Counting Rooted and Unrooted Triangular Maps

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***Abstract*—In this paper, we describe a new way to count isomorphism classes of rooted triangular maps and unrooted triangular maps. We point out an explicit connection with the asymptotic expansion of the Airy function. The analysis presented here is used in a recent paper "*Vidal (2007)*" to present an algorithm that generates in optimal amortized time an exhaustive list of triangular maps of a given size.**

***Index Terms*—rooted triangular maps, unrooted triangular maps, generating functions, Airy funtion, cycle index series**

Introduction

Triangulations of surfaces constitute an important data structure in computer graphics as they provide a handy discrete model of surfaces. It has proven invaluable for instance to model the shape of objects in computed graphics. From the point of view of computer science, the applications of surface triangulations are well known and numerous, they touch both practical and theoretical aspects of the discipline and they range from computer graphics to discret methods of solving partial differential equations. They also play a central role in many algorithms of computational geometry, a fast growing subject having an heavy industrial impact as it is used in computer aided design.

One particularly interesting treat of the subject, apart from its broad range of applications, is precisely its ubiquity both in computer science, mathematical physics and even pure mathematics, providing generous range of fruitful exchange between seemingly remote parts of science. From the point of view of mathematics, the theory of combinatorial maps is also a venerable subject dating back to Cayley and Hamilton. Since those times, it generated an impressive amount of results of all sorts concerning the particular enumeration problem of counting the *rooted* combinatorial maps. Those results came from various communities of researchers, each with its own methods and tradition. Among them, *enumerative combinatorists* of course played a significant role, starting with pioneering works by Tutte [17] on rooted planar maps. Those works where at first motivated by the four color problem. *Theoretical physicists* also played a significant role, starting with the work by t'Hooft [16] on integration on random matrix spaces and Feynman diagrams. Pure mathematicians like Harer and Zagier [5] also have contributed to the theory in connection with cutting edge algebraic geometry problems concerning moduli spaces of Riemann surfaces. Last but not least, one must mention in mathematical physics the Witten-Kontsevich model of quantum gravity [7] using in a central fashion the higher combinatorics of triangular maps and trivalent diagrams.

Although *a lot* is known concerning the theory of *rooted* combinatorial maps, *very little* is currently known about the outstanding problem of enumeration of *unrooted* combinatorial maps up to isomorphism except for planar maps with the pioneering work of Liskovets [9]. It appears as a very difficult problem of combinatorics, which stayed barely untouched for almost 150 years. As a matter of fact, the only general result on those important objects were up to now contained in the recent paper by Mednykh and Nedela [14]. In section II-D of this paper, we give our first contribution to this problem, namely in the form of a generating series giving the number of *unrooted* triangular maps (*c.f.* series (20) on page 4).

In this paper, a *triangular map* is a triangulation of a (not necessarily connected) oriented surface without boundary, and its *size* is an integer divisible by 6, hence of the form $n = 6k$, such that the triangulation has $2k$ triangular faces and $3k$ edges. Apart from those unrooted map enumeration results the most interesting theorems of this article are the following.

Theorem 1: Let a_n be the number of *labelled* triangular maps of size n. Then, $a_n = 0$ if n isn't a multiple of 6 and,

$$a_{6k} = \frac{(6k)!}{k!}\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k 6^k \tag{1}$$

where $(x)_k = x\,(x+1)\ldots(x+k-1)$ is the Pochhammer symbol. Therefore, the exponential generating series of the a_n is hypergeometric and divergent. We have,

$$\sum_{n\geq 0}\frac{a_n}{n!}\,z^n = {}_2\mathrm{F}_0\left(\begin{matrix}\frac{1}{6},\frac{5}{6}\\ -\end{matrix}\;\middle|\;6\,z^6\right) \tag{2}$$

$$= \sum_{k\geq 0}\frac{(\frac{1}{6})_k(\frac{5}{6})_k}{k!}\,6^k\,z^{6k} \tag{3}$$

Theorem 2: Let b_n be the number of *pointed connected unlabelled* triangular maps of size $6n$. Then b_n satisfies the following recurrence equation,

$$b_{n+1} = (6n+6)\,b_n + \sum_{k=1}^{n-1} b_k\,b_{n-k} \tag{4}$$

with $n \geq 1$ and $b_1 = 5$.

Those two theorems provide a connection with the asymptotic expansion of the Airy function as explained in section II-C.

I. Counting principle for triangular maps

An *oriented* surface without boundary is described by a finite set of triangular faces. The orientation of the surface is given by an normal unitary vector on each point of the surface, the normal vector varying continuously with respect to the point. This normal vector induce a cyclic order on the tree edges belonging to a same triangle.

Any triangulation is transformed by a classical duality (due to Poincaré), to a triangular diagram (see figures 1 and 2). Faces of the triangulation become black vertices of

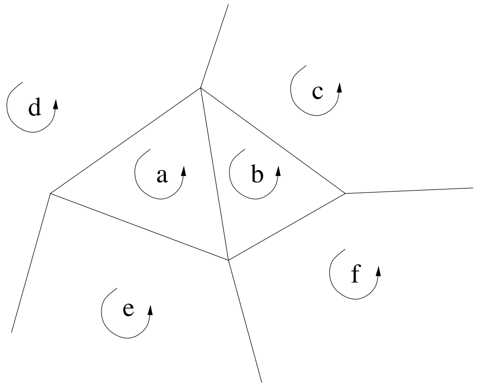

Fig. 1. A triangulation of the Riemann sphere made of $2k$ triangles (a, b, c, d, e, f). $\boxed{\text{case } k = 3}$

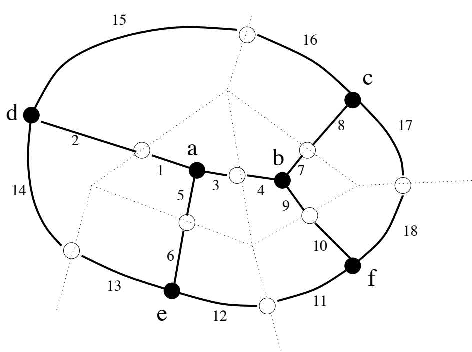

Fig. 2. The associated triangular diagram made of $n = 6k$ edges, $2k$ black vertices and $3k$ white vertices.

the diagram while edges of the triangulation become white vertices of the diagram. In such a diagram, every edges is adjacent to one black vertex and one white vertex. Black vertices have degree 3 (i.e. are adjacent to three edges). White vertices have degree 2 (i.e. are adjacent to 2 edges). Every three edges adjacent to a same black vertex are cyclically ordered. An isomorphism between two diagrams is given by a bijection which transforms edges and vertices of one of the diagrams into edges and vertices of the second, preserving color of the vertices and cyclic orientation of the edges.

In what follows, we shall describe a triangulations by a couple of two permutations (σ_0, σ_1). The permutation σ_0 exchanges the two edges adjacent to a same white vertex while the σ_1 permutation cyclically permutes the three edges adjacent to a same black vertex. For example the two permutations corresponding to figure 2, are the following,

$$\begin{aligned}\sigma_0 &= (1,2)(3,4)(5,6)(7,8)(9,10)\\ &\quad (11,12)(13,14)(15,16)(17,18),\\ \sigma_1 &= (1,5,3)(2,15,14)(4,9,7)\\ &\quad (6,13,12)(10,11,18)(8,17,16).\end{aligned}$$

Definition 1 (triangular map): A triangular map of size $n \in \mathbb{N}$ is given by a finite set of edges labelled from 1 to n and by a pair of permutations (σ_0, σ_1) belonging to $\mathfrak{S}_n$ such that the cycles of σ_0 are only of length 2 and that the cycles of σ_1 are only of length 3.

Two maps (σ_0, σ_1) and $(\bar{\sigma}_0, \bar{\sigma}_1)$ both of size n are called *isomorphic* if there exists a permutation $\tau \in \mathfrak{S}_n$ such that, $\bar{\sigma}_0 = \tau \circ \sigma_0 \circ \tau^{-1}$ and $\bar{\sigma}_1 = \tau \circ \sigma_1 \circ \tau^{-1}$, this is an exact translation of the notion of diagram isomorphism defined above.

A. Species used in this paper and related generating series

In the sequel, we use the following species (see appendix A) :

1) The species of *sets*, denoted by E, which associates to any labelling set U the singleton $E[U] := \{U\}$. For every set U there is a unique structure which is U itself.
2) The species of permutations, denoted by S, which associates to any labelling set U the set $S[U]$ composed of permutations of U. For every relabelling $\sigma : U \to U$ and every permutation $\tau \in S[U]$, one put $S[\sigma](\tau) := \sigma\tau\sigma^{-1}$.
3) The species of cycles of length n, denoted by C_n, which associates to any labelling set U the set $C_n[U]$ composed of cyclic permutations of U of length exactly n. One put $C_n[U] = \varnothing$ when $\operatorname{card} U \neq n$.
4) The species S_n of permutations having only cycles of length n.
5) The species of triangular maps, denoted by T^*, which associates to any labelling set U, the set $T^*[U]$ of triangular maps whose edges are labelled by U.
6) The species of connected triangular maps, denoted by T, which associates to any labelling set U the set $T[U]$ of connected triangular maps whose edges are labelled by U.
7) The species of pointed connected triangular maps, denoted by $T^\bullet$, having a distinguished edge.

The considered species are related by the following relations,

$$\begin{cases} S_2 = E(C_2) \\ S_3 = E(C_3) \\ T^* = S_2 \odot S_3 \\ T^* = E(T) \end{cases} \tag{5}$$

In intuitive terms, this means,

- A permutation of S_2 decompose uniquely in a set of cycles of length 2.
- A permutation of S_3 decompose uniquely in a set of cycles of length 3.
- A triangular map is uniquely determined by a couple of permutation $(\sigma_0, \sigma_1) \in S_2 \times S_3$ acting simultaneously on a set of labels.
- A triangular map is uniquely decomposed in a set of connected triangular map.

The set of species permits to derivate in an automatic fashion from the equations (5), the following relations between generating series,

$$\begin{cases} S_2(z) = \exp C_2(z) \\ S_3(z) = \exp C_3(z) \\ T^*(z) = S_2(z) \odot S_3(z) \\ T(z) = \log T^*(z) \\ T^\bullet(z) = z \dfrac{d}{dz}\, T(z) \end{cases} \tag{6}$$

II. Counting triangular maps

A. Labelled triangular maps

In this section, we are counting the triangular maps having n edges labelled by numbers from 1 to n.

Proof of theorem 1 – There are $(n-1)!$ labelled cycles of length n then $C_n(t) = \frac{z^n}{n}$. This enables to compute the following generating series,

$$S_2(z) = \exp \frac{z^2}{2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^k} \frac{1}{k!}, \tag{7}$$

$$S_3(z) = \exp \frac{z^3}{3} = \sum_{k=0}^{\infty} \frac{z^{3k}}{3^k} \frac{1}{k!}. \tag{8}$$

Let's study the coefficients of the series $T^*(z) = S_2(z) \odot S_3(z)$. In $T^*(z)$, the coefficient of z^n vanish when n in not a multiple of 6. One has,

$$[\, z^{6n} \,]\, T^* = \frac{1}{2^{3n} 3^{2n}} \frac{(6n)!}{(2n)!\; (3n)!}, \qquad (n \geq 0).$$

Let's put $T^*(z) = f(6z^6)$. Let's show that the series $f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$ is hypergeometric. Using formula (21), the computation of f_n yields,

$$f_n = \frac{1}{6^n 2^{3n} 3^{2n}} \frac{n!\; (6n)!}{(2n)!(3n)!}, \qquad (n \geq 0). \tag{9}$$

By a straightforward computation one deduces that the f_n coefficients satisfie the following linear recurrence,

$$\frac{f_{n+1}}{f_n} = (n + a_1)(n + a_2) \text{ where } a_1 = \frac{1}{6},\, a_2 = \frac{5}{6}. \tag{10}$$

As $f(0) = 1$, one deduces that $f(x) = {}_2\mathrm{F}_0 \left(\begin{matrix} a_1, a_2 \\ - \end{matrix} \middle| \, x \right)$ which ends the demonstration. □

B. Pointed connected triangular maps

Proposition 1 ([23] lemma 1.4): The species $T^\bullet$ of pointed connected triangular maps is rigid (see definition 3), or equivalently $\widetilde{T}^\bullet(z) = T^\bullet(z)$.

Proof: Let ϕ be an automorphism of a pointed connected diagram (Γ, a) preserving basepoint, *i.e.* such that $\phi(a) = a$. Let A be the set of edges of Γ left invariant under ϕ *i.e.* edges x such that $\phi(x) = x$. Our proposition is proved if we show that every edge of Γ belongs to A. As ϕ preserves adjacency and cyclic orientations around vertices, if $x \in A$ then every edges adjacent to x must be preserved by ϕ. Hence by induction the condition propagates to the full connected component of x. As $a \in A$ then A is nonempty, and as Γ is connected then every edges of Γ belongs to A, which ends the proof. ■

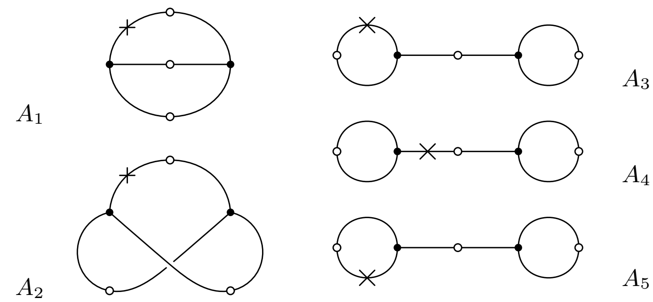

Fig. 3. As an example, this figure gives the 5 pointed connected triangular maps having 6 edges. This corresponds to $b_1 = 5$.

Proof of theorem 2 – From the linear recurrence 10, the hyper geometric series $f(x)$ satisfies the differential equation $Lf(x) = 0$ where,

$$L := \theta - x(\theta + a_1)(\theta + a_2), \qquad \theta = x \frac{d}{dx}. \tag{11}$$

Using the change of variables $x = 6z^6$, the equations (6) implies,

$$\begin{aligned} T^\bullet(z) &= z \frac{d}{dz} T(z) = z \frac{d}{dz} \log T^*(z) \\ &= 6x \frac{d}{dx} \log f(x) = 6v(x). \end{aligned} \tag{12}$$

the function $v(x) := x f'(x)/f(x)$ is solution to the Riccati equation associated to the linear equation (11),

$$v - x\big[\theta v + v^2 + (a_1 + a_2)v + a_1 a_2\big] = 0. \tag{13}$$

From this equation, one deduce that the coefficients v_n of the series $v(x) = \sum_{n \geq 0} v_n x^n$ satisfies the following quadratic recurrence (where δ is the Kronecker symbol),

$$v_{n+1} = (a_1 + a_2 + n)\, v_n + \sum_{k=0}^{n} v_k v_{n-k} + a_1 a_2\, \delta_{n,0}$$

with $n \geq 0$. Replacing a_1 et a_2 by their values yields,

$$v_{n+1} = (n+1)\, v_n + \sum_{k=0}^{n} v_k v_{n-k} + \frac{5}{36} \delta_{n,0}, \tag{14}$$

with $n \geq 0$ and initial condition $v_0 = 0$. One deduce $v_1 = 5/36$.

The sequence b_n one gets from the sequence v_n while putting One then have $b_n = 6^{n+1} v_n$. The recurrence (14) defining the sequence v_n is equivalent to the recurrence,

$$b_{n+1} = 6\,(n+1)\, b_n + \sum_{k=0}^{n} b_k b_{n-k} + 5\delta_{n,0} \tag{15}$$

with $n \geq 0$ and initial condition $b_0 = 0$ which defines the sequence b_n of theorem 2. □

That recurrence (15) gives,

$$\begin{aligned} \widetilde{T^\bullet}(z) = T^\bullet(z) \; &= 5\, z^6 + 60\, z^{12} + 1105\, z^{18} + 27120\, z^{24} \\ &+ 828250\, z^{30} + 30220800\, z^{36} + \cdots \end{aligned}$$

C. Connection with Airy asymptotics

The following special function,

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^{+\infty} \cos\Big(\frac{1}{3}t^3 + xt\Big)dt \tag{16}$$

due to the british astronomer Airy, find applications in optics. It solves the linear differential equation $y''(x) = x\,y(x)$. In ordre to numerically compute the zeros of this function, Stokes in 1857, used the following asymptotic expansion

$$\mathrm{Ai}(x) \underset{x\to+\infty}{\sim} \frac{\exp(-2/3\,x^{3/2})}{2\sqrt{\pi}\,x^{1/4}}\ {}_2\mathrm{F}_0\left(\begin{array}{c}\frac{1}{6},\frac{5}{6}\\ -\end{array}\middle|\, -\frac{3}{4}x^{-3/2}\right), \tag{17}$$

wich is divergent. It establishes the connection with the generating series (2) of $T^*(z)$.

Reworking the computation of the proof of theorem 2, one proves the following asymptotic expansion,

$$z^{2/3}\frac{\mathrm{Ai}'\big(1/4\,z^{2/3}\big)}{\mathrm{Ai}\big(1/4\,z^{2/3}\big)} + \frac{1}{2}z + 1 \underset{z\to+\infty}{\sim} \frac{b_1}{z} - \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots = -T^\bullet(-1/z), \tag{18}$$

where Ai' is the derivative of the Airy function.

D. Connected triangular maps

In this section, we are computing the generating series $\widetilde{T}(z)$ which counts connected triangular maps having n edges. Two isomorphic maps are counted once. The species T isn't rigid ; we shall thus use the cycle index series of T through the formula $\widetilde{T}(z) = \mathcal{Z}_T(z, z^2, z^3, \ldots)$. To that purpose we shall first compute the cycle index series of the specie C_2, C_3 and E which are,

$$\mathcal{Z}_{C_2}(z_1,\, z_2,\, \ldots) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2$$
$$\mathcal{Z}_{C_3}(z_1,\, z_2,\, \ldots) = \frac{1}{3}z_1^3 + \frac{2}{3}z_3$$
$$\mathcal{Z}_{E}(z_1,\, z_2,\, \ldots) = \exp\Big(z_1 + \frac{z_2}{2} + \frac{z_3}{3} + \cdots\Big)$$

For the species $S_2 := E(C_2)$, one gets using (23),

$$\begin{aligned}\mathcal{Z}_{S_2}(z_1,\, z_2,\, \ldots) &= \mathcal{Z}_E\Big(\frac{1}{2}(z_1^2 + z_2),\ \frac{1}{2}(z_2^2 + z_4),\ \frac{1}{2}(z_3^2 + z_6),\ \ldots\Big)\\ &= \exp\Big(\frac{1}{2}(z_1^2 + z_2) + \frac{1}{4}(z_2^2 + z_4) + \frac{1}{6}(z_3^2 + z_6) + \cdots\Big)\\ &= A_1(z_1)\ A_2(z_2)\ A_3(z_3)\ \cdots\end{aligned}$$

where for all integer $n \geq 1$,

$$A_n(z_n) = \begin{cases}\exp\Big(\dfrac{z_n^2}{2n} + \dfrac{z_n}{n}\Big) & n \equiv 0 \mod 2\\ \exp\Big(\dfrac{z_n^2}{2n}\Big) & n \equiv 1 \mod 2\end{cases}$$

A similar computation gives for the species $S_3 := E(C_3)$,

$$\mathcal{Z}_{S_3}(z_1,\, z_2,\, \ldots) = B_1(z_1)\ B_2(z_2)\ B_3(z_3)\ \cdots$$

where for all integer $n \geq 1$,

$$B_n(z_n) = \begin{cases}\exp\Big(\dfrac{z_n^3}{3n} + \dfrac{2z_n}{n}\Big) & n \equiv 0 \mod 3\\ \exp\Big(\dfrac{z_n^3}{3n}\Big) & n \not\equiv 0 \mod 3\end{cases}$$

The cycle index series $\mathcal{Z}_{S_2}(z_1,\, z_2,\, \ldots)$ and $\mathcal{Z}_{S_3}(z_1,\, z_2,\, \ldots)$ are separated (see definition 4). The cycle index series of the species $T^* := S_2 \odot S_3$ is computed using lemma 1,

$$\begin{aligned}\mathcal{Z}_{T^*}(z_1,\, z_2,\, \ldots) = (A_1 \odot B_1)(z_1)\ (A_2 \odot B_2)(z_2)\\ (A_3 \odot B_3)(z_3)\ \cdots\end{aligned}$$

The cycle index series of the connected species T is computed using formula (24),

$$\mathcal{Z}_T(z_1,\, z_2,\, \ldots) = \sum_{k\geq 1}\frac{\mu(k)}{k}\sum_{n\geq 1}\log(A_n \odot B_n)(z_{nk}). \tag{19}$$

The series $\widetilde{T}(z) := \mathcal{Z}_T(z, z^2, z^3, \ldots)$ is eventually computed using a computer algebra package. One gets,

$$\begin{aligned}\widetilde{T}(z) = 3\,z^6 &+ 11\,z^{12} + 81\,z^{18} + 1228\,z^{24}\\ &+ 28174\,z^{30} + 843186\,z^{36} + 30551755\,z^{42}\\ &+ 1291861997\,z^{48} + 62352938720\,z^{54}\\ &+ 3381736322813\,z^{60} + \ldots\end{aligned} \tag{20}$$

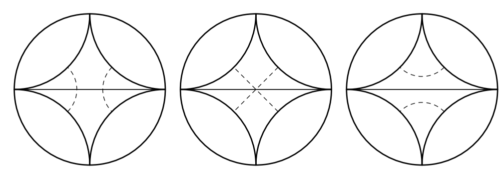

Fig. 4. The three triangular maps with two faces. Using the notion of Poincaré duality described in section I, the diagrams A_1 and A_2 of figure 3 corresponds to the leftmost map and to the middle map respectively while the three diagrams A_3, A_4 and A_5 being conjugated, altogether correspond to the same triangular map on the right.

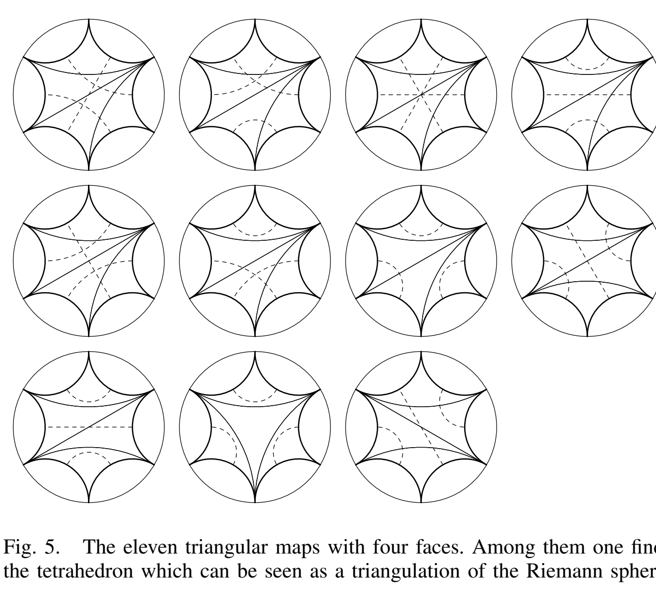

Fig. 5. The eleven triangular maps with four faces. Among them one finds the tetrahedron which can be seen as a triangulation of the Riemann sphere.

APPENDIX

A. Definitions

The goal of species theory is to provide a simple way to describe *labelled* structures that are met in computer science. That description style is broader than the pure syntactic style for it explicitly takes account of the relabelling process. It enables to count structures up to isomorphism, which means that two structures are counted as one if they are equal modulo a permutation of their labels.

Definition 2 (species): A species of structure [6], [1] is a functor F which produces

(i) for all finite label set U, a finite set of structures denoted by $F[U]$,
(ii) for all bijections $\sigma : U \to V$, a bijection

$$F[\sigma] : F[U] \longrightarrow F[V].$$

$F[U]$ denotes the set of F-structures labelled by U, σ is any permutation of the labels and $F[\sigma]$ is the related permutation of labelled structures. One assumes that F is a functor, which means that for all permutation $\sigma : U \to V$ and $\sigma' : V \to W$, one has $F[\sigma' \circ \sigma] = F[\sigma'] \circ F[\sigma]$. Moreover, for all set U, one has $F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$.

B. Generating series

Two F-structures $s_1 \in F[U_1]$ and $s_2 \in F[U_2]$ are called *isomorphic* if there exists a bijection $\sigma : U_1 \to U_2$ such that $s_2 = F[\sigma](s_1)$.
One usualy consider the following series:

1) the *exponential* generating series which counts the number $a_n(F)$ of *labelled* F-structures of size n:

$$F(z) = \sum_{n\geq 0} a_n(F)\, \frac{z^n}{n!}$$

2) the *ordinary* generating series which counts the number $\tilde{a}_n(F)$ of *unlabelled* F-structures of size n *i.e.* counted up to isomorphism:

$$\widetilde{F}(z) = \sum_{n\geq 0} \tilde{a}_n(F)\, z^n$$

3) the *cycle index* series which counts the number $f_{n_1,n_2,\ldots,n_k}(F)$ of F-structures left invariant under a permutation $F[\sigma]$ when the cycle type of the permutation σ is the partition $\lambda := (n_1, n_2, \ldots, n_k)$. Equivalently, σ decomposes in n_1 cycles of length 1, n_2 cycles of length 2 etc. One sets for $n = n_1 + 2n_2 + \cdots + kn_k$:

$$\begin{aligned}\mathcal{Z}_F(z_1,\, z_2,\, \ldots) &= \sum_{\sigma} f_{n_1,\ldots,n_k}(F) \frac{z_1^{n_1} \ldots z_k^{n_k}}{n!} \\ &= \sum_{\lambda} f_{n_1,\ldots,n_k}(F) \frac{z_1^{n_1} \ldots z_k^{n_k}}{1^{n_1} n_1! \ldots k^{n_k} n_k!}\end{aligned}$$

One gets the second formula from the first collecting in the sum all the permutations σ having the same cycle type λ.

One can recover the two other generating series from the cycle index series using,

$$\begin{aligned} F(z) &= \mathcal{Z}_F(z, 0, 0, \ldots) \\ \widetilde{F}(z) &= \mathcal{Z}_F(z, z^2, z^3, \ldots)\end{aligned}$$

1) Rigidity: The automorphism group (*i.e.* symetry group) of a F-structure $s \in F[U]$ is the set of permutation σ of U such that $F[\sigma](s) = s$.

Definition 3 (rigid species): A species F is called *rigid* whenever the group of automorphism of any F-structure is reduced to the identity transformation.
A F-structure is rigid if and only if $a_n(F) = n! \cdot \tilde{a}_n(F)$ for all integer n, which is equivalent to the equality of the two generating series $F(z)$ and $\widetilde{F}(z)$.

C. Two basic operations on structure species

1) Cartesian product (superposition): The cartesian product $F \odot G$ of two species F and G corresponds to a superposition of two structures labelled over a same alphabet. More precisely, to any set of labels U, one sets $(F \odot G)[U] := F[U] \times G[U]$ and $(F \odot G)[\sigma] := F[\sigma] \times G[\sigma]$ for all permutation $\sigma : U \to U$.

By definition of the superposition and following the notations above, for all natural number n one has $a_n(F \odot G) = a_n(F) \cdot a_n(G)$ and for all partition $\lambda := (n_1, n_2, \ldots, n_k)$ one has $f_\lambda(F \odot G) = f_\lambda(F) \cdot f_\lambda(G)$. At the opposite, for a non-rigid species $\tilde{a}_n(F \odot G) \neq \tilde{a}_n(F) \cdot \tilde{a}_n(G)$, which justifies the use of cycle index series. Translation in the language of generating series is straightforward ; this yields a variation of the Hadammard product,

$$(F \odot G)(z) = \sum_{n\geq 0} a_n(F)\, a_n(G)\, \frac{z^n}{n!} \tag{21}$$

$$\mathcal{Z}_{F\odot G}(z_1,\, z_2,\, \ldots) = \sum_{\lambda} f_\lambda(F) f_\lambda(G) \frac{z_1^{n_1} \ldots z_k^{n_k}}{1^{n_1} n_1! \ldots k^{n_k} n_k!} \tag{22}$$

2) Composition of two species: Compostion of two species F et G will be denoted by $F \circ G$ or else $F(G)$. A $(F \circ G)$-structure labelled by a set U is given by (see book [2, page 41])

(i) a partition $\pi = \{U_1, U_2, \ldots, U_k\}$ with $U = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$,
(ii) a F-structure labelled by the set $\{U_1, U_2, \ldots, U_k\}$,
(iii) a list $(s_1, s_2, \ldots, s_k)$ of G-structures labelled by $U_1, U_2, \ldots, U_k$ respectively,

which can be summed up by,

$$(F \circ G)[U] = \sum_{\pi \text{ partition of } U} F[\pi] \times \prod_{p\in\pi} G[p].$$

This construction translates to the following relations on generating series,

$$\left\{ \begin{aligned} (F \circ G)(z) &= F(G(z)), \\ (\widetilde{F \circ G})(z) &= \mathcal{Z}_F\big(\tilde{g}_1,\, \tilde{g}_2,\, \tilde{g}_3, \ldots\big), \\ \mathcal{Z}_{F\circ G}(z_1,\, z_2,\, \ldots) &= \mathcal{Z}_F(g_1,\, g_2,\, \ldots), \end{aligned}\right. \tag{23}$$

using the following conventions $\tilde{g}_k := \widetilde{G}(z^k)$ and $g_k := \mathcal{Z}_G(z_k, z_{2k}, z_{3k}, \ldots)$ for all integers $k \geq 1$.

The application we have in mind here is the decomposition of a structure in its connected components. One may write the following relation between species $F = E(F^c)$, this is a particular case of the composition of species : E is the species of sets F is a given species and F^c is the species of connected F-structures. In this particular situation, one may use the following formulae (see book [2, page 54]) where μ is the Mobius function,

$$\begin{cases} F^c(z) = \log F(z) \\ \widetilde{F^c}(z) = \sum_{k\geq 1} \frac{\mu(k)}{k} \cdot \log \widetilde{F}(z^k) \\ \mathcal{Z}_{F^c}(z_1, z_2, \ldots) = \sum_{k\geq 1} \frac{\mu(k)}{k} \cdot \log \mathcal{Z}_F(z_k, z_{2k}, \ldots) \end{cases} \tag{24}$$

3) Separated series:

Definition 4 (separated series): A cycle index series is said to be separable if

(i) it can be decomposed in a product of series each one on a single indeterminate, equivalently, if it is of the following form,

$$\mathcal{Z}_F(z_1, z_2, \ldots) = F_1(z_1) \cdot F_2(z_2) \cdot F_3(z_3) \ \cdots$$

(ii) for all partitions $\lambda := (n_1, n_2, \ldots, n_\ell)$, one has

$$f_\lambda(F) = f_{\lambda_1}(F) \cdot f_{\lambda_2}(F) \ \cdots \ f_{\lambda_\ell}(F),$$

using the following partitions $\lambda_k = (0, \ldots, 0, n_k, 0, \ldots, 0)$ for $1 \leq k \leq \ell$.

Lemma 1: Let

$$\mathcal{Z}_F(z_1, z_2, \ldots) = F_1(z_1) \cdot F_2(z_2) \cdot F_3(z_3) \ \cdots,$$
$$\mathcal{Z}_G(z_1, z_2, \ldots) = G_1(z_1) \cdot G_2(z_2) \cdot G_3(z_3) \ \cdots,$$

be two separated cycle index series. Then the series $\mathcal{Z}_{F\odot G}(z_1, z_2, \ldots)$ is also separated and we have,

$$\mathcal{Z}_{F\odot G}(z_1, z_2, \ldots) = (F_1 \odot G_1)(z_1) \cdot (F_2 \odot G_2)(z_2) \cdots \tag{25}$$

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