On approximation of nonlinear generating series by rational series

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***Abstract*—In his article we propose an improvement of the method of identification and modeling of dynamical systems of Hespel–Jacob in the case of a single input. Our new method allows one to construct, whenever possible, the unique bilinear system of minimal rank satisfying all conditions obtained during the process of identification of the coefficients of the generating series of the dynamical system. Most importantly, we can unambiguously recognize a rational power series of rank r from the information obtained from the first $2r-1$ derivatives of the output of the dynamical system.**

***Index Terms*—Formal power series, Hankel matrices, dynamical systems, finite analysis of dynamical systems, generating series, identification of dynamical systems, modeling of dynamical systems.**

I. INTRODUCTION

The causal input/output functionals can be described by a certain noncommutative formal power series: the generating (or Fliess) series. The generating series is a canonical representation of the causal functional, in the sense that different functionals have different generating series. The functional corresponding to a generating series is obtained as a product with another noncommutative power series depending on the input: the Chen series.

If the system of equations defining a causal functional is not known, we may consider it as a black box [12], [14] and identify the coefficients of the generating series from the input/output behavior. It was shown by Hespel and Jacob that it is possible to identify the coefficients of the generating series G using a sufficient number of appropriate correlated input/output sets and their derivatives, up to an arbitrary order k [10], [11]. The proof is of a combinatorial nature, the coefficient of the generating series being binomial coefficients.

Once a generating series is identified up to order k, it is possible to construct a rational series of minimal rank that coincides with it up to order k [7], [8], [9]. A rational series corresponds to a bilinear dynamical system that can be constructed using the dependencies between the columns of its Hankel matrix. As a result, the method of Hespel–Jacob allows one to construct a bilinear system that approximates an unknown system with an error of $O(t^k)$.

Since the combinatorial explosion makes it difficult to identify the coefficients of high order, it would be quite interesting to better profit from all the information available to us and identify more generating series coefficients. Moreover, in general, there may be more than one rational series of minimal rank that coincides with a given nonlinear generating series up to order k.

In this article, we thus propose to reduce as much as possible the problem of choosing one of those rational series, for the case of systems with a single input. The main idea is to use the partial information about the coefficients of orders greater then k that was obtained during the identification. Indeed, during the modeling step, one uses only the values of the coefficients of orders up to k. However, some of linear combinations of the coefficients of higher order were also identified at the identification step. We propose thus an algorithm that uses this additional information in order to give the rational series that fits best to the known data. In the cases when the series is rational of rank r such that the output derivatives of orders up to $2r-1$ were used during the identification, we show that this rational series can be uniquely determined.

II. PRELIMINARIES

By a dynamical system we will mean an affine system of ordinary differential equations of the form

$$(\Sigma) \qquad \begin{cases} \dot{\mathbf{q}}(t) = \mathbf{v}_0(\mathbf{q}) + \displaystyle\sum_{j=1}^{m} \mathbf{v}_j(\mathbf{q}) u_j(t), \\ y(t) = h(\mathbf{q}(t)), \end{cases} \qquad \text{(II.1)}$$

where

1) $\mathbf{u}(t) = (u_1(t), \ldots, u_m(t))$ is the input vector,
2) $\mathbf{q}(t) \in \mathcal{M}$ is the current state, where $\mathcal{M}$ is a real differential manifold,
3) $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is a family of smooth vector fields on $\mathcal{M}$,
4) $h : \mathcal{M} \to \mathbb{R}$ is a smooth function called the observation map,
5) $y(t) \in \mathbb{R}$ is the output function.

We will be working with the *causal functional* that associates to the set of m input functions (commands) $\mathbf{u}(t)$ the corresponding output function $y(t)$. To the commands $u_1(t), u_2(t), \ldots, u_m(t)$ we associate an *alphabet* $\mathcal{Z} = \{z_0, z_1, \ldots, z_m\}$ of $(m+1)$ letters, z_0 being associated to the drift (which we will represent as an additional constant input function $u_0(t) \equiv 1$). To every multi-index $I = (i_1, i_2, \ldots, i_k)$ we associate a word $w = z_I = z_{i_1} z_{i_2} \cdots z_{i_k}$. These words form $\mathcal{Z}^*$, the free monoid over $\mathcal{Z}$. (The empty word is denoted by ε.)

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The behavior of causal functionals is uniquely described by two noncommutative power series: the generating series and the Chen series.

The *generating series* $G = \sum_{w\in\mathcal{Z}^*}\langle G|z_I\rangle z_I$ [4] is the geometric contribution and it is independent of the input. Its coefficients $\langle G|z_I\rangle$ are obtained by iteratively applying Lie derivatives corresponding to the vector fields to the observation map and evaluating the resulting expression at the initial state $\mathbf{q}_0$:

$$\langle G|z_I\rangle = \langle G|z_{i_1}z_{i_2}\cdots z_{i_k}\rangle = \mathbf{v}_{i_1}\circ\mathbf{v}_{i_2}\circ\cdots\circ\mathbf{v}_{i_k}\circ h\Big|_{\mathbf{q}_0}.$$

The generating series completely describes the causal functional. More precisely, two formal power series define the same functional if and only if they are equal [5], [16].

The *Chen series* $\mathcal{C}_u(t) = \sum_{w\in\mathcal{Z}^*}\langle\mathcal{C}_u(t)|z_I\rangle z_I$ measures the input contribution [1], [2], and is independent of the system. The coefficients of the Chen series are calculated recursively by integration using the following two relations:

- $\langle\mathcal{C}_u(t)|\varepsilon\rangle = 1$,
- $\langle\mathcal{C}_u(t)|w\rangle = \int_0^t \langle\mathcal{C}_u(\tau)|v\rangle u_j(\tau)d\tau$ for a word $w = vz_j$.

The causal functional $y(t)$ is then obtained locally as the product of the generating series and the Chen series [6]:

$$y(t) = \langle G||\mathcal{C}_u(t)\rangle = \sum_{w\in\mathcal{Z}^*}\langle G|w\rangle\langle\mathcal{C}_u(t)|w\rangle \qquad \text{(II.2)}$$

This formula is known as the *Peano–Baker formula*, as well as the *Fliess' fundamental formula*. Differentiating (II.2), we obtain

$$\frac{d^n y(t)}{dt^n} = \sum_{w\in\mathcal{Z}^*}\langle G|w\rangle\Big\langle\frac{d^n}{dt^n}\mathcal{C}_u(t)\Big|w\Big\rangle. \qquad \text{(II.3)}$$

Note that only the time derivatives of the Chen series appear in this expression. Their exact or at least recursive formula is needed. It can be shown in a straightforward way that [15]

$$\frac{d^n}{dt^n}\mathcal{C}_u(t) = \mathcal{C}_u(t)A_n(t), \qquad \text{(II.4)}$$

where the noncommutative polynomials $A_n(t)$ are recursively defined by the following relations

$$A_0(t) = 1, \qquad A_{n+1}(t) = \mathcal{L}_u(t)A_n(t) + \frac{d}{dt}A_n(t), \quad \text{(II.5)}$$

where $\mathcal{L}_u(t) = \sum_{z_i\in\mathcal{Z}} u_i(t)z_i$. Thus we finally obtain that

$$\frac{d^n}{dt^n}y(t)\Big|_{t=0} = \sum_{w\in\mathcal{Z}^*}\langle G|w\rangle\langle A_n(0)|w\rangle \qquad \text{(II.6)}$$

The *Hankel matrix* of a formal power series G is an infinite matrix with columns and rows indexed by the monomials from $\mathcal{Z}^*$ ordered lexicographically, such that the entry on the intersection of the row u and the column v is $\langle G|uv\rangle$.

Theorem II.1. *A (real–valued) formal power series is recognizable if and only if its Hankel matrix has finite rank [3].*

Theorem II.2. *A (real–valued) formal power series is rational if and only if it is recognizable [13].*

The following result due to Fliess is also important in this article.

Theorem II.3. *If all the rows (columns) corresponding to monomials of a certain fixed length are linear combinations of previous rows (columns), then all the following rows (columns) are also linear combinations thereof.*

Corollary II.4. *If a rational series in two variables is of rank n, then the upper left block of the Hankel matrix of size $(2^n-1)\times(2^n-1)$ is of rank n.*

Algorithm II.5. *The method of Hespel–Jacob consists in two steps: identification of the coefficients of the generating series and construction of a bilinear model. A short description follows. For a complete description, see [7], [8], [9], [10], [11].*

- **Identification.**
  The derivatives of the output are linear in the generating series coefficients and polynomial in the inputs u_j and their derivatives (cf. (II.6)). Choosing appropriate input/output sets, certain linear combinations of the generating series coefficients can be identified (those are exactly the coefficients of the monomials in the inputs and their derivatives).
  On the second stage of identification, the identified linear combinations are used to find the generating series coefficients themselves. Identification of coefficients of orders up to k can be done using the output derivatives of orders up to $m = k + \lfloor k/2\rfloor(k-\lfloor k/2\rfloor)$, in the case of a series in two letters. However, not all the linear combinations are used during this step, but only those involving the coefficients of orders $\leq k$. The remaining linear combinations give only partial information about the individual coefficients.
- **Modeling.**
  During the identification, the coefficients of the generating series were identified up to order k. These values are inserted into the Hankel matrix whose column basis is then calculated. The bilinear model is of the form
  $$\begin{cases}\dot{\mathbf{x}}(t) = (M_0 + \mathbf{u}(t)M_1)\mathbf{x}(t),\\ \mathbf{x}(0) = x_0,\end{cases}$$
  where M_0 and M_1 are matrices that are computed by expressing, in terms of the basis vectors, the left–multiplicative action of the letters of $\mathcal{Z}$ on the basis vectors. As the Hankel matrix is not completely determined, one obtains multi–parameter families of linear combinations of the basis vectors. The algorithm proposes to choose the linear combination that depends on the leftmost basis vectors.

We note that some partial information obtained during the identification is not used in the modeling. The main goal of this article is to fill this gap and to try to construct the unique model that fits best to all the available information. However, the unique model exists only if the generating series is rational of an appropriate rank. In the other cases, the modeling method we present here still has an advantage, as it allows one to construct a bilinear approximating system of rank r using the output derivatives of orders up to $2r-1 \leq k+\lfloor k/2\rfloor(k-\lfloor k/2\rfloor)$. Thus the identification can be done using fewer input/output sets, which is quite imporatant he since the necessary number of input/output sets grows exponentially with the increase of the order of differentiation of the output.

In order to simplify the explications, we will only deal with column operations. However, in practice, a more efficient strategy is to mix the column and row operations. In other words, our MAPLE package constructs both a row basis and a column basis of the Hankel matrix and uses both column and row dependencies during its "filling in". Even though it is possible to rely exclusively on column operations, the algorithm is simpler for a mixed strategy.

III. IDENTIFICATION OF FORMAL POWER SERIES OF RANK 2

Corollary III.1. *(of theorem (II.3)) If the Hankel matrix is of rank 2, then either* $\{C_\varepsilon, C_{z_0}\}$ *or* $\{C_\varepsilon, C_{z_1}\}$ *span the space of column vectors.*

Theorem III.2. *A rational generating series of rank 2 can be uniquely identified from the output derivative conditions (II.6) of orders up to 3.*

Proof:
Without loss of generality, we can assume that $\{C_\varepsilon, C_{z_0}\}$ span the column space. If not, we can interchange z_0 and z_1. Let $\lambda = (g_\varepsilon, g_0)$, $M_0 = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$ and $M_1 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$. Let us consider the following system of equations, equivalent to the conditions obtained from the output derivative conditions of orders up to 3:

$$\begin{cases} \lambda M(z)\gamma = g_i (i = \varepsilon, z_0, z_1, z_0^2, z_0z_1, z_1z_0, z_1^2, z_0^3, z_1^3), \\ \lambda(M(z_0^2z_1)+M(z_0z_1z_0)+M(z_1z_0^2))\gamma = g_{001}+g_{010}+g_{100}, \\ \lambda(M(z_0z_1^2)+M(z_1z_0z_1)+M(z_1^2z_0))\gamma = g_{011}+g_{101}+g_{110}, \end{cases} \tag{III.1}$$

Here $M(z_{i_0}\cdots z_{i_k}) = M_{i_0}\cdots M_{i_k}$, the coefficient of the rational series corresponding to a bilinear system of the type (II.7).

Appending to it the conditions of vanishing of all 3×3 minors of the Hankel matrix involving the coefficients of orders up to 3, we obtain a system in the unknowns $\{a, b, c, d, e, f\}$. Using Gröbner basis techniques, it is possible to show (it takes a significant amount of time) that it admits a solution for any value of the coefficients g_i on the right-hand side of the equations (of course satisfying the minor vanishing conditions). Moreover, the solution is unique under an additional assumption that the rank of the Hankel matrix is strictly greater than 1. ∎

Solving complicated systems of polynomial equations is a very useful tool from a theoretical point of view, but it becomes practically unfeasible as the number of equations and unknowns increases. However, we do not need to solve the most general system in every case. Firstly, one does not need to consider the most general case. Most of the generating series coefficients involved in the system are found during the identification. The system (III.1) can then be solved almost instantaneously. Secondly, the algorithm II.5 allows one to find the two–parameter family of rational series of rank 2 having the given coefficients of orders up to 2. Substituting it into the system (III.1) eliminates even more variables and any example of rank 2 can be easily solved this way. However, the systems one has to solve become more difficult for higher–rank cases. We would like thus to propose a different method that involves solving mostly linear equations.

Algorithm III.3. *The algorithm consists in a loop that includes the following three main steps.*

1) *Identify dependencies between the columns and use them to fill in a part of the matrix.*
2) *Solve the system of linear equations obtained during the identification, and substitute the solution into the Hankel matrix (thus some entries will be linear expressions of other entries).*
3) *(necessary only at ranks 4 and higher) Find a parameter (or a linear combination of parameters) in such a way that the rank of the Hankel matrix is "too high" for all but one value of this parameter (or linear combination of parameters). "Too high" means that is is greater then* $\lfloor (m+1)/2 \rfloor$ *when* m *is the maximal order of output differentiation used during identification.*

For generating series of rank 2, there are 3 different possible scenarios :

- $\{C_\varepsilon, C_{z_0}\}$ and $\{C_\varepsilon, C_{z_1}\}$ both form bases of the column space <u>and</u> $\{R_\varepsilon, R_{z_0}\}$ and $\{R_\varepsilon, R_{z_1}\}$ both form bases of the row space. In this case, filling in the matrix using the column dependencies allows one to find both unknown parameters of the two-parameter family.
- only one of $\{C_\varepsilon, C_{z_0}\}$ and $\{C_\varepsilon, C_{z_1}\}$ is a base of the column space <u>or</u> only one of $\{R_\varepsilon, R_{z_0}\}$ and $\{R_\varepsilon, R_{z_1}\}$ is a base of the row space. In this case, filling in the matrix allows one to find one of the unknown parameters. The other one is found from the equations (II.6).
- only one of $\{C_\varepsilon, C_{z_0}\}$ and $\{C_\varepsilon, C_{z_1}\}$ is a base of the column space <u>and</u> only one of $\{R_\varepsilon, R_{z_0}\}$ and $\{R_\varepsilon, R_{z_1}\}$ is a base of the row space. In this case, filling in the matrix is not sufficient to find any unknown parameters. But it allows one to diminish the number of unknown coefficients of the generating series and to find the unknown parameters one by one.

The algorithm is very easy in each case. But since complete explanations will not be feasible in higher–rank cases, we will illustrate in detail our techniques here, for cases 1 and 3.

Example III.4. *Let us first consider the following (bilinear) dynamical system which is an example of case 1.*

$$\begin{cases} \dot q_1 = -q_2, \\ \dot q_2 = q_1 + q_2 + (2q_1 + q_2)u, & q_1(0) = 1, \\ & q_2(0) = 0. \\ y(t) = q_1(t) + 2q_2(t). \end{cases} \tag{III.2}$$

Using the algorithm II.5, we obtain the following information at order 3 of differentiation:

$$\langle \mathsf{G} | \varepsilon \rangle = 1, \langle \mathsf{G} | z_0 \rangle = 2, \langle \mathsf{G} | z_1 \rangle = 4, \langle \mathsf{G} | z_0^2 \rangle = 1,$$
$$\langle \mathsf{G} | z_0z_1 \rangle = 2, \langle \mathsf{G} | z_1z_0 \rangle = 2, \langle \mathsf{G} | z_1^2 \rangle = 4, \langle \mathsf{G} | z_0^3 \rangle = -1,$$
$$\langle \mathsf{G} | z_0^2z_1 + z_0z_1z_0 + z_1z_0^2 \rangle = -3,$$
$$\langle \mathsf{G} | z_0z_1^2 + z_1z_0z_1 + z_1^2z_0 \rangle = 0, \langle \mathsf{G} | z_1^3 \rangle = 4.$$

Its Hankel matrix is thus as follows (with monomials ordered lexicographically) :

$$\begin{pmatrix} 1 & 2 & 4 & 1 & 2 & 2 & 4 & \cdots \\ 2 & 1 & 2 & -1 & -3-x_1-x_2 & x_1 & -y_1-y_2 & \\ 4 & 2 & 4 & x_2 & y_1 & y_2 & 4 & \\ 1 & -1 & -3-x_1-x_2 & & & & & \\ 2 & x_1 & -y_1-y_2 & & & & & \\ 2 & x_2 & y_1 & & & & & \\ 4 & y_2 & 4 & & & & & \\ \cdots & & & & & & & \end{pmatrix}$$

(where x_1, x_2, y_1, y_2 are yet unknown values). The rank of this matrix should be 2. Using the algorithm II.5, we obtain $\{C_\varepsilon, C_{z_0}\}$ as the basis of the column space, as well as

$$\lambda = (1 \quad 2), \mu(z_0) = \begin{pmatrix} 0 & 1-2a \\ 1 & a \end{pmatrix}, \mu(z_1) = \begin{pmatrix} 0 & 2-2b \\ 2 & b \end{pmatrix},$$

where a and b are unknown parameters (recall that the algorithm does not use any coefficients of order 3 since most of them were not identified yet). The available information on the third–order terms allows us to conclude immediately that we also have $C_{z_0^2} = C_{z_1} - C_\varepsilon$ and $C_{z_1^2} = 2C_\varepsilon$.

Using the known values of the Hankel matrix, these two relations between the columns together with $C_{z_1} = 2C_{z_0}$ and their consequences, we obtain additional relations $-3 - x_1 - x_2 = 4$, $-y_1 - y_2 = 2x_1$, $y_1 = 2x_2$, $x_2 = -2$ and $8 = 2y_2$. Solving these equations, we obtain all the coefficients of order 3. The Hankel matrix is now

$$\begin{pmatrix} 1 & 2 & 4 & 1 & 2 & 2 & 4 & -1 & -2 & 1 & 2 & -2 & -4 & 2 & 4 \\ 2 & 1 & 2 & -1 & -2 & 1 & 2 & & & & & & & & \\ 4 & 2 & 4 & -2 & -4 & 2 & 4 & & & & & & & & \\ 1 & -1 & -2 & & & & & & & & & & & & \\ 2 & 1 & 2 & & & & & & & & & & & & \\ 2 & -2 & -4 & & & & & & & & & & & & \\ 4 & 2 & 4 & & & & & & & & & & & & \\ \cdots & \cdots & \cdots & & & & & & & & & & & & \end{pmatrix}$$

giving us $a = 1$ and $b = 1$. The rank 2 rational series is thus completely determined. Constructing the bilinear system corresponding to it, we obtain the system (III.2). Let us remark that it was a different system that was found by the original method, using the same information. The rational generating series corresponding to (III.2) is

$$G = 1 + (2 - z_0)\Big(z_0 + z_1 - (z_0 + 2z_1)z_0\Big)^*(z_0 + z_1)$$

Remark. Of course, if the additional relations were contradictory, we would conclude that the rank of the series was greater than 2, and use only some additional conditions to find the values of a and b.

Example III.5. *Let us now consider an example of case 3:*

$$\left\{ \begin{array}{c} \dot{q}_1 = q_1 + q_2, \\ \dot{q}_2 = q_2 + (q_1 + 2q_2)u, \\ y(t) = q_1(t) + 2q_2(t). \end{array} \right. \quad \begin{array}{l} q_1(0) = 1, \\ q_2(0) = 0. \end{array} \qquad \text{(III.3)}$$

At order 3 of differentiation of the output, we have identified the following:

$$\langle \mathsf{G} | \varepsilon \rangle = 1, \langle \mathsf{G} | z_0 \rangle = 1, \langle \mathsf{G} | z_1 \rangle = 2, \langle \mathsf{G} | z_0^2 \rangle = 1,$$
$$\langle \mathsf{G} | z_0 z_1 \rangle = 3, \langle \mathsf{G} | z_1 z_0 \rangle = 2, \langle \mathsf{G} | z_1^2 \rangle = 4, \langle \mathsf{G} | z_0^3 \rangle = 1,$$
$$\langle \mathsf{G} | z_0^2 z_1 + z_0 z_1 z_0 + z_1 z_0^2 \rangle = 9,$$
$$\langle \mathsf{G} | z_0 z_1^2 + z_1 z_0 z_1 + z_1^2 z_0 \rangle = 16, \langle \mathsf{G} | z_1^3 \rangle = 8.$$

Its Hankel matrix is thus of the following form

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 3 & 2 & 4 & \cdots \\ 1 & 1 & 3 & 1 & 9-x_1-x_2 & x_1 & 16-y_1-y_2 & \\ 2 & 2 & 4 & x_2 & y_1 & y_2 & 8 & \\ 1 & 1 & 9-x_1-x_2 & & & & & \\ 3 & x_1 & 16-y_1-y_2 & & & & & \\ 2 & x_2 & y_1 & & & & & \\ 4 & y_2 & 8 & & & & & \\ \cdots & \cdots & \cdots & & & & & \end{pmatrix}$$

By the algorithm II.5 we obtain, taking the basis $\{C_\varepsilon, C_{z_1}\}$

$$\lambda = (1 \quad 2), \mu(z_0) = \begin{pmatrix} 1 & 3-2a \\ 0 & a \end{pmatrix}, \mu(z_1) = \begin{pmatrix} 0 & 4-2b \\ 1 & b \end{pmatrix},$$

where a and b are again some unknown parameters. Using the only known relation $C_{z_0} = C_\varepsilon$ and its consequences, we obtain additional equations $x_1 = 3$, $x_2 = 2$ and $y_2 = 4$. This allows us to conclude that that $C_{z_0 z_1} = C_\varepsilon + C_{z_1}$ and thus $a = 1$. This last relation between the columns implies in its turn that $y_1 = 6$ and thus $b = 2$. The rank 2 rational series is now completely determined:

$$G = \Big(1 + 2(z_0 + 2z_1)^* z_1\Big)\Big(z_0 + z_0(z_0 + 2z_1)^* z_1\Big)^*.$$

IV. Identification of formal power series of rank 3

The rank 3 case would be rather similar to the rank 2 case, except for the fact that we may not know the complete basis of the column space at the beginning. However, the following easy proposition guarantees that at least 2 basis vectors are known.

Proposition IV.1. *The part of the Hankel matrix of a rational series of rank 3, constructed using the coefficients obtained from derivatives of orders up to 5 of the output, cannot be of rank 1.*

Theorem IV.2. *All the coefficients of a rational power series of rank 3 can be uniquely determined from the information obtained from the output derivative conditions (II.6) of orders up to 5.*

Proof:

This theorem, which is a counterpart to (III.2), cannot be realistically proven by solving a system of polynomial equations, since the corresponding system is quite complicated in this case. Solving this polynomial system is still feasible for a given series if we use the information obtained from the algorithm II.5.

However, the general proof can rely on the new techniques described in this paper and it can be done by considering separately different cases arising during the identification. We give its outline here. Further details can be easily filled in.

At fifth-order differentiation of the output, the known part of the Hankel matrix is of the form

$$\begin{pmatrix}
y & y & y & y & y & y & y & y & y & y & y & y & y & y & y & \cdots \\
y & y & y & y & y & y & y & y & & & & & & & & \\
y & y & y & y & y & y & y & & & & & & & & y & \\
y & y & y & y & & & & y & & & & & & & & \\
y & y & y & & & & & & & & & & & & & \\
y & y & y & & & & & & & & & & & & & \\
y & y & y & & & & y & & & & & & & & y & \\
y & y & & y & & & & & & & & & & & & \\
y & & & & & & & & & & & & & & & \\
y & & & & & & & & & & & & & & & \\
y & & & & & & & & & & & & & & & \\
y & & & & & & & & & & & & & & & \\
y & & & & & & & & & & & & & & & \\
y & & & & & & & & & & & & & & & \\
y & & y & & & & y & & & & & & & & &
\end{pmatrix}$$

where y denotes (different) known values. There are 3 possibilities.

1) The first three columns and the first three rows are independent. The algorithm II.5 gives us all the coefficients of M_1 and M_2. The rest of the matrix can then be then unambiguously found.
2) The known part of the Hankel matrix is of rank 3, but the upper-left 3×3 determinant vanishes. Without loss of generality, we can assume that $C_\varepsilon, C_{z_0}, C_{z_1}$ span the column space. (If not, the same argument works by considering the rows instead of the columns.) Since there is a linear relationship between the first 3 rows, there is a linear relationship between the coefficients of orders 3 and 4, of the form $a\langle \mathsf{G} | z_0 z_I\rangle + b\langle \mathsf{G} | z_1 z_I\rangle + c\langle \mathsf{G} | z_I\rangle = 0$, where I is a multi-index of length 3. These equations are independent, which allows us to reduce the number of unknown coefficients of order 4 from 8 to 3.
   Now, among the rows R_4 through R_7, one is part of the basis, two are linear combinations of the other ones (consequence of linear dependence of the first 3 rows). The remaining one is also a linear combination of the basis rows. This relationship gives us another 4 relations between the coefficients of order 4, which allows us to identify them completely. The image of the basis columns under the left multiplication is now identified and the rational series uniquely determined.
3) The known part of the Hankel matrix has rank 2. That is, the third basis column vector has to be somewhere among columns C_4 through C_7 and third basis row vector somewhere among rows R_4 through R_7.
   As in the previous case, we obtain the relations of the form $a\langle \mathsf{G} | z_0 z_I\rangle + b\langle \mathsf{G} | z_1 z_I\rangle + c\langle \mathsf{G} | z_I\rangle = 0$ for the fourth-order terms. However, since we do not know the whole basis, we do not have an immediate extra relation between the rows. However, we can use the similar relationship among the first 3 columns to find more equations for the coefficients of order 4. This allows us to find the third row completing the row basis as well as the third column completing the column basis. Doing another round of equating the coefficients of the dependent columns and rows allows us to find all the coefficients of orders 4 and 5, thus determining the missing coefficients of the matrices M_1 and M_2.

$\blacksquare$

Example IV.3. *Examples of the three above-mentioned cases can be the series obtained using $\gamma = (1\ 0\ 0)^\perp$ and the following matrices :*

$$\lambda = (1\ 2\ 3),\quad M_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad M_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda = (1\ 1\ 1),\quad M_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\lambda = (1\ 1\ 2),\quad M_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\quad M_1 = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The corresponding generating series are:

$$G_1 = \Big(1+3z_1+2z_1^*z_0+3z_0z_1^*z_0\Big)\Big((z_0+2z_1)(z_1+z_0z_1^*z_0)\Big)^*$$

$$\begin{aligned} G_2 = \Big(&1 + (1+z_0+z_1)\big((1+z_0z_1^*)(z_0+z_1)\big)^* z_1 + \\ &+ (1+z_0)\big(z_1+(z_0+z_1)^+ z_0\big)^* z_0\Big) \times \\ &\times \Big(\big(z_0+z_1\big)\big(z_1+(z_0+z_1)^+ z_0\big)\big)^*\big((z_0+z_1)^+ z_1 + z_0\big)\Big)^* \end{aligned}$$

$$\begin{aligned} G_3 = \Big(&1 + (1+2z_0z_1^*)\big((z_0+z_1)z_1^*z_0\big)^* z_1\Big) \times \\ &\times \Big(z_0 + 2z_1\big((z_0+z_1)z_1^*z_0\big)^* z_1 + \\ &+ (z_0-z_1)\big(z_1+z_0(z_0+z_1)\big)^* z_0 z_1\Big)^* \end{aligned}$$

V. Identification of formal power series of rank 4

At rank 4, we meet additional difficulties. The main one is that we want to identify a rational series of rank 4 before the complete identification of coefficients of order 4. Thus, the known part of the Hankel matrix can be of rank 1. However, using the technique of minimization of the rank of the Hankel matrix, we can still find the unique rational series of rank 4 whenever the generating series of rank ≤ 4.

Theorem V.1. *A rational series of rank ≤ 4 can be uniquely identified from the information obtained from the output derivative conditions up to order 7.*

Proof:

The proof is done by considering every possible case, as for the Theorem IV.2. However, there are many more cases and it is impossible to present a complete proof here. We will only illustrate in detail the new technique of minimizing the rank. This technique is applied whenever the first two techniques are insufficient for "filling in" the whole matrix.

Its principle is based on the observation that the rank of a parametric matrix may vary is a function of its parameters. The goal is to find a square submatrix depending on at least one parameter p whose rank is greater than 4 unless the parameter p is equal to a certain value k. Since we want to find the rational series of minimal rank (i.e. less than or equal to 4), we can take $p = k$ and repeat the algorithm's loop from the beginning. Lemma V.2 guarantees that it is always possible to find such a value.

Now, considering separately all the different cases (for different possibilities of basis vectors and basis rows), we see that at most 1 minimization is necessary in order to complete the identification of the parameters. Indeed, the only coefficients of order 4 that were not identified at order 7 are those of the monomials involving 2 occurrences of z_0 and 2 of z_1. There are 6 such coefficients in all, but only 5 equations to find them. However, the minimization technique allows us to find one of these coefficients. The remaining ones are then immediately found from the 5 identified linear combinations. Once this step is finished, the remaining steps are quite similar to the rank 3 case and are executed without difficulties. ■

Lemma V.2. *Let H be the Hankel matrix corresponding to a rational series of rank 4, whose coefficients were identified from the output derivative condition of orders up to 7. Then there exists a yet unidentified coefficient g_i with the property that the rank of H is greater or equal to 5, unless g_i is equal to a certain value k.*

Proof:
The lemma is again proven on a case-by-case basis. ■

Remark. The disadvantage of the minimization part is its nonlinearity, but it can still be efficiently implemented on computer, since it involves only one symbolic parameter.

Example V.3. *Consider the rational generating series for the bilinear system with $\lambda = (1\ 1\ 1\ 1)$,*

$$\gamma = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, M_0 = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&2\end{pmatrix}, M_1 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

At the order 7 of differentiation of the output, all the known coefficients are equal to 1. The yet unknown conditions for the coefficients of the fourth–order terms give $g_{0011} = a$, $g_{0101} = 7 - 5a$, $g_{0110} = 4 - 3a$, $g_{1001} = 4 - 3a$, $g_{1010} = 6 - 5a$, $g_{1100} = a$, where a is an unknown parameter.

Since all the known coefficients are equal to 1, no dependency between the columns can be identified at this stage. As a consequence, we cannot obtain any other coefficient yet. Studying the rank of the Hankel matrix, we see that its rank is $r \geq 4$ for $a = 1$, $r \geq 8$ for $a \neq 1$ and $r \geq 9$ for $a = (9 \pm \sqrt{1105})/32$ (no matter what are the values of the coefficients of orders 5 and higher). Therefore, since we are trying to construct a rational series of rank ≤ 4, $a = 1$ is the only value that could eventually allow us to obtain a matrix of rank 4. Once we choose $a = 1$, four independant columns are immediately found and the rest of the matrix is easily filled in.

VI. Identification of series of rank greater than 4

The identification algorithm works essentially in the same way as in the rank 4 case, except that there are no direct counterparts of the lemma V.2. However, it can be replaced with a following strategy, applied as many times as all the other strategies fail. It is based on the following conjecture based on strong experimental evidence :

Conjecture VI.1. *Let H be the Hankel matrix corresponding to a rational series of rank $r \geq 5$, whose coefficients were identified from the output derivative conditions of orders up to $2r - 1$. Then there exists a combination of unidentified coefficients $s = \sum_j g_{I_j}$ with the property that the rank of H is greater than r, unless s is equal to a certain value k.*

The rank minimization algorithm is as follows :

1) Let n be the length of the shortest word in the generating series (the smallest according to the lexicographical ordering) whose coefficient is unknown.
2) Let $m = \sum_i a_i g_i$ be the sum of the coefficients of order n that are not identified yet and a_i unknown constants.
3) We initialize the stack with $(\{m = 0\}, C_\varepsilon)$, the basis with C_ε (since it can be always considered as a part of the basis) and the current column with $\tilde{C} = C_{z_0}$.
4) During each iteration of the loop, one considers the column $\tilde{C}$.
    - If the basis contains more than r columns, pop the stack and obtain the last–in element (s, C_K). Solve the system s, replace the obtained values in the Hankel matrix and exit the loop.
    - If $\tilde{C}$ is the column corresponding to a word of length $> r$, we encountered a linear combination that does not work. Let (s, C_K) be the element on top of the stack. We pop the stack and reset $\tilde{C}$ to the column that follows C_K.
    - We write the (truncated) column C_J as a linear combination (with arbitrary coefficients) of the basis columns and solve the corresponding system of equations augmented with the system s, where (s, C_K) is the element on the top of the stack.
        – If the system is incompatible, we add this column to the basis.
        – If there is one solution m' that involves only linear combinations of g_i and such that not all a_i vanish, we append m' to s and push (s, C_J) onto the stack.
        – In all the other cases, no action needs to be done on this stage.

    Finally, we reset $\tilde{C}$ to the column that follows C_J.

Conjecture VI.2. *A rational series of rank r (or less) can be uniquely identified from the information obtained from the output derivative conditions (II.6) up to order $2r - 1$.*

Proof:
Due to a large number of different cases, it is not possible to clearly identify all the different possible scenarios and give a complete proof similar to the rank 2, 3 and 4 cases. Extensive experimental evidence shows however clearly that the system of rank k can be identified in all the cases. ■

VII. Conclusion

In this article, we propose an improvement of the method of Hespel–Jacob for modeling nonlinear dynamical systems in the case of a single input. Among its main advantages are: the use of all the information obtained during the identification, which implies a better approximation and the possibility to precisely identify a rational series whenever there is sufficient data. Moreover, we have bounded by $2k-1$ the order of differentiation of the output derivatives which is necessary for identifying a rational series of order k. This algorithm was successfully programmed and tested in MAPLE.

An interesting direction to pursue is to generalize this algorithm to the case of several inputs.

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