Generating series : a combinatorial computation

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I. Introduction

The purpose of this paper is to apply combinatorial techniques for computing coefficients of rational formal series (G_k) in two noncommutative variables and their differences at order k and k-1. This in turn may help one to study validation of a family (B_k) of bilinear systems, described by the series (G_k) and global modeling of an unknown dynamical system (Σ).
The model validation is a central problem in system identification [2]. In almost cases, the model validation consists, in a test that falsifies or not falsifies the model, using a validation data set.
Computing and bounding these differences, we propose an estimation of the error due to approximations by (B_k). This error computation is a sum of differential monomials in the input functions and behavior system. We identify each differential monomial with its colored multiplicity and analyse our computation in the light of the free differential calculus.
We propose also a combinatorial interpretation of coefficients of (G_k), according to [12]. These coefficients are powers of an operator Θ which is in the monoid generated by two linear differential operators Δ and Γ.
The n-th power of Θ is equal to the sum of the labels of all forests of colored increasing trees.
This error computation allows one to better measure the impact of noisy inputs on the convergence of (B_k). Indeed, one can determine the contribution of the inputs and of the system in the error computation.

II. A local modeling of the unknown system

The problem consists in modeling an unknown dynamic system (Σ) for $t \in [0,T] = \bigcup_{i\in I}[t_i, t_i + d]$, when knowing some correlated sets of input/output.
We construct a behavioral model, based on the identification of its input/output functional (the generating series), in a neighborhood of every t_i, up to a given order k [1], [4]. At once a local modeling by a bilinear system $(B_i)_k$ around every t_i is provided. Then a family $((B_i)_{i\in I})_k$, global modeling of the unknown system is produced, such that the outputs of (Σ) and $((B_i)_{i\in I})_k$ coincide up to order k.

III. The bilinear system

We consider a certain class $(\mathcal{G}P)$ enclosing the electric equation

$$y^{(1)}(t) = f(y(t)) + u(t) \tag{1}$$

where $u(t)$ is the input function
Σ, the unknown system is an affine system.
In this case, equation (1) can be written

$$(\Sigma) \qquad \left\{ \begin{array}{rcl} \dot{x} &=& A_0(x) + A_1(x)u(t) \\ y(t) &=& x(t) \end{array} \right.$$

- u(t) is the real input
- x(t) is the current state
- $A_0 = a^{(0)}\frac{\partial}{\partial x}$ where $a^{(0)} = f(x)|_{x(0)}$
- $A_1 = \frac{\partial}{\partial x}$

The class $(\mathcal{G}P)$ encloses the nonlinear differential equation relating the current excitation i(t) and the voltage v(t) across a capacitor [9]

$$v^{(1)} + k_1 v + k_2 v^2 = i(t)$$

Let $a^{(i)} = f^{(i)}(x)|_{x(0)}$
We notice that the fundamental formula [9]provides the following bilinear system (B_k), approximating at order k :

$$\left\{ \begin{array}{rcl} \dot{x}_k(t) &=& (M_0 + M_1 u(t))x_k(t) \\ \overline{y}_k(t) &=& \lambda x_k(t) \end{array} \right.$$

where $\lambda = (x(0) \quad 1 \quad 0\cdots 0)$

$$x_k(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$M_0 = (C_{z_0 z_1^k})$ (resp $M_1 = (C_{z_1^{k+1}})$) expressed in basis $(C_{z_1^k})$.

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a^{(0)} & a^{(1)} & a^{(2)} & \cdots & a^{(k)} \\ 0 & a^{(0)} & 2a^{(1)} & \cdots & 0 \\ 0 & 0 & a^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

So, at order k, we obtain the ith derivative of the state vector x as a function of the previous ones. Our solution consists of two steps : to compute $x_{(k-n)k}^{(k-n+i)}(0)$ and to compute the difference of the ith derivative $x_{2k}^{(i)}(0) - x_{2(k-1)}^{(i)}(0)$.

IV. First step : Computation of $x_{(k-n)k}^{(k-n+i)}(0)$

By derivating and term's regrouping, we can show that :

$$x_{(k-n)k}^{(k-n+i)}(0)$$

$$=$$

$$\sum_{m=1}^{min(i+1,k-1)} a^{(m)} \sum_{l=1}^{k-n-1} \binom{k-n-l+m-1}{m}$$

$$(a^{(0)} + u^{(0)})^{l-1} x_{(k-n-l+m)k}^{(k-n-l+m+i-m)}(0)$$

$$+$$

$$\sum_{m=1}^{i+1} u^{(m)} \sum_{l=1}^{k-n-2} \binom{k-n-l+i}{m}$$

$$(a^{(0)} + u^{(0)})^{l-1} x_{(k-n-l)k}^{(k-n-l+i-m)}(0) + 1$$

$$(if \quad m = i+1)$$

We analyze now these equations in the light of the free differential calculus. Considering the derivative $a^{(i)}$ and $u^{(i)}$ specialized in time t=0 as differential letters, it is clear that our computation is a sum of differential monomials in a and u.

A. Colored partitions and multiplicities

A number partition or multiplicity is a sequence $\mu = (\mu_1, \mu_2, \mu_3, \cdots)$ (often written as $1^{\mu_1} 2^{\mu_2} 3^{\mu_3} \cdots$) of nonnegative integers. On a single letter a, the differential monomials become :

$$a^\mu = (a^{(i_1)})^{e_1} (a^{(i_2)})^{e_2} \cdots (a^{(i_q)})^{e_q},$$

$$1 \le i_1 < i_2 < \dots i_q$$

Such a monomial is indexed by the following partition [10] :

$$\mu = (i_1^{\mu_{i_1}} i_2^{\mu_{i_2}} \cdots i_q^{\mu_{i_q}})$$

Let $C = \{a, u\}$ be a set of two colors. We call colored partition on C an element of the free monoid generated by the cartesian product $N \times N$ i.e. any finite sequence of couples of nonnegative integers

$$\mu = ((\mu_1^a, \mu_1^u), (\mu_2^a, \mu_2^u), \cdots)$$

So, a colored partition μ will denote the differential monomial

$$a^\mu = (a^{(i_1)})^{e_1} \cdots (a^{(i_p)})^{e_p} (u^{(j_1)})^{f_1} \cdots (u^{(j_q)})^{f_q}$$

$$1 \le i_1 < i_2 < \dots i_p, \quad 1 \le j_1 < i_2 < \dots j_q$$

where e_l (resp f_l) = $\mu_{i_l}^a$ (resp $\mu_{j_l}^u$). The weight and the size of μ are defined as follows :

$$wgt(\mu) = \sum_c \sum_k k.\mu_k^c$$

$$size(\mu) = \sum_c \sum_k \mu_k^c$$

The empty partition is noted ϵ.
If L is the set of colored partitions, we define a partial order $\ll$ on L :

$$\nu = \{(\nu_i^a, \nu_i^u)\} \ll \mu = \{(\mu_i^a, \mu_i^u)\}$$

if

$$\nu_i^a \le \mu_i^a \quad and \quad \nu_i^u \le \mu_i^u \quad \forall i$$

L, with this partial ordering forms a Young lattice. [11]
We consider now B_i a subset of L defined by :

$$\{\mu / wgt(\mu) = i\}$$

and we note $I(\mu_{max})$ the order ideal generated by μ_{max}, if

$$\mu_{max} = max(\mu / \mu \in B_i)$$

B. Combinatorial analysis of our computation

Let us now interpret combinatorially our computation by identifying each differential monomial with its colored multiplicity. The recursive relation is captured by the operation :

$$\mu_{max} \odot c = \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu) = j \le i}} c^{(i-j+1)}.\nu$$

By factorizing according to the colored partitions, we get :

$$x_{(k-n)k}^{(k-n+i)} = \sum_c \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu) = j \le i}} c^{(i-j+1)}.\nu.g^1_{(c^{(i-j+1)}\nu)}$$

where :

$$g^l_{a^{(m)}\nu} = (a^{(0)} + u^{(0)})^{m+1} \sum_{p=m}^{n_l+m} \binom{l}{m} g_v^p$$

and

$$g^l_{u^{(m)}\nu} = (a^{(0)} + u^{(0)})^{m-1} \sum_{p=1}^{n_l} \binom{l+i+1}{m} g^p_v$$

with $n_1 = k-n-1, \quad n_l = l \quad \forall l > 1$

$$g_\epsilon = 1$$

C. Computation of $x^{(k-n+i)}_{(k-n)k}(0)$

We consider now permutations of a colored partition μ on an alphabet $X = \bigcup_{c\in C} X_c$. A permutation [11] of μ is a word in which each letter belongs to X and for each $x_i \in X$, the total number of appearances of x_i in the word is μ^c_i, for some $c \in C$
Let us note $\pi = \xi_1\xi_2\cdots\xi_{size(\mu)}$ a permutation of μ and σ_μ the set of permutations of μ.
Since, our alphabet

$$X_a = \{a^{(p)}|p = 1, min(k-1, i+1)\}$$

and

$$X_u = \{u^{(p)}|p = 1, i+1)\}$$

$$\xi_j = c^{(i_j)}$$

, for some (c, i_j).
$x^{(k-n+i)}_{(k-n)k}$ is a linear combination of monomial $y_1^{\lambda_1}\cdots y_n^{\lambda_n}(y_i \in X_a \bigcup X_u)$ and all distinct monomials obtained from it by a permutation of variables.
We get finally , if $s = (\sum_j j| \quad \mu^u_j \neq 0)$ and r =size(μ)

$$x^{(k-n+i)}_{(k-n)k} = \sum_{wgt(\mu)=i+1} \mu.(a^{(0)}+u^{(0)})^{k-n+i-r-s} g^n_\mu$$

$$g^n_\mu = \sum_{\pi\in\sigma_\mu} A_1 \prod_{j=2}^{r} A_j + b$$

where:

$$A_j = \begin{cases} \sum_{m_j=i_j}^{m_{j-1}+i_j} \binom{m_j}{i_j} & if \quad \xi_j = a^{(i_j)} \\ \sum_{m_j=1}^{m_{j-1}} \binom{m_j+i-j+2}{i_j} & if \quad \xi_j = u^{(i_j)} \end{cases}$$

$$A_1 = \begin{cases} \sum_{m_1=m}^{k-n-2+m} \binom{m_1}{i_1} & if \quad \xi_1 = a^{(i_1)} \\ \sum_{m_j=1}^{k-n-2} \binom{m_1+i+1}{i_1} & if \quad \xi_1 = u^{(i_1)} \end{cases}$$

and b = 1 if $\xi_1 = u^{(i+1)}$, 0 otherwise.

Remark : $x^{(k-n+i)}_{(k-n)k}$ is not a symmetric polynomial even if its structure is the same, because input and system contributions are different.

V. Second step: Computation of $x^{(k+i)}_{2k}(0) - x^{(k+i)}_{2(k-1)}(0)$

The first derivative coincide up to order k-2, but at order k-1, we have
$x^{(k-1)}_{2k} - x^{(k-1)}_{2(k-1)} = 0$ and $x^{(k-1)}_{jk} - x^{(k-1)}_{j(k-1)} \neq 0$.
Let M (resp P) the set of partitions on the single letter a (resp u)
W_i a subset of M defined by

$$\{\nu | 1 \leq size(\nu) \leq i+2\}$$

V_i a subset of P defined by

$$\{\lambda | size(\lambda) = \lfloor\frac{i}{2}\rfloor, wgt(\lambda) \leq i-2 \quad or \quad \lambda = u^{(i-2)} \quad or \quad \lambda = u^{(i-1)}\}$$

and S_l a subset of L defined by

$$\{\mu | wgt(\mu) = l\}$$

We define now an operation $\nabla : M \times P \times L \mapsto L$

$$\nabla(\nu, \lambda, \mu) = ((\nu_i + \mu^a_i, \lambda_i + \mu^c_i))_i$$

and a subset P_t of $L \quad \forall 0 \leq t \leq i$

$$P_t = \{\tau = \nabla(\nu, \lambda, \mu)| \quad \mu \in S_t, \lambda \in V_i, \nu \in W_i, wgt(\tau) = k+i-1\}$$

We obtain, by a straightforward computation :

$$x^{(k+i)}_{2k} - x^{(k+i)}_{2(k-1)} = \sum_{\substack{\nabla(\nu,\lambda,\mu)\in P_t \\ 0\leq t\leq i}} \nabla(\nu, \lambda, \mu) \quad h_\nu.f_\lambda.g^1_\mu.(a^{(0)}+u^{(0)})^{k+i-2-r}$$

where

$$f_\lambda = \sum_{\pi\in\sigma_\lambda} \prod_{l=1}^{size(\lambda)} \binom{k+i-2l}{k+i-2l-i_j}$$

$$h_\nu = \begin{cases} \sum_{\pi\in\sigma^1_\nu} \prod_{j=1}^{r-2} \binom{i_j + i_{j+1} - 1}{i_{j+1}} \binom{k-2}{i_r} & if \quad size(\nu) \neq 1 \\ 1 & if \quad size(\nu) = 1 \end{cases}$$

with

$$r = size(\nu$$

)

$$r_1 = r + size(\mu)$$

$$s = (\sum_j j | \mu^u_j \neq 0)$$

$$\pi = \xi_1\xi_2\cdots\xi_r$$

$$\xi_j = c^{(i_j)}$$

g^1_μ defined previously.

$$\sigma^1_\nu = \{\pi \in \sigma_\nu | \pi \neq \nu_1.\mu, size(\nu_1) < size(\nu)$$
$$and \quad \pi \neq (a^{(1)})^{r-1}.\xi_r\}$$

Taking into account that $\overline{y}^{(i)}_k(0) = x^{(i)}_{2k}(0)$, we obtain a right computation of the output's difference at order k and k-1. By majorization of these output's differences, and when k tends towards infinity, we get an overestimation of the error due to approximation by the (B_k)

VI. Coefficients of the generating series

We give, in this section, a combinatorial interpretation of coefficients of generating series. In [12], the author define increasing trees, model used to describe powers of a differential linear operator. We extend this concept to multiple operators by introducing colored increasing trees.

A. Forest of colored increasing trees

A forest of increasing trees on $\{1, \cdots, n\}$, according to [12], is a set of rooted increasing trees, the set of vertices of which is exactly [n] and such each vertex is smaller than all its successors. To take into account the multiplicity of operators,, we use a notion of a "colored partition" ([11]). For each vertex i, we color i any one of c_i colors. Let C the set of colors. We define colored increasing trees on cartesian product $\{1, \cdots, n\} \times C$.

B. Combinatorial interpretation

The author shows that the n-power of a linear differential operator is equal to the sum of the labels of all forests of increasing trees on $\{1, \cdots n\}$. So, in our case, the label of a forest on $\{1, \cdots, n\} \times C$ is a noncommutative monomial and is defined as :

$$\Pi_{(i,c) \in \{1, \cdots, n\} \times C} P^{(\alpha(i,c))} \frac{\partial}{\partial q}^{k}$$

where
$P(q) = 1$ or $P(q) = a^{(0)}$
$\alpha(i, c)$ is the number of sons of the node (i,c)
k is the number of trees of the forest.

C. Application

We consider the class $(\mathcal{G}P)$ given in the previous section. According to ([5]), the coefficients of the generating series are :

$$\langle G \mid z_{i_1} z_{i_2} \cdots z_{i_k} \rangle = [A_{i_1} \circ A_{i_2} \circ \cdots \circ A_{i_k} \circ h(q)]_0$$

where :

$$A_{i_j} = a^{(0)} \frac{\partial}{\partial q}$$

or

$$A_{i_j} = \frac{\partial}{\partial q}$$

Let us define two differential operators

$$\Delta = a^{(0)} \frac{\partial}{\partial q}$$

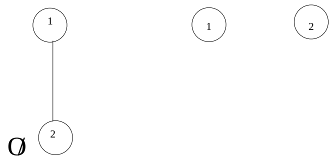
$$\Gamma = \frac{\partial}{\partial q}$$

These coefficients are powers of an operator Θ which is in the monoid generated by the two linear differential operators Δ and Γ . $C = \{c_1, c_2\}$

The 2-power of operator Θ is :

$$\Theta^2 = \langle G \mid z_1 z_0 \rangle + \langle G \mid z_0 z_1 \rangle + \langle G \mid z_0 z_0 \rangle + \langle G \mid z_1 z_1 \rangle$$

The colored increased trees are :



The labels of these trees are monomials $P^{(0)} P^{(1)} \frac{\partial}{\partial q}$, $P^{(0)2} \frac{\partial}{\partial q}^2$
Each colored vertex is associated to $P(q) = 1$ or $P(q) = a^{(0)}$
We note that, since the observation function h(q) is the identity function, all the powers of $\frac{\partial}{\partial q}^n$, $n \geq 2$ are zero.

VII. Conclusion

The validation which is presented in this paper is not statistical. It consists in valuing the convergence of a bilinear models family (B_k) on the unknown system (Σ) by an effective symbolic computation. It displays the respective contributions of the input and of the system itself.
More than a symbolic validation, these computing tools are parameterized by the input and the system's behavior. They can particularly provide a valuation process for rough and oscillating inputs as well as for smooth inputs.

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