, we deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.
Convention : Since we will deal with a partially defined function, we adopt the following convention. If f is a partial function, then "$f(x) = f(y)$" means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

Objective

Following the examples of graphs and square-free integers, we introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.
Regarding the previous examples, we deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.
Convention : Since we will deal with a partially defined function, we adopt the following convention. If f is a partial function, then "$f(x) = f(y)$" means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \rightarrow M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \rightarrow M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \rightarrow M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Partial commutative monoids

A partial commutative monoids is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Notations

A sum $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is written as $\bigoplus_{i=1}^{n} x_i$, and, $n.x := \underbrace{x \oplus \cdots \oplus x}_{n \text{ factors}}$ for an integer n.

Notations

A sum $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is written as $\bigoplus_{i=1}^{n} x_i$, and, $n.x := \underbrace{x \oplus \cdots \oplus x}_{n \text{ factors}}$ for an integer n.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then we shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

Now, a question arises : what are the properties of partial commutative monoids that characterize monoids with the unique decomposition property ?

1. Cancellation ;
2. Well-founded divisibility relation ;
3. Indecomposable elements are “primes” with respect to divisibility.

Now, a question arises : what are the properties of partial commutative monoids that characterize monoids with the unique decomposition property ?

1. Cancellation ;
2. Well-founded divisibility relation ;
3. Indecomposable elements are "primes" with respect to divisibility.

Now, a question arises : what are the properties of partial commutative monoids that characterize monoids with the unique decomposition property ?

1. Cancellation ;
2. Well-founded divisibility relation ;
3. Indecomposable elements are “primes” with respect to divisibility.

Now, a question arises : what are the properties of partial commutative monoids that characterize monoids with the unique decomposition property ?

1. Cancellation ;
2. Well-founded divisibility relation ;
3. Indecomposable elements are "primes" with respect to divisibility.

Cancellation

A partial commutative monoid M is cancellative if $x \oplus y = x \oplus z$ implies that $y = z$. The partial monoid of graphs with the direct sum is cancellative.

Cancellation

A partial commutative monoid M is cancellative if $x \oplus y = x \oplus z$ implies that $y = z$. The partial monoid of graphs with the direct sum is cancellative.

Divisibility relation

Let $(M, \oplus)$ be a partial commutative monoid, and $x, y \in M$. We say that y divides x, denoted by $y|x$, if there is $y' \in M$, such that $x = y \oplus y'$.

Divisibility relation

Let $(M, \oplus)$ be a partial commutative monoid, and $x, y \in M$. We say that y divides x, denoted by $y|x$, if there is $y' \in M$, such that $x = y \oplus y'$.

Divisibility relation

Let $(M, \oplus)$ be a partial commutative monoid, and $x, y \in M$. We say that y divides x, denoted by $y|x$, if there is $y' \in M$, such that $x = y \oplus y'$.

Divisibility relation

Let $(M, \oplus)$ be a partial commutative monoid, and $x, y \in M$. We say that y divides x, denoted by $y|x$, if there is $y' \in M$, such that $x = y \oplus y'$.

Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

1. M is cancellative ;
2. The divisibility relation of M is well-founded ;
3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

1. M is cancellative ;
2. The divisibility relation of M is well-founded ;
3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

1. M is cancellative ;
2. The divisibility relation of M is well-founded ;
3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

1. M is cancellative ;
2. The divisibility relation of M is well-founded ;
3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

1. M is cancellative ;
2. The divisibility relation of M is well-founded ;
3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

1. M is cancellative ;
2. The divisibility relation of M is well-founded ;
3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

Remark : Points (1) and (2) ensure the existence of a decomposition for every elements. Unicity is given by point (3).

Square-free partial commutative monoids

A partial commutative monoid M with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, then $\partial_x(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The graphs and the square-free integers are examples of square-free partial commutative monoids.

Square-free partial commutative monoids

A partial commutative monoid M with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, then $\partial_x(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The graphs and the square-free integers are examples of square-free partial commutative monoids.

Square-free partial commutative monoids

A partial commutative monoid M with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, then $\partial_x(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The graphs and the square-free integers are examples of square-free partial commutative monoids.

Square-free partial commutative monoids

A partial commutative monoid M with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(P)$, then $\partial_x(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The graphs and the square-free integers are examples of square-free partial commutative monoids.

Set-theoretical support (1/2)

Let $(M, \oplus)$ a square-free partial commutative monoid with D for the domain of $\oplus$. Now M is considered as a class of structures, *i.e.*, there exists a set X and a set-theoretical mapping $\sigma : M \to \mathcal{P}_{fin}(X)$, called support mapping, such that

$$\begin{array}{lcll} \sigma(x) & = & \emptyset & \text{iff} \quad x = 0 \,, \\ D & = & \{(x, y) \in M^2 : \sigma(x) \cap \sigma(y) = \emptyset\} \,, & \\ \sigma(x \oplus y) & = & \sigma(x) \cup \sigma(y) \,. & \end{array} \tag{2}$$

Set-theoretical support (1/2)

Let $(M, \oplus)$ a square-free partial commutative monoid with D for the domain of $\oplus$. Now M is considered as a class of structures, *i.e.*, there exists a set X and a set-theoretical mapping $\sigma : M \to \mathcal{P}_{fin}(X)$, called support mapping, such that

$$\begin{array}{lcll} \sigma(x) & = & \emptyset & \text{iff} \quad x = 0\ , \\ D & = & \{(x,y) \in M^2 : \sigma(x) \cap \sigma(y) = \emptyset\}\ , & \\ \sigma(x \oplus y) & = & \sigma(x) \cup \sigma(y)\ . & \end{array} \tag{2}$$

Set-theoretical support (2/2)

For instance, let us consider the square-free partial commutative monoid $\mathcal{G}(\mathbb{N})$ of graphs with integer numbers as vertices. Then, the mapping $V : \mathcal{G}(\mathbb{N}) \rightarrow \mathcal{P}_{fin}(\mathbb{N})$ which maps a graph G to its set of vertices $V(G)$ is a support mapping.

A 3-tuple (M, X, σ) defined as in the previous slide is called a square-free partial commutative monoid with support in (the finite subsets of) X.

Set-theoretical support (2/2)

For instance, let us consider the square-free partial commutative monoid $\mathcal{G}(\mathbb{N})$ of graphs with integer numbers as vertices. Then, the mapping $V : \mathcal{G}(\mathbb{N}) \to \mathcal{P}_{fin}(\mathbb{N})$ which maps a graph G to its set of vertices $V(G)$ is a support mapping.

A 3-tuple (M, X, σ) defined as in the previous slide is called a square-free partial commutative monoid with support in (the finite subsets of) X.

Set-theoretical support (2/2)

For instance, let us consider the square-free partial commutative monoid $\mathcal{G}(\mathbb{N})$ of graphs with integer numbers as vertices. Then, the mapping $V : \mathcal{G}(\mathbb{N}) \to \mathcal{P}_{fin}(\mathbb{N})$ which maps a graph G to its set of vertices $V(G)$ is a support mapping.

A 3-tuple (M, X, σ) defined as in the previous slide is called a square-free partial commutative monoid with support in (the finite subsets of) X.

Locally finite square-free monoids

Let (M, X, σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} . \quad (3)$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M, X, σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

Locally finite square-free monoids

Let (M, X, σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} . \quad (3)$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M, X, σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

Locally finite square-free monoids

Let (M, X, σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} . \tag{3}$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M, X, σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

Locally finite square-free monoids

Let (M,X,σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} . \qquad (3)$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M,X,σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

Locally finite square-free monoids

Let (M, X, σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} . \tag{3}$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M, X, σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

Locally finite square-free monoids

Let (M,X,σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} \ . \tag{3}$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M,X,σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \ . \quad (4)$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \; . \tag{4}$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) . \qquad (4)$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) . \quad (4)$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \ . \tag{4}$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) . \quad (4)$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \ . \qquad (4)$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \ . \tag{4}$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then
$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \ . \tag{4}$$
(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)
2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) \ . \tag{4}$$

   (where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

2. μ is multiplicative, *i.e.*, $\mu(x \oplus y) = \mu(x)\mu(y)$.

Proposition : Equivariance property on M_Y

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let Y, Y' be two finite subsets of X of same cardinality n. Then,

$$\mu(M_Y) = \mu(M_{Y'}) := \mu(M[n]) \ . \tag{5}$$

Proposition : Equivariance property on M_Y

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let Y, Y' be two finite subsets of X of same cardinality n. Then,

$$\mu(M_Y) = \mu(M_{Y'}) := \mu(M[n]) \ . \quad (5)$$

Proposition : Equivariance property on M_Y

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let Y, Y' be two finite subsets of X of same cardinality n. Then,

$$\mu(M_Y) = \mu(M_{Y'}) := \mu(M[n]) \ . \tag{5}$$

Proposition : Equivariance property on M_Y

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let Y, Y' be two finite subsets of X of same cardinality n. Then,

$$\mu(M_Y) = \mu(M_{Y'}) := \mu(M[n]) \ . \tag{5}$$

Exponential generating function of M and $I(M)$

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let $N \in \{M, I(M)\}$. We define the exponential generating function of N by

$$\mathsf{EGF}(N; z) := \sum_{n=0}^{\infty} \mu(N[n]) \frac{z^n}{n!} . \quad (6)$$

(Recall that $\mu(N[n])$ is the common value of $\mu(N_Y)$ for every finite subset Y of X of cardinality n.)

Exponential generating function of M and $I(M)$

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let $N \in \{M, I(M)\}$. We define the exponential generating function of N by

$$\mathsf{EGF}(N; z) := \sum_{n=0}^{\infty} \mu(N[n]) \frac{z^n}{n!} \, . \quad (6)$$

(Recall that $\mu(N[n])$ is the common value of $\mu(N_Y)$ for every finite subset Y of X of cardinality n.)

Exponential generating function of M and $I(M)$

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let $N \in \{M, I(M)\}$. We define the exponential generating function of N by

$$\mathsf{EGF}(N; z) := \sum_{n=0}^{\infty} \mu(N[n]) \frac{z^n}{n!} . \quad (6)$$

(Recall that $\mu(N[n])$ is the common value of $\mu(N_Y)$ for every finite subset Y of X of cardinality n.)

Exponential generating function of M and $I(M)$

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let $N \in \{M, I(M)\}$. We define the exponential generating function of N by

$$\mathsf{EGF}(N; \mathrm{z}) := \sum_{n=0}^{\infty} \mu(N[n]) \frac{\mathrm{z}^n}{n!} \ . \tag{6}$$

(Recall that $\mu(N[n])$ is the common value of $\mu(N_Y)$ for every finite subset Y of X of cardinality n.)

Exponential formula for M

We have

$$\mathsf{EGF}(M;\mathrm{z}) = \mu(0) - 1 + e^{\mathsf{EGF}(I(M);\mathrm{z})} \ . \tag{7}$$

In particular, if $\mu(0) = 1$,

$$\mathsf{EGF}(M;\mathrm{z}) = e^{\mathsf{EGF}(I(M);\mathrm{z})} \ . \tag{8}$$

Exponential formula for M

We have

$$\mathsf{EGF}(M;\mathrm{z}) = \mu(0) - 1 + e^{\mathsf{EGF}(I(M);\mathrm{z})} \ . \tag{7}$$

In particular, if $\mu(0) = 1$,

$$\mathsf{EGF}(M;\mathrm{z}) = e^{\mathsf{EGF}(I(M);\mathrm{z})} \ . \tag{8}$$

Exponential formula for M

We have

$$\mathsf{EGF}(M;\mathtt{z}) = \mu(0) - 1 + e^{\mathsf{EGF}(I(M);\mathtt{z})}\ . \tag{7}$$

In particular, if $\mu(0) = 1$,

$$\mathsf{EGF}(M;\mathtt{z}) = e^{\mathsf{EGF}(I(M);\mathtt{z})}\ . \tag{8}$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$:
$E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k\geq 0} S_2(n,k)\mathrm{x}^k \ . \tag{9}$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E});\mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \tag{10}$$

and therefore,

$$\mathsf{EGF}(\mathfrak{E};\mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} \ . \tag{11}$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k \geq 0} S_2(n,k)\mathrm{x}^k \ . \tag{9}$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E}); \mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \tag{10}$$

and therefore,

$$\mathsf{EGF}(\mathfrak{E}; \mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} \ . \tag{11}$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k \geq 0} S_2(n,k) \mathrm{x}^k . \quad (9)$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E}); \mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \quad (10)$$

and therefore,

$$\mathsf{EGF}(\mathfrak{E}; \mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} . \quad (11)$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k \geq 0} S_2(n,k)\mathrm{x}^k \,. \tag{9}$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E});\mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \tag{10}$$

and therefore,

$$\mathsf{EGF}(\mathfrak{E};\mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} \,. \tag{11}$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k \geq 0} S_2(n,k)\mathrm{x}^k \ . \tag{9}$$

Then, we can prove that

$$\mathrm{EGF}(\mathcal{I}(\mathfrak{E}); \mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \tag{10}$$

and therefore,

$$\mathrm{EGF}(\mathfrak{E}; \mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} \ . \tag{11}$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k\geq 0} S_2(n,k)\mathrm{x}^k \ . \tag{9}$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E});\mathrm{z}) = \mathrm{x}(e^{\mathrm{z}}-1) \tag{10}$$

and therefore,

$$\mathsf{EGF}(\mathfrak{E};\mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} \ . \tag{11}$$

Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling numbers of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k \geq 0} S_2(n,k)\mathrm{x}^k \ . \tag{9}$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E});\mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \tag{10}$$

and therefore,

$$\mathsf{EGF}(\mathfrak{E};\mathrm{z}) = e^{\mathrm{x}(e^{\mathrm{z}}-1)} \ . \tag{11}$$