Synchronization of countable cellular systems, localization of quasi-periodic solutions of autonomous differential systems

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***Abstract*—We address the question of frequencies locking in coupled differential systems, related to the existence of quasi-periodic solutions of differential systems. Our tool is what we call "cellular systems", quite general as it deals with countable number of coupled systems in some general Banach spaces. Moreover, the inner dynamics of each subsystem does not have to be specified. We reach some general results about how the frequencies locking phenomenon is related to the structure of the coupling map, and therefore about the localization of quasi-periodic solutions of some differential systems that may be seen as cellular systems. This paper gives some explanations about how and why synchronized behaviors naturally occur it a wide variety of complex systems.**

***Index Terms*—coupled differential systems, synchronization, frequencies locking.**

I. INTRODUCTION

SYNCHRONIZATION is an extremely important and interesting emergent property of complex systems. The first example found in literature goes back to the 17th century with Christiaan Huygens' work [11], [2]. This kind of emergent behavior can be found in artificial systems as well as in natural ones and at many scales (from cell to whole ecological systems). Biology abounds with periodic and synchronized phenomena and the work of Ilya Prigogine shows that such behaviors arise within specific conditions: a dissipative structure generally associated to a nonlinear dynamics [20]. Biological systems are open, they evolve far from thermodynamic equilibrium and are subject to numerous regulating processes, leading to highly nonlinear dynamics. Therefore periodic behaviors appear (with or without synchronization) at any scale [21]. More generally, life itself is governed by circadian rhythms [9]. Those phenomena are as much attractive as they are often spectacular: from cicada populations that appear spontaneously every ten or thirteen years [10] or networks of heart cells that beat together [17] to huge swarms in which fireflies, gathered in a same tree, flash simultaneously [3]. This synchronization phenomenon occupies a privileged position among emergent collective phenomena because of its various applications in neuroscience, ecology, earth Science, for instance [27], [25], [16], as well as in the field of coupled dynamical systems, especially through the notion of chaotic systems' synchronization [18], [7] and the study of coupled-oscillators [13]. This wide source of examples leads the field of research to be highly interdisciplinary, from pure theory to concrete applications and experimentations.

The classical concept of synchronization is related to the locking of the basic frequencies and instantaneous phases of regular oscillations. One of the most successful attempts to explore this emergent property is due to Kuramoto [14], [15]. As in Kuramoto's work, those questions are usually addressed by studying specific kinds of coupled systems (see for instance [5], [22], [8]). Using all the classical methods available in the field of dynamical systems, researchers study specific trajectories of those systems in order to get information on possible attracting synchronized state [28], [13], [22], [19], [8], [12].

The starting point of this work was the following question : "Why synchronization is such a widely present phenomena ?" In order to give some mathematical answer to this question, the first step is to build a model of coupled systems that is biologically inspired. This is what is done in the second section, after having described some basic material, we define what we call cellular systems and cellular coupler. If one would summarize the specificities of cellular system, one could say that each cell (subsystem) of a cellular system receives information from the whole population (the coupled system) according to some constraints:

- a cell has access to linear transformations of all the others cell's states
- the way this information is gathered depends (not linearly) on the cell's state itself

In other words, a cell interprets its own environment via the states of the whole population and according to its own state.

It's a bit surprising that despite this model arising very naturally, it gives a good framework to address the main question. Indeed, in the third section we expose a localization result concerning periodic trajectories of cellular systems, according to some sub-periods dependencies. It exhibits some links between the coupler's properties and the structure of periodic trajectories.

The fourth section gives some example of general results that may be proved using the localization lemma. Moreover, it goes out of the scope of coupled systems as synchronization is strongly related to the more abstract field of dynamical systems. If one thinks about presence of regular attractors (in opposition with strange attractors) in a differential system, one may for example classify those as:

- point attractor

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- limit cycle
- limit torus

Those attractors can be related to coupled systems in an obvious way: roughly speaking, a point attractor may be seen as a solution of coupled systems for which each of the subsystems has a constant behavior. Similarly, a limit cycle may be thought as the situation where every subsystem oscillate, all frequencies among the whole system being locked. A limit torus is a similar situation which differs from the previous one by the fact that the frequencies are not locked (non commensurable periods of a quasi-periodic solution of the whole coupled system). Hence, the three previous cases may be translated into the coupled dynamical systems context:

- point attractor ↔ constant trajectories
- limit cycle ↔ periodic trajectories, locked frequencies
- limit torus ↔ periodic trajectories, unlocked frequencies

Therefore, we deduce some results about the localization of solutions of the third type, quasi-periodic solutions, using the point of view of coupled dynamical systems. The results of this third section may help to understand why the second case is the most observed in natural systems, which may be seen as coupled dynamical systems, at many levels. Indeed, the section ends with a sketch of how the cellular systems point of view may be applied to a wide class of differential systems in order to systematically address those questions with algebraic tools.

II. BASIC MATERIAL AND NOTATIONS

As our model is inspired by cellular tissues, some terms clearly come from the vocabulary used to describe those kinds of complex systems.

A. Model of population behavior

Here are the basic compounds and notations of our model:

A *population* $\mathcal{I}$ is a countable set, so we may consider it as a subset $\mathcal{I} \subset \mathbb{N}$. Moreover, as it's only the cardinality of $\mathcal{I}$ that's important, $\mathcal{I}$ may be chosen as an interval of integer. Elements of $\mathcal{I}$ are called *cells*.

We suppose that the systems we want to study are valued in some Banach spaces. Thus, for any $i \in \mathcal{I}$, $(E_i, \|.\|_i)$ is a Banach space, and the *state space* of $\mathcal{I}$ is the vector space $\mathcal{S} = \displaystyle\prod_{i \in I} E_i$.

We will sometimes identify E_i with

$$\prod_{j<i} \{0\} \times E_i \times \prod_{j>i} \{0\} \subset \mathcal{S}$$

and then consider it as a subspace of $\mathcal{S}$.

We denote $\mathcal{S}_b$ the space of uniformly bounded states:

$$\mathcal{S}_b = \left\{ x \in \mathcal{S},\ \sup_{i \in \mathcal{I}} \|x_i\|_i < \infty \right\}$$

This subspace will sometimes be useful as, embodied with the norm $\|x\|_\infty = \sup_{i \in \mathcal{I}} \|x_i\|_i$, it's a Banach space, allowing the classic PicardLindelöf theorem to be valid.

Given an interval $\Omega \subset \mathbb{R}$, a *trajectory* x of $\mathcal{I}$ is an element of $\mathcal{C}^\infty(\Omega, \mathcal{S})$. Such x is then described by a family of $\mathcal{C}^\infty$ applications $(x_i)_{i \in \mathcal{I}}$ such that $\forall i \in \mathcal{I}$:

$$\begin{array}{rccc} x_i : & \Omega & \longrightarrow & E_i \\ & t & \longmapsto & x_i(t) \end{array}$$

The space of trajectories on I is denoted $\mathcal{T}$.

A *period* on $\mathcal{I}$ is a map $\tau : \mathcal{I} \to \mathbb{R}_+^*$. A trajectory $x \in \mathcal{T}$ is said to be *τ-periodic* if for any $i \in \mathcal{I}$, x_i is $\tau(i)$-periodic and non constant. $\tau(i)$ is then said to be *a period of the cell* i. The space of such trajectories is written $\mathcal{T}_\tau$.

Each cell i is supposed to behave according to an autonomous differential system given by a vector field $F_i : E_i \to E_i$. Thus, given a family of functions $\{F_i\}_{i \in \mathcal{I}}$ we define the vector field $F_\mathcal{I}$ on $\mathcal{S}$:

$$\begin{array}{rccc} F_\mathcal{I} : & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & x & \longmapsto & F_\mathcal{I}(x) \end{array}$$

with, for any $i \in \mathcal{I}$:

$$[F_\mathcal{I}(x)]_i = F_i(x_i)$$

Remark. The definition of periodic trajectory handle both classical concepts of periodic and quasi-periodic solutions of a differential system. From the point of view of coupled systems, it describes the situation in which each subsystem of the whole system oscillates. We stress the point that a period of a periodic trajectory needs not to be a minimal period ($\tau(i)$ isn't necessarily a generator of the group of periods of x_i). Nevertheless, our definition of $\mathcal{T}_\tau$ avoid any trajectory which contains some constant component (none of the x_i can be a constant map) as they may be seen as degenerate (localized into an "hyperplane" of $\mathcal{S}$).

We recall that a (finite) subset $\{\tau_1, \ldots, \tau_k\}$ of $\mathbb{R}$ is said to be **rationally dependent** if there exists some integers $l_1, \ldots, l_k$ non all zero and such that:

$$l_1\tau_1 + \ldots + l_k\tau_k = 0$$

Then there exists a unique lowest common multiple (*lcm*) τ_0 for which there exits $n_1, \ldots, n_k$ such that:

$$n_1\tau_1 = \ldots = n_k\tau_k = \tau_0$$

An infinite set of real numbers is said to be rationally dependent if any finite subset is rationally dependent.

Now, any period τ on $\mathcal{I}$ (or, equivalently, any periodic trajectory) defines a equivalence relation on $\mathcal{I}$ as:

$$i \sim j \iff \{\tau(i), \tau(j)\} \text{ is a dependent set}$$

Hence we may consider the (countable) partition $\mathcal{I}(\tau)$ of $\mathcal{I}$ into equivalence classes (K countable):

$$\mathcal{I}(\tau) = \{\mathcal{I}_k\}_{k \in K}$$

B. Cellular coupler and cellular systems

In this section we build what we call *cellular systems* by way of *cellular coupler*. Most of the works always deal with a specific way of coupling dynamical systems: one adds a quantity (that models interactions between subsystems) to the derivative of the systems. This leads to equations with the following typical shape (here, there are only two coupled systems):

$$\begin{aligned} x_1'(t) &= F\big(x_1(t)\big) + G_1\big(x_1(t), x_2(t)\big) \\ x_2'(t) &= F\big(x_2(t)\big) + G_2\big(x_1(t), x_2(t)\big) \end{aligned}$$

The functions G_1 and G_2 are the coupling functions. The problem is then restated in terms of phase-shift variables and efforts are made to detect stable states and to prove their stability.

Our approach to coupling is different. We study exclusively a way of coupling where the exchanges are made on the current state of the system. This means that the coupling quantity applies inside the map F, which leads us to the following type of equation:

$$\begin{aligned} x_1'(t) &= F\big(x_1(t) + H_1(x_1(t), x_2(t))\big) \qquad (1) \\ x_2'(t) &= F\big(x_2(t) + H_2(x_1(t), x_2(t))\big) \end{aligned}$$

Remark. We stress the point that those two different ways of handling coupled systems are quite equivalent in most cases. Indeed, starting with the first two equations, as soon as G_1 and G_2 stay in the range of F (which is likely if the coupling functions are small), we can rewrite them in the second shape involving H_1 and H_2.

The last kind of coupled systems is sometimes studied (for instance in [12]) but never broadly (indeed, if one wants some quantitative results about convergence of trajectories, one must work with specific equations and dynamical systems). Even in a few papers that are quite general (as the very interesting [24]) some strong assumptions are made (in [24] authors deal with symmetric periodic solutions). The kind of coupled systems we handle are a generalization of the one describe in equation (1). Its general shape is:

$$x_i'(t) = F_i\left(\sum_{j\in\mathcal{I}} c_{ij}(x_i(t))x_j(t)\right)$$

Each cell $i \in \mathcal{I}$ holds it's own differential system represented by a map F_i (hence, all the dynamical systems are not forcibly identical nor have the same shape, nor that they are weakly coupled (as in the classical paper of Art Winfree [26]). A cell i "interprets" it's own environment by mean of the functions c_{ij}.

Before giving the exact definition of a cellular coupler, we recall that, as $\mathcal{S}$ may be seen as a module on the ring $\prod_{i\in\mathcal{I}} \mathcal{L}(E_i)$, $\mathcal{L}(\mathcal{S})$ has to be understood as the space of linear operators on $\mathcal{S}$ with coefficients in the $\mathcal{L}(E_i, E_j)$. Any $M \in \mathcal{L}(\mathcal{S})$ may then be written as an infinite (if $\mathcal{I}$ isn't finite) matrix:

$$M = [m_{ij}]_{(i,j)\in I^2}\ ,\ m_{ij} \in \mathcal{L}(E_j, E_i)$$

In this context, here is the definition of a cellular coupler on $\mathcal{I}$:

Definition 1. A *cellular coupling map* on $\mathcal{I}$ is a map c:

$$\begin{array}{rccc} c: & \mathcal{S} & \longrightarrow & \mathcal{L}(\mathcal{S}) \\ & x & \longmapsto & c(x) \end{array}$$

such that the matrix $[c_{ij}]_{(i,j)\in\mathcal{I}^2}$ satisfies:

1) $\forall (i,j) \in \mathcal{I}^2,\ \forall x \in \mathcal{S},\ c_{ij}(x)$ depends only on x_i (so that we may consider it as a map $c_{ij} : E_i \to \mathcal{L}(E_j, E_i)$);
2) $\forall i \in \mathcal{I},\ \forall x_i \in E_i,\ \sum_{j\in\mathcal{I}} \|c_{ij}(x_i)\|_i < +\infty$

Then, c defines a *cellular coupler* $\widetilde{c}$ on $\mathcal{I}$ in the following way:

$$\begin{array}{rccc} \widetilde{c}: & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & x & \longmapsto & c(x).x \end{array}$$

In other words (for the sake of simplicity, we only take examples with a finite population), for any $x \in \mathcal{S}$, the matrix $c(x)$ has the following shape:

$$c(x) = \begin{bmatrix} c_{11}(x_1) & \cdots & c_{1k}(x_1) \\ \vdots & \ddots & \vdots \\ c_{k1}(x_k) & \cdots & c_{kk}(x_k) \end{bmatrix} \in \mathcal{L}(\mathcal{S})$$

And then :

$$\widetilde{c}(x) = c(x).x = \begin{bmatrix} c_{11}(x_1).x_1 + \ldots + c_{1k}(x_1).x_k \\ \vdots \\ c_{k1}(x_k).x_1 + \ldots + c_{kk}(x_k).x_k \end{bmatrix} \in \mathcal{S}$$

Now we can define a cellular system:

Definition 2. Let $F_\mathcal{I}$ be a vector field on $\mathcal{S}$ given by a family $\{F_i\}_{i\in\mathcal{I}}$ of vector fields on the E_i. Let $\widetilde{c}$ a cellular coupler on $\mathcal{I}$. $(\mathcal{I}, F_\mathcal{I}, \widetilde{c})$ is called a *cellular system*. A trajectory of this system is a trajectory $x \in \mathcal{T}$ that satisfies:

$$x' = F_\mathcal{I} \circ \widetilde{c}(x) = F_\mathcal{I}\big(c(x).x\big)$$

in other words:

$$\forall i \in \mathcal{I},\ \forall t \in \Omega,\ x_i'(t) = F_i\left(\sum_{j\in\mathcal{I}} c_{ij}(x_i(t)).x_j(t)\right)$$

This equation may be naturally interpreted in biological terms: the cell i behaves according to a mean of the states of all other cells x_j, but only its state defines how this mean is computed (the cell interprets its own environment), and this link *state*↔ *interpreting function* has no reason to be linear in x_i.

Remark. In order to avoid any confusion, we stress the differences between trajectory and solution regarding periodic behaviors. In this paper, periodic trajectory has a specific meaning related to the cells. A periodic trajectory of a cellular system is a trajectory for which each cell has a periodic behavior. From the classic point of view of differential equations, periodic and quasi-periodic solutions of the cellular system are periodic trajectories. In the case of a periodic solution, $\tau(\mathcal{I})$ admit a *lcm*, which false in the case of a quasi-periodic solution.

In the next section we start of by exposing algebraic links between a cellular coupler and a periodic trajectory, and then we turn to our localization lemma.

III. LOCALISATION LEMMA

If M is a matrix indexed on $\mathcal{I}^2$, and if $J \subset \mathcal{I}$, we write $\overline{J} = \mathcal{I} - J$ and we define M^J as the matrix:

$$M = (m_{ij})_{(i,j)\in J\times\overline{J}}$$

For $x \in \mathcal{S}$ (resp. $x \in \mathcal{T}$) we denote x^J the vector (resp. the map) $[x_i]_{i\in J}$ (see figure 1).

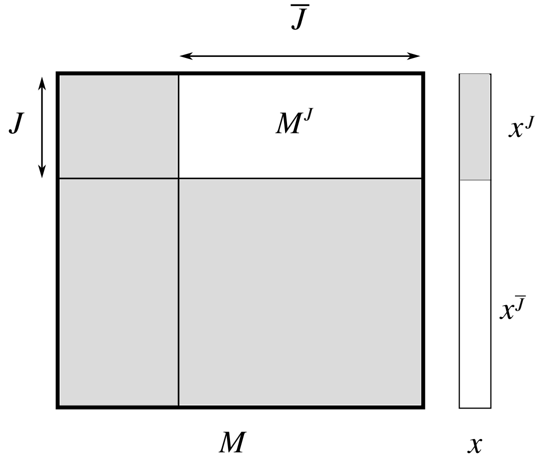

Fig. 1. Matrix and vectors associated to a subset J of $\mathcal{I}$.

If $\mathcal{I}(\tau) = \{I_1, ..., I_K\}$ is a partition of $\mathcal{I}$, we define the matrix $M^{\mathcal{I}(\tau)}$ as (see figure 2):

$$m_{ij}^{\mathcal{I}(\tau)} = \begin{cases} 0 & \text{if } (i,j) \in I_1 \times I_1 \cup \ldots \cup I_K \times I_K \\ m_{ij} & \text{if not} \end{cases}$$

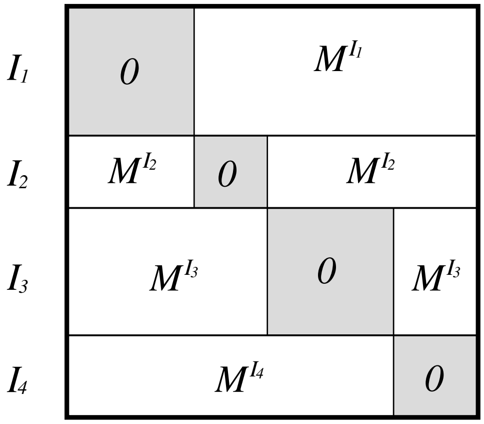

Fig. 2. Matrix associated to a partition of $\mathcal{I}$.

We can then go further and begin to work on the heart of our matter. The forthcoming result that can be used in many ways and generalized as, for the sake of simplicity, we did not use the weakest assumptions under which it holds (for example, the series convergence in the proof can be insured in many other contexts).

Lemma 1. *Let $(\mathcal{I}, F_\mathcal{I}, \widetilde{c})$ be a cellular system and τ a period on $\mathcal{I}$. Let $U \subset \mathcal{S}$ on which $F_\mathcal{I}$ is injective. If $x \in \mathcal{T}^\tau$ is a periodic trajectory of cellular system that satisfies:*

1) $x(\Omega) \subset \mathcal{S}_b$ *;*
2) $\widetilde{c}(x)(\Omega) \subset U$

then there exists $b \in \mathcal{S}_b$ such that for any $t \in \Omega$:

$$x\left(t\right) - b \in \ker\left(c(x(t))^{\mathcal{I}(\tau)}\right)$$

Remark. Note that the first condition on x is useless if $\mathcal{I}$ is finite.

The previous result is not very practical as it involves the trajectory x itself, which is unknown. As there is no ambiguity, we define the kernel of $c^{\mathcal{I}(\tau)}$ as:

$$\ker\left(c^{\mathcal{I}(\tau)}\right) = \bigcup_{x\in\mathcal{S}} \ker\left(c(x)^{\mathcal{I}(\tau)}\right)$$

Hence we may give a weaker version of the previous lemma

Corollary 1. *Under the conditions of lemma 1 there exists $b \in \mathcal{S}$ such that:*

$$x\left(\Omega\right) - b \in \ker\left(c^{\mathcal{I}(\tau)}\right)$$

Proof: (of lemma 1) First of all, let's check that $\widetilde{c}(x)$ is τ-periodic.
For any $i \in \mathcal{I}$, x_i' is $\tau(i)$-periodic and non constant for x_i is so. Let's $U_i = U \cap E_i$, F_i has to be injective on U_i. Hence, as x is a trajectory of the cellular system, $F_i\left(\widetilde{c}(x)_i\right)$ must be periodic and then $\widetilde{c}(x)_i$ is $\tau(i)$-periodic. Therefore, $c(x)$ is τ-periodic.

Now, according to the partition $\mathcal{I}(\tau) = \{\mathcal{I}_k\}_{k\in K}$ defined by τ (see section II-A), let $k \in K$ and $i \in \mathcal{I}_k$. For any $M \in \mathbb{N}$ we define the following set:

$$\mathcal{I}_k^M = \mathcal{I}_k \cap [\![0, M]\!]$$

The set $\tau\left(\mathcal{I}_k^M\right)$ is now a finite dependent set, so that we can consider its *lcm* τ_k^M. Now, for any $j \in \mathcal{I}_k^M$, x_j and $\widetilde{c}(x)_j$ are

τ_j^M-periodic, so that, for any integer N:

$$
\begin{aligned}
\widetilde{c}(x)_i(t) &= \frac{1}{N+1}\sum_{l=0}^{N}\widetilde{c}(x)_i\left(t+l\tau_k^M\right)\\
&= \frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\mathcal{I}} c_{ij}\big(x_i\left(t+l\tau_k^M\right)\big).x_j\left(t+l\tau_k^M\right)\\
&= \frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\mathcal{I}} c_{ij}\big(x_i(t)\big).x_j\left(t+l\tau_k^M\right)\\
&= \frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big).x_j(t)\\
&\quad+\frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\mathcal{I}_k-\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big).x_j\left(t+l\tau_k^M\right)\\
&\quad+\frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big).x_j\left(t+l\tau_k^M\right)\\
&= \sum_{j\in\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big).x_j(t)\\
&\quad+\frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\mathcal{I}_k-\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big).x_j\left(t+l\tau_k^M\right)\\
&\quad+\frac{1}{N+1}\sum_{l=0}^{N}\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big).x_j\left(t+l\tau_k^M\right)
\end{aligned}
$$

As x is uniformly bounded, using the second property of a coupling map (def. 1), we may invert the summation orders in the previous equation and compute the limits when $M\to+\infty$ and $N\to+\infty$ in any order. Thus we have:

$$
\begin{aligned}
\widetilde{c}(x)_i(t) = &\sum_{j\in\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big).x_j(t)\\
&+\sum_{j\in\mathcal{I}_k-\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big)\left[\frac{1}{N+1}\sum_{l=0}^{N}x_j\left(t+l\tau_k^M\right)\right]\\
&+\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big)\left[\frac{1}{N+1}\sum_{l=0}^{N}x_j\left(t+l\tau_k^M\right)\right]
\end{aligned}
$$

for the same reasons, it's easy to show that: and

$$
\lim_{M,N\to+\infty}\sum_{j\in\mathcal{I}_k-\mathcal{I}_k^M} c_{ij}\big(x_i(t)\big)\left[\frac{1}{N+1}\sum_{l=0}^{N}x_j\left(t+l\tau_k^M\right)\right]=0
$$

Now, as for all $j\in\overline{I_k}$, τ_k^M and $\tau(j)$ are non commensurable, if we denote τ_j' the generator of x_j group of period, as $\tau(j)=n_j\tau_j'$ for a certain integer n_j, τ_k^M and τ_j' as well are non commensurable. Therefore, the sequence $\left(\dfrac{t+l\tau_k^M}{\tau_j'}\right)_{l\in\mathbb{N}}$ is equidistributed mod 1, and we may apply some classic ergodic theorem (see for instance [23], [4]) and write:

$$
\begin{aligned}
\lim_{N\to+\infty}\frac{1}{N+1}\sum_{l=0}^{N}x_j\left(t+l\tau_k^M\right) &= \frac{1}{\tau_j'}\int_0^{\tau(j)}x_j(s)ds\\
&= \frac{n_j}{\tau(j)}\int_0^{\tau(j)}x_j(s)ds
\end{aligned}
$$

We can now define the state b as:

$$
b=[b_j]_{j\in\mathcal{I}},\ b_j=\frac{n_j}{\tau(j)}\int_0^{\tau(j)}x_j(s)ds
$$

so that:

$$
\begin{aligned}
\lim_{N\to+\infty}\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big)&\left[\frac{1}{N+1}\sum_{l=0}^{N}x_j\left(t+l\tau_k^M\right)\right]\\
&=\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big)b_j
\end{aligned}
$$

hence, we have shown that:

$$
\widetilde{c}(x)_i(t)=\sum_{j\in\mathcal{I}_k} c_{ij}\big(x_i(t)\big).x_j(t)+\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big).b_j
$$

But, obviously, from the beginning we had:

$$
\widetilde{c}(x)_i(t)=\sum_{j\in\mathcal{I}_k} c_{ij}\big(x_i(t)\big).x_j(t)+\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big).x_j(t)
$$

So that:

$$
\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big).x_j(t)=\sum_{j\in\overline{\mathcal{I}_k}} c_{ij}\big(x_i(t)\big).b_j
$$

The previous work can be done for any i which belongs to $\mathcal{I}_k$, thus we can summarize in the following way (see previously defined notations):

$$
c\big(x(t)\big)^{\mathcal{I}_k}.\left(x(t)-b\right)^{\overline{\mathcal{I}_k}}=0
$$

Again, the previous conclusion still holds for any $k\in K$, hence we may conclude using our notations:

$$
c\big(x(t)\big)^{\mathcal{I}(\tau)}.\left(x(t)-b\right)=0
$$

■

In the next section we give some examples of results based upon this lemma. We will mainly show how lemma 1 may be applied to turn synchronization issues (and existence of quasi-periodic solutions to a differential system) into algebraic problems. One of the main argument is that one wants to avoid periodic trajectories for which one cell is inert (a constant map), as it may be discarded from the population (in the case of an infinite population, this may lead to recurrence reasoning).

IV. APPLICATIONS

A. Weakly injective coupler

In this example we just write down an elementary property of $\widetilde{c}$ which ensures that a periodic trajectory must have an inert cell.

Definition 3. Let $\widetilde{c}$ be a cellular coupler on $\mathcal{I}$. $\widetilde{c}$ is said to be weakly injective if for any non trivial partition $\mathcal{I}(\tau)$ of $\mathcal{I}$ there exist $i \in \mathcal{I}$ such that:

$$\forall x \in \mathcal{S},\ \ker\left(c(x)^{\mathcal{I}(\tau)}\right) \cap E_i = \{0\}$$

Now we can state a simple result:

Proposition 1. *Under the conditions of lemma 1, if $\widetilde{c}$ is weakly injective and if x is a τ-periodic trajectory of the cellular system, then $\tau(\mathcal{I})$ is a dependent set.*

Proof: Let suppose that $\mathcal{I}(\tau)$ is not trivial, applying lemma 1 we know that:

$$c\big(x(t)\big)^{\mathcal{I}(\tau)}.\,(x(t) - b) = 0$$

As $\widetilde{c}$ is weakly injective, there exists $i \in \mathcal{I}$ such that:

$$\forall t \in \Omega,\ x(t)_i = b_i$$

which contradicts the definition of a periodic trajectory. ■

This result may be restated in terms of quasi-periodic solution of the cellular system:

Proposition 2. *Under the conditions of lemma 1, if $\widetilde{c}$ is weakly injective and if τ is bounded, the cellular system has no quasi-periodic solution.*

The next example deals with some topological properties of a coupler (how it connects cells together).

B. Chained cellular system

In this section, for the sake of simplicity, all the vector spaces E_i have finite dimension.

We first study the case of differential systems for which the spaces E_i have same dimension and are coupled with k-nearest neighbors (the finite dimension condition isn't necessary, but it makes the exposure simpler). This case is formally described by a cellular system $(\mathcal{I}, F_{\mathcal{I}}, \widetilde{c})$ where $\mathcal{I}$ is countable, all $\dim(E_i) = n$ and $\widetilde{c}$ satisfies:

$$\forall i, j \in \mathcal{I}, |j - i| > k \Rightarrow c_{ij} = 0$$

This is what we call a *chained cellular system*. Adding the following condition on the coupler, we may reach a general result:

Definition 4. A cellular coupler $\widetilde{c}$ is said to have *full rank* if for any $i, j \in \mathcal{I}$ and $x \in \mathcal{S}$ the map $c_{ij}(x)$ has full rank

Proposition 3. *Let $(\mathcal{I}, F_{\mathcal{I}}, \widetilde{c})$ be a chained cellular system coupled with k-nearest neighbors (all E_i having same finite dimension). Let $F_{\mathcal{I}}$ be injective on $U \subset \mathcal{S}$ and x a τ-periodic trajectory that stays in U. If $\widetilde{c}$ has maximal rank and if there exists $I \in \mathcal{I}(\tau)$ which contains $2k$ consecutive cells, i.e. there exists $i \in \mathcal{I}$ such that:*

$$[\![i, i + 2k - 1]\!] \subset I$$

Then $\mathcal{I}(\tau) = \{\mathcal{I}\}$ (equivalently, $\tau(\mathcal{I})$ is a dependent set).

Proof: Let suppose that $I \neq \mathcal{I}$. There must exist $[\![i, i + 2k]\!] \subset I$, such that $i - 1 \notin I$. Then, line $i + k - 1$ of the matrix $c(x(t))^{\mathcal{I}(\tau)}$ contains only one non zero element $c_{i+k-1,i-1}$. As this linear map is injective for any $t \in \Omega$, we know that:

$$\ker(c(x(t))^{\mathcal{I}(\tau)}) \bigcap E_{i-1} = \{0\}$$

Applying lemma 1 we know that there exists $b_{i-1} \in E_{i-1}$ such that for any $t \in \Omega$:

$$x_{i-1}(t) - b_{i-1} \in \ker\left(c(x(t))^{\mathcal{I}(\tau)}\right) \bigcap E_{i-1}$$

i.e. $x_{i-1}(t) = b_{i-1}$ is a constant map, which contradicts the definition of a periodic trajectory. So we can conclude that $I = \mathcal{I}$. ■

If we assume that τ is bounded, this result may be restated as: "as soon as k consecutive cells are synchronized (locked frequencies), then all the population is syncrohnized".

Moreover, we may drop some assumptions made on the indentical dimension of the E_i and reach an interesting connecting result concerning the case when $k = 1$.

Proposition 4. *Let $(\mathcal{I}, F_{\mathcal{I}}, \widetilde{c})$ be a chained cellular system coupled with 1-nearest neighbor. Let $F_{\mathcal{I}}$ be injective on $U \subset \mathcal{S}$ and x a τ-periodic trajectory that stays in U. If $\widetilde{c}$ has maximal rank and if there exists two sets I_1 and I_2 in $\mathcal{I}(\tau)$ such that:*

$$[\![i, i + 1]\!] \subset I_1 \quad [\![i + 2, i + 3]\!] \subset I_2$$

Then $I_1 = I_2$.

Proof: Let suppose that the cells $i + 1$ have non commensurable periods with those of the cells $i + 2$ (*i.e.* $I_1 \neq I_2$). Following the previous proof, we know that the lines $i + 1$ and $i + 2$ of the matrix $c(x(t))^{\mathcal{I}(\tau)}$ contains only one non zero element, respectively $c_{i+1,i+2}$ and $c_{i+2,i+1}$. But, we recall that for any $t \in \Omega$:

$$c_{i+1,i+2}(x_{i+1}(t)) : E_{i+2} \to E_{i+1}$$

and

$$c_{i+2,i+1}(x_{i+2}(t)) : E_{i+1} \to E_{i+2}$$

As the coupler has maximal rank, one of the previous map must be injective for all $t \in \Omega$. Using the same argument we may conclude that either x_{i+1} is a constant map, either it's x_{i+2}, leading to a contradiction. ■

Moreover, one could restate those results in terms of quasi periodic solutions of differential systems, but it may sound less intuitive. We will do it in the next sections.

For the next example, we add some regularity conditions on the cellular system which lead to a interesting description of $\mathcal{S}$.

C. Localization results with bounded states

As $(\mathcal{S}_b, \|.\|_\infty)$ is a Banach space, the classic PicardLindelöf theorem is valid and we can give a version adapted to cellular systems (we stress the point that in this section, any vector field $F_{\mathcal{I}}$ has to be a vector field on $\mathcal{S}_b$, as well for any cellular coupler $\widetilde{c}$, which brings some constraint on the families $(F_i)_{i\in\mathcal{I}}$ and $(c_{ij})_{(i,j)\in\mathcal{I}^2}$).

Proposition 5. *If $F_{\mathcal{I}} : \mathcal{S}_b \to \mathcal{S}_b$ and $\widetilde{c}$ are locally lipshcitz, which is the case if for any $x \in \mathcal{S}_b$ there exists a neighborhood $V = \prod_{i \in \mathcal{I}} V_i$, a positive number k and a sequence $(k_j)_{j \in \mathcal{I}}$ of positive numbers such that:*

1) $\forall y, z \in V, \forall i \in \mathcal{I},\ \|F_i(y_i) - F_i(z_i)\|_i \leq k\|y_i - z_i\|_i$
2) $\forall y, z \in V, \forall i \in \mathcal{I},\ \|c_{ij}(y_i) - c_{ij}(z_i)\|_{(E_j,E_i)} \leq k_j\|y_i - z_i\|_i$
3) $\sum_{j \in \mathcal{I}} k_j < +\infty$

then, given any initial condition (t^0, x^0) in $\mathbb{R} \times \mathcal{S}_b$, the cellular coupling admits a unique maximal solution x that satisfies $x(t^0) = x^0$.

Before stating our localization result, we need to define the sets that any non synchronized periodic trajectory of the cellular system must avoid (or, with the classical point of view, any quasi periodic solution).

Definition 5. Let $\widetilde{c}$ be a cellular coupler on $\mathcal{I}$. The set of regular points for $\widetilde{c}$ is defined as:

$$R(\widetilde{c}) = \left\{ x \in \mathcal{S},\ \forall \mathcal{I}(\tau) \text{ partition of } \mathcal{I},\ c(x)^{\mathcal{I}(\tau)} \text{ is injective} \right\}$$

We say that $\widetilde{c}$ is regular if $R(\widetilde{c}) = \mathcal{S}$.

Now we can state a localization result:

Proposition 6. *Under the conditions of lemma 1 and proposition 5, if there exists a infinite compact subset $V \subset \Omega$ such that:*

$$\forall t \in V,\ x(t) \in R(\widetilde{c})$$

then $\tau(\mathcal{I})$ is a dependent set.

One can rewrite this result in terms of differential systems:

Proposition 7. *Under the conditions of lemma 1 and proposition 5, and if τ is bounded, a quasi-periodic trajectory must "avoid" $R(\widetilde{c})$ (it can't cross this set on an infinite compact subset of Ω).*

Proof: (of proposition 6) Let suppose that $\mathcal{I}(\tau)$ is not trivial, applying lemma 1 we know that:

$$c\big(x(t)\big)^{\mathcal{I}(\tau)}.\,(x(t) - b) = 0$$

the assumptions made on $\widetilde{c}$ ensure that:

$$\forall t \in V,\ x(t) = b$$

As V has an accumulation point, we may conclude that there exists $t_0 \in V$ such that:

$$x'(t_0) = 0$$

Proposition 5 may be applied, hence we know that $t \mapsto x(t)$ is a constant map, which contradicts the definition of a periodic trajectory. ■

The next example gives a more precise result in the case where the maps c_{ij} don't depend on the state of the system (homogeneous coupler).

D. Exact frequencies locking with homogeneous cellular coupler

If $x \in \mathcal{T}_\tau$, for any $i \in \mathcal{I}$ the map x_i equals its Fourier's series. We write:

$$e^k_{\tau(i)}(t) = \exp\left(\frac{2i\pi kt}{\tau(i)}\right)$$

and we define :

$$\widehat{x}_i(k) = \frac{1}{\tau(p)} \int_0^{\tau(p)} x_i(t) e^k_{\tau(i)}(t)dt$$

so that we have :

$$x = \sum_{k \in \mathbb{Z}} \widehat{x}(k) e^k$$

i.e. $\forall i \in \mathcal{I}$:

$$x_i(t) = \sum_{k \in \mathbb{Z}} \widehat{x}_i(k) e^k_{\tau(i)}(t)$$

with normal convergence (note that $\widehat{x}_i(k)$ is E_i-valued).

Theorem 1. *Under the conditions of lemma 1, let $\widetilde{c}$ be homogeneous and regular. If τ is a bounded period on $\mathcal{I}$ and x a τ-periodic trajectory of the cellular system then τ is constant on $\mathcal{I}$.*

Remark. As this result is true as soon as τ is a period of x, it may be applied to the minimal periods of each x_i, then its conclusion is that all cells have exactly the same minimal period.

Proof: As $\widetilde{c}$ is homogeneous, we may identify it with c. Moreover, applying lemma 1 we know that $\tau(\mathcal{I})$ is a dependent set (unless at least one of the x_i would be a constant map). We now have to prove that τ is constant on $\mathcal{I}$.

Let's write a partition of $\mathcal{I}$ according to τ's values on $\mathcal{I}$ (we must recall that τ is supposed bounded):

$$\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_K\}$$

such that

$$\forall 1 \leq k \leq K,\ \tau(\mathcal{I}_k) = \tau_k$$

and $\tau_l \neq \tau_k$ if $l \neq k$.

We now suppose that $K > 1$.

As $\tau(\mathcal{I})$ is a finite dependent set, there exists $n_1, \ldots, n_K$ integers and τ_0 (the *lcm*) such that:

$$\tau_0 = n_1\tau_1 = n_2\tau_2 = \ldots = n_K\tau_K$$

The trajectory x is τ_0-periodic. We may therefore write its Fourier's series:

$$x(t) = \sum_{l \in \mathbb{Z}} \widehat{x}(l) e^l_{\tau_0}(t)$$

and as well for $c.x$:

$$(c.x)(t) = \sum_{l \in \mathbb{Z}} \widehat{c.x}(l) e^l_{\tau_0}(t)$$

uniqueness of Fourier coefficients forces them to satisfy:

$$\widehat{c.x}(l) = c\,\widehat{x}(l)$$

So that, for any $i \in \mathcal{I}$:

$$\widehat{c.x}_i(l) = \sum_{j=1}^{k} c_{ij}\widehat{x}_j(l)$$

Now, let $i \in \mathcal{I}_k$, the properties of Fourier decomposition ensure that $\widehat{x}_i(l)$ and $c.\widehat{x}_i(l)$ are zero as soon as n_k does not divide l (as $(c.x)_i$ and x_i are τ_k-periodic and $\tau_0 = n_k\tau_k$). So, if $l \in \mathbb{Z}$, let's define $I(l)$ as:

$$I(l) = \{k \in \{1, \dots, K\},\ n_k \not| \, l\}$$

For any integer l, if $k \in I(l)$ and $i \in \mathcal{I}_k$, then $\widehat{x}_i(l) = c.\widehat{x}_i(l) = 0$, so that (with similar convergence arguments that in the proof of lemma 1):

$$\begin{aligned}
c.\widehat{x}_i(l) &= \sum_{j=1}^{k} c_{ij}\widehat{x}_j(l) \\
0 &= \sum_{j\in I(l)}^{k} c_{ij}\widehat{x}_j(l) + \sum_{j\notin I(l)}^{k} c_{ij}\widehat{x}_j(l) \\
0 &= \sum_{j\notin I(l)}^{k} c_{ij}\widehat{x}_j(l)
\end{aligned}$$

This last property, (observable on figure 3), can be writen as:

$$\forall l \in \mathbb{Z} \quad c^{I(l)}\,\widehat{x}(l)^{\overline{I(l)}} = 0$$

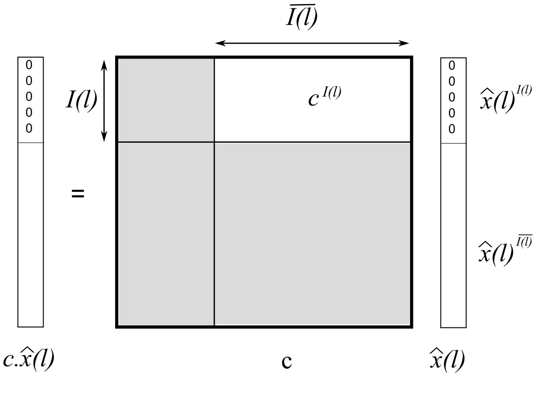

Fig. 3. Constraints on the Fourier's coefficients $\hat{x}(l)$.

This property holds for any integer l, and is empty when l is a multiple of all the n_i. So that, if $\mathcal{I}(l)$ is the partition of $\mathcal{I}$ defined as:

$$\mathcal{I}(l) = \left\{I(l), \overline{I(l)}\right\}$$

we can re-write it as:

$$\forall l \in \mathbb{Z} \quad c^{\mathcal{I}(l)}\,\widehat{x}(l) = 0$$

Let's now consider $\mathcal{I}_1 \neq \mathcal{I}_2$ (this is possible as $K > 1$). As those two classes are distinct, there exists l such that n_1 does not divide l and n_2 divides l. As c is regular, $c^{\mathcal{I}(l)}$ is thereby injective. We deduce that:

$$\widehat{x}(l)^{\overline{I(l)}} = 0$$

This proves that for any l divisible by n_2 and not by n_1, $\widehat{x}(l)^{\overline{I(l)}}$ is zero. Thus, for any coefficient of $\widehat{x}(l)^{\overline{I(l)}}$ to be non zero, n_1 must divide l, and consequently (as none of the x_i is a constant map) for all $i \in \mathcal{I}_2$, $x_i(t)$ is $n_1\tau_0$ periodic. This is incompatible with the partition of $\mathcal{I}$. Thus, $K = 1$ and thereby τ is a constant map (in other words, $\mathcal{I}$ is synchronized). ■

E. Perspectives of application to classical differential systems

In this last section, we show how the cellular systems point of view may be applied to classic differential systems and how dealing with different Banach spaces E_i may be useful. This discussion will be enlightened with a really simple example (finite population).

Let E be a Banach space and F a vector field on E. We want to see how this differential equation may be seen as a cellular system. For instance, one could consider a simple conservative system on $E = \mathbb{R}^4$ with an Hamilton's equation given by (see [1])

$$\begin{array}{rcl}
x_1' &=& y_1 \\
y_1' &=& \alpha x_1 - \beta x_1^3 + \varepsilon x_2 \\
x_2' &=& y_2 \\
y_2' &=& -\gamma x_2 + \varepsilon x_1
\end{array}$$

The first step is to identify the different cells of $\mathcal{I}$. The first step is to factorize each term in the equations according to the different variables. For example, the second equation may be seen as:

$$y_1' = (\alpha - \beta x_1^2)x_1 + \varepsilon x_2$$

So that the term $(\alpha - \beta x_1^2)$ has to be a part of the coupler we are building. Moreover, as it's the equation giving y_1', and as the way a cell computes how it interprets the population's state depends only on its own state, x_1 and y_1 have to belong to the same cell. In this simple example it's the only case where to variables have to be gathered in the same cell. In the end, this leads to the following structure of cellular system:

$$\mathcal{I} = \{1, 2, 3\}$$

with the Banach spaces:

$$E_1 = \mathbb{R}^2, E_2 = E_3 = \mathbb{R}$$

As it should often be the case, the associated vector fields are just identity maps on E_i, and the coupler is then:

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

with

$$\begin{array}{rccc}
c_{11}: & E_1 & \longrightarrow & \mathcal{L}(E_1) \\
& (x_1, y_1) & \longmapsto & \begin{bmatrix} 0 & 1 \\ \alpha - \beta x_1^2 & 0 \end{bmatrix}
\end{array}$$

$$\begin{array}{rccl} c_{12}: & E_2 & \longrightarrow & \mathcal{L}(E_2, E_1) \\ & x_2 & \longmapsto & \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{13}: & E_3 & \longrightarrow & \mathcal{L}(E_3, E_1) \\ & y_2 & \longmapsto & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{21}: & E_1 & \longrightarrow & \mathcal{L}(E_1, E_2) \\ & (x_1, y_1) & \longmapsto & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{22}: & E_2 & \longrightarrow & \mathcal{L}(E_2) \\ & x_2 & \longmapsto & \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{23}: & E_3 & \longrightarrow & \mathcal{L}(E_3, E_2) \\ & y_2 & \longmapsto & \begin{bmatrix} 1 \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{31}: & E_1 & \longrightarrow & \mathcal{L}(E_1, E_3) \\ & (x_1, y_1) & \longmapsto & \begin{bmatrix} \varepsilon & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{32}: & E_2 & \longrightarrow & \mathcal{L}(E_2, E_3) \\ & x_2 & \longmapsto & \begin{bmatrix} -\gamma \end{bmatrix} \end{array}$$

$$\begin{array}{rccl} c_{33}: & E_3 & \longrightarrow & \mathcal{L}(E_3) \\ & y_2 & \longmapsto & \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

Now, before applying some of the previous techniques, we may compute the different decomposition of c upon different non trivial partitions of $\mathcal{I}$. Those partitions are:

$$P_1 = \big\{\{1\}, \{2\}, \{3\}\big\},\ P_2 = \big\{\{1, 2\}, \{3\}\big\}$$

$$P_3 = \big\{\{1, 3\}, \{2\}\big\},\ P_4 = \big\{\{1\}, \{2, 3\}\big\}$$

which gives:

$$c^{P_1} = \begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix} \quad c^{P_2} = \begin{bmatrix} 0 & 0 & c_{13} \\ 0 & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

$$c^{P_3} = \begin{bmatrix} 0 & c_{12} & 0 \\ c_{21} & 0 & c_{23} \\ 0 & c_{32} & 0 \end{bmatrix} \quad c^{P_4} = \begin{bmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & 0 \\ c_{31} & 0 & 0 \end{bmatrix}$$

Now, in order to simplify, we replace the c_{ij} that are identically zero by 0, we obtain the following different matrices:

$$c^{P_1} = \begin{bmatrix} 0 & c_{12} & 0 \\ 0 & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix} \quad c^{P_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{bmatrix}$$

$$c^{P_3} = \begin{bmatrix} 0 & c_{12} & 0 \\ 0 & 0 & c_{23} \\ 0 & c_{32} & 0 \end{bmatrix} \quad c^{P_4} = \begin{bmatrix} 0 & c_{12} & 0 \\ 0 & 0 & 0 \\ c_{31} & 0 & 0 \end{bmatrix}$$

In the end, writing the coupler as an application from $\mathcal{S}$ to $\mathcal{L}(\mathcal{S})$, one finds those four matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & -\gamma & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & -\gamma & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\gamma & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix}$$

At this point, we just have to check that the coupler is weakly injective:

$$\ker\left(c^{P_1}\right) \cap E_2 = \ker\left(c^{P_4}\right) \cap E_2 = \{0\}$$

$$\ker\left(c^{P_2}\right) \cap E_3 = \ker\left(c^{P_3}\right) \cap E_3 = \{0\}$$

So, we may apply the proposition 1 and without any analytic calculus, state that this differential system may not admit any quasi-periodic solution. In other words, in case there exists periodic trajectories (which is well known to be true) they must be synchronized.

Moreover, those conclusions may hold in a more general case were the c_{ij} are less simple, and we can easily produce a result without any effort:

Proposition 8 (Generalized coupled pendulum)**.** *Let's consider a differential system which is driven by the following equations:*

$$\begin{array}{rcl} x_1' & = & a_1(x_1, y_1)x_1 + a_2(x_1, y_1)y_1 + a_3(x_1, y_1)x_2 \\ & & +a_4(x_1, y_1)y_2 \\ y_1' & = & a_5(x_1, y_1)x_1 + a_6(x_1, y_1)y_1 \\ x_2' & = & a_7(x_2)x_2 + u(x_2)y_2 \\ y_2' & = & \varepsilon(y_2)x_1 + a_8(y_2)y_1 - \gamma(y_2)x_2 + a_9(y_2)y_2 \end{array}$$

If the maps u and ε never vanish, then the systems has no quasi-periodic solution.

This result doesn't have to be deep in itself, neither has it to be the most general one we could have deduced from the previous discussion. It's just a sketch of how one can handle some structure properties of a differential system, applying 1, without going into deep and specific calculus.

V. CONCLUSION

In this work we have built a general framework of cellular systems in order to handle a wide variety of coupled systems, and therefore a wide class of complex systems. We focused on an emergent property of those dynamical systems: the frequencies locking phenomenon. Usually one observes solutions of particular coupled systems and shows that within suitable conditions synchronization must occur. Those results are qualitatively dependent on the systems of interest and do not stand in the general cases. We tried to change our point of view and to bring out completing results. As we choose not to address the problem of the convergence to a periodic solution, we don't prove that synchronization ultimately happens. Instead, we consider the problem at its end: if one supposes that some coupled systems "truly" oscillates, then they must be synchronized, regardless to the individual dynamical systems as soon as the maps which define each of them are injective nearby the trajectories. The loss in time evolution information is compensated by very general results, almost independent from the individual differential systems to be coupled. For example, we proved results concerning countable coupled systems, each of them needing not to be finite dimensional. In most papers (see for instance [13]) this population of coupled systems is implicitly defined and has only two cells (sometimes a finite number N, and more rarely an infinity). Moreover, on the contrary of what most studies about synchronization issues state, we do not assume anything concerning the cells dynamics (especially, we don't assume that they are oscillators). We only assume that they exhibit periodic behaviors under the coupling effects (the first assumption implies the second, but the opposite is clearly false).

We believe that this way of reaching general results about cellular systems gives some explanations about why the frequencies locking phenomenon emerges naturally in a large variety of coupled dynamical systems. Our results show that the following alternative is natural in many cases: either the whole population is synchronized, or its cells can't all have periodic behaviors.

Another interesting perspective is to apply this strategy to differential systems, as we outlined in the end of the fourth section. We think that it could prove useful to understand the stability or instability of limit torus when one adds some perturbations to a differential system. For example, if one already knows that there's no periodic solution, even with perturbations, and if on is able to prove, using our strategy, that a quasi-periodic solutions disappears under the effects of those perturbation, some conclusions about strange attractor emergence may be reachable.

Moreover, we have achieved some similar work on a natural generalization of this strategy to non countable population (in order to model natural systems, it's often necessary to handle continuous populations). We truly think that all those results are only a part of what can be done using cellular systems and that this work enlarges the possibilities of studying synchronization issues. But the scope of those kind of cellular systems may be beyond synchronization questions, as it's quite general and allows some theoretical studies. It could be a promising theoretical tool to model complex systems by the way of coupled differential systems.

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