DECOMPOSABLE FUNCTORS AND THE EXPONENTIAL PRINCIPLE, II

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Abstract. We develop a new setting for the exponential principle in the context of multisort species, where indecomposable objects are generated intrinsically instead of being given in advance. Our approach uses the language of functors and natural transformations (composition operators), and we show that, somewhat surprisingly, a single axiom for the composition already suffices to guarantee validity of the exponential formula. We provide various illustrations of our theory, among which are applications to the enumeration of (semi-)magic squares and (higher-dimensional) cubes.

1. Introduction

One of the corner stones of combinatorial enumeration is a theory which runs under several different names, for instance *theory of species* [7], *theory of exponential structures* [25, Ch. 5], *theory of exponential families* [26], *symbolic method* [12, Part A], *théorie du composé partitionnel* [13], *theory of prefabs* [5], all of which are more or less equivalent. It is probably fair to say that the most elaborate of these theories is the theory of species, as formulated by Joyal [16] (with the functorial concept of species of structures going back to Ehresmann [11]) and further developed by many other authors. It provides the most general framework for such a theory, at the expense of employing a rather abstract language, namely the language of category theory.

A fundamental theorem in each of these theories is the so-called *exponential formula*. Roughly speaking, given a family $\mathcal{G}$ of labelled combinatorial objects ("components"), one produces a larger family $\mathcal{F}$ ("composite objects") whose objects are obtained by

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putting together various elements of $\mathcal{G}$. The theorem then states that the (exponential) generating function for $\mathcal{F}$ equals the exponential of the (exponential) generating function for $\mathcal{G}$.

The aim of the present article is to develop a setting, where one starts with a family $\mathcal{F}$ of labelled combinatorial objects ("composite objects") and a composition of such objects, and then identifies, in an intrinsic way, a subfamily $\mathcal{G}$ of indecomposable objects ("components"), such that each element of $\mathcal{F}$ can be decomposed into objects from $\mathcal{G}$, and such that the exponential formula holds for $\mathcal{F}$ and $\mathcal{G}$. The main point here is that, in contrast to the usual set-up for the exponential formula, indecomposable objects are *not* given in advance, but are defined *inherently* via the composition operation. In particular, our theory leads to a uniform definition of the property to be *indecomposable* for arbitrary labelled combinatorial objects equipped with a composition operator. Interestingly, we show that a single axiom for the composition operator suffices to guarantee validity of the exponential formula. The natural language for formulating a corresponding theory is that of functors and natural transformations. Consequently, our presentation will be in the context of species theory.

For "ordinary" species (1-sort species), such a theory has been presented in [10] on the basis of two axioms for the composition operator. In the present article, we extend this approach to *weighted multisort species*. Moreover, we show that, actually, one of the axioms in [10] can be derived from the other, and that also in our multivariate setting a single axiom suffices. Strictly speaking, our presentation does not cover weighted species (in the sense of [16, Sec. 6], [7, p. 104]) in full generality; rather, we restrict ourselves to the case where the defining functor maps to a category of finite sets, thus avoiding unnecessary technicalities. However, extension to the general case of weighted multisort species is completely straightforward, and is left to the interested reader (see also Footnote 3).

In the next section, we develop the general set-up for our theory. It is formulated within the theory of multisort species, for which we define certain composition operators $\boldsymbol{\eta}$ that are subject to a single axiom, which, in order to be consistent with [10], we call (D1). Furthermore, in the same section, we present our main results. These are two exponential formulae, one being the refinement of the other, see Theorems 1 and 4. The finer result, Theorem 4, requires two general facts about our composition operators $\boldsymbol{\eta}$, which are presented in Propositions 2 and 3. (It is the latter, which, in the less general context of [10], had been assumed as a separate axiom, (D2). As our proof of Proposition 3 shows, this was actually not necessary since, within the general framework, (D2) follows from (D1).) The proofs of Theorems 1 and 4, and of Propositions 2 and 3, are given in Sections 4 and 6, respectively. They require a number of auxiliary results, which are established in Sections 3 and 5, respectively.

Sections 7–9 offer illustrations for the theory developed in Sections 2–6. Section 7 presents three simple examples highlighting different aspects of combinatorial situations covered by Theorems 1 and 4. In Section 8, we show how to apply our results to obtain generating function identities for (semi-)magic squares, thereby generalising previous results in the literature. Furthermore, in Section 9, we discuss an enumeration problem for (semi-)magic cubes of zeroes and ones.

The final section, Section 10, aims to throw light on the question how far a family $\mathcal{F}$ of labelled combinatorial objects, equipped with a composition operator in the sense indicated above, can be from the species $E(\mathcal{G})$ of sets of indecomposable objects in $\mathcal{F}$. We find that, roughly speaking, whenever the composition operator is pointwise associative and commutative, it can be re-constructed, in a sense made precise in Theorem 24, from the standard operation of forming the disjoint union in $E(\mathcal{G})$.

2. Set-up and main results

Denote by $\widehat{\mathbf{Set}}$ the category of finite sets and injective mappings, and by $\mathbf{Set}$ the subcategory consisting of finite sets and bijective maps. Moreover, for a positive integer r, let $\mathfrak{D}_r$ be the full subcategory of $\mathbf{Set}^r \times \mathbf{Set}^r$ whose objects are given by

$$\mathrm{Ob}(\mathfrak{D}_r) = \Big\{(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathbf{Set}^r \times \mathbf{Set}^r) : \ \mathbf{\Omega}_1 \cap \mathbf{\Omega}_2 = \boldsymbol{\emptyset}\Big\},$$

where $\boldsymbol{\emptyset} = (\emptyset, \dots, \emptyset)$ is the element of $\mathrm{Ob}(\mathbf{Set}^r)$ all of whose components are empty, and intersection is componentwise.

The ingredients needed for our theory are *r-sort species* and certain *composition operators* defined on them. Recall from [16] (or see [7, Def. 4 on p. 102] for a definition avoiding the language of category theory) that, for a positive integer r, an r-sort species is a covariant functor $F : \mathbf{Set}^r \to \mathbf{Set}$. Given r and an r-sort species F, the composition operators we have in mind are certain natural transformations $\boldsymbol{\eta}$ from the functor[1]

$$F \times F : \mathfrak{D}_r \xrightarrow{(F,F)} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set} \xrightarrow{\iota} \widehat{\mathbf{Set}}$$

to the functor

$$F \circ \amalg : \mathfrak{D}_r \xrightarrow{\amalg} \mathbf{Set}^r \xrightarrow{F} \mathbf{Set} \xrightarrow{\iota} \widehat{\mathbf{Set}};$$

that is, families $\boldsymbol{\eta} = (\eta_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)})_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)}$ of injective maps,

$$\eta_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)} : F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2] \hookrightarrow F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2],$$

such that, for every morphism $\boldsymbol{f} : (\mathbf{\Omega}_1, \mathbf{\Omega}_2) \longrightarrow (\widetilde{\mathbf{\Omega}}_1, \widetilde{\mathbf{\Omega}}_2)$ of $\mathfrak{D}_r$, the diagram

$$\begin{array}{ccc} F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2] & \xrightarrow{\eta_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)}} & F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2] \\ {\scriptstyle (F\times F)[\boldsymbol{f}]}\big\downarrow & & \big\downarrow{\scriptstyle (F\circ\amalg)[\boldsymbol{f}]} \\ F[\widetilde{\mathbf{\Omega}}_1] \times F[\widetilde{\mathbf{\Omega}}_2] & \xrightarrow[\eta_{(\widetilde{\mathbf{\Omega}}_1, \widetilde{\mathbf{\Omega}}_2)}]{} & F[\widetilde{\mathbf{\Omega}}_1 \amalg \widetilde{\mathbf{\Omega}}_2] \end{array} \tag{2.1}$$

commutes. Here, $\times$ is the natural product (Cartesian product) in the category of sets, $\amalg$ is the natural coproduct (componentwise disjoint union) in the category $\mathbf{Set}^r$ and in the category $\mathbf{Set}$ (relying on the context to clarify the intended meaning), and $\iota : \mathbf{Set} \to \widehat{\mathbf{Set}}$ is the inclusion functor. In what follows, the set-theoretic operations $\cap, \cup, -$ as well as the inclusion relation $\subseteq$ and $|$ (restriction of morphisms) in $\mathbf{Set}^r$ are all understood to be componentwise.[2] We shall most of the time drop the indices of $\boldsymbol{\eta}$-maps when they are clear from the context, thus writing $\eta(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2])$ instead of $\eta_{(\mathbf{\Omega}_1, \mathbf{\Omega}_2)}(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2])$, for example. We shall think of the elements of a set $\eta(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}_2])$ as *composite objects* within $F[\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2]$.

Given an r-sort species F and a composition operator $\boldsymbol{\eta}$ as above, the next step is to identify the subset $F_{\boldsymbol{\eta}}[\mathbf{\Omega}]$ of *"indecomposable" elements* of a set $F[\mathbf{\Omega}]$. It is most

[1]The introduction of the category $\mathfrak{D}_r$ corrects a slight imprecision in the set-up of [10].

[2]Throughout this paper, we use the symbol $-$ to denote the difference of sets.

natural to define $F_{\boldsymbol{\eta}} : \mathrm{Ob}(\mathbf{Set}^r) \to \mathrm{Ob}(\mathbf{Set})$ via

$$F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}] := \begin{cases} F[\boldsymbol{\Omega}] - \bigcup\limits_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r) \\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega} \\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), & \boldsymbol{\Omega} \neq \boldsymbol{\emptyset} \\ \emptyset, & \boldsymbol{\Omega} = \boldsymbol{\emptyset} \end{cases}, \qquad \boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r).$$

At this point, $F_{\boldsymbol{\eta}}$ is just defined as a *map* from $\mathrm{Ob}(\mathbf{Set}^r)$ to $\mathrm{Ob}(\mathbf{Set})$. In Lemma 14 in Section 3 we shall show that $F_{\boldsymbol{\eta}}$ is in fact a functor, that is, an r-sort species.

Not every natural transformation $\boldsymbol{\eta}$ is suited for giving rise to an exponential principle. We present the single axiom which is needed for this purpose next. Given F, we call a natural transformation $\boldsymbol{\eta} : F \times F \to F \circ \amalg$ a *composition operator* of F, if Axiom (D1) below holds.

(D1) *For each* $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$ *and any two partitions* $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2), (\widetilde{\boldsymbol{\Omega}}_1, \widetilde{\boldsymbol{\Omega}}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$ *of* $\boldsymbol{\Omega}$ *into disjoint parts,*

$$\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 = \boldsymbol{\Omega} = \widetilde{\boldsymbol{\Omega}}_1 \amalg \widetilde{\boldsymbol{\Omega}}_2,$$

we have that

$$\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \cap \eta\big(F[\widetilde{\boldsymbol{\Omega}}_1] \times F[\widetilde{\boldsymbol{\Omega}}_2]\big) = \eta\Big(\eta\big(F[\boldsymbol{\Omega}_{11}] \times F[\boldsymbol{\Omega}_{12}]\big) \times \eta\big(F[\boldsymbol{\Omega}_{21}] \times F[\boldsymbol{\Omega}_{22}]\big)\Big), \tag{2.2}$$

where $\boldsymbol{\Omega}_{ij} := \boldsymbol{\Omega}_i \cap \widetilde{\boldsymbol{\Omega}}_j$ *for* $i, j \in \{1, 2\}$.

An r-sort species F will be called *decomposable*, if $F \neq \emptyset$ (that is, $F[\boldsymbol{\Omega}] \neq \emptyset$ for some $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$), and if F admits some composition operator $\boldsymbol{\eta}$.

Next, we define *weights* on $(F, \boldsymbol{\eta})$. Fix a commutative ring Λ which contains the rational numbers. A family $\boldsymbol{w} = (w_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)}$ of maps $w_{\boldsymbol{\Omega}} : F[\boldsymbol{\Omega}] \to \Lambda$ is termed a Λ-*weight* on $(F, \boldsymbol{\eta})$, if the following three conditions hold:

(W0) *For all* $x \in F[\boldsymbol{\emptyset}]$, *we have* $w_{\boldsymbol{\emptyset}}(x) = 1$.

(W1) *For each morphism* $\boldsymbol{f} : \boldsymbol{\Omega}_1 \to \boldsymbol{\Omega}_2$ *of* $\mathbf{Set}^r$, *the diagram*

$$\begin{array}{ccc} F[\boldsymbol{\Omega}_1] & \xrightarrow{w_{\boldsymbol{\Omega}_1}} & \Lambda \\ {\scriptstyle F[\boldsymbol{f}]}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_\Lambda} \\ F[\boldsymbol{\Omega}_2] & \xrightarrow[w_{\boldsymbol{\Omega}_2}]{} & \Lambda \end{array}$$

commutes.

(W2) *For each pair* $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, *the diagram*

$$\begin{array}{ccc} F_\eta[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2] & \xrightarrow{\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}|_{F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]}} & F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \\ {\scriptstyle w_{\boldsymbol{\Omega}_1}|_{F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]} \times w_{\boldsymbol{\Omega}_2}}\big\downarrow & & \big\downarrow{\scriptstyle w_{\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2}} \\ \Lambda \times \Lambda & \xrightarrow[\text{multiplication in } \Lambda]{} & \Lambda \end{array}$$

commutes.

Here, (W0) and (W1) make F a *weighted* r-sort species (cf. [7, p. 104]), whereas (W2) demands (in a weak form) the Λ-weight $\boldsymbol{w}$ to be compatible with the composition operator $\boldsymbol{\eta}$.[3] In Section 10, we shall also need the concept of a *weak Λ-weight*, by which we mean a collection $\boldsymbol{w} = (w_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)}$ of mappings as above satisfying (W0) and (W1), but not necessarily (W2).

Given a Λ-weight $\boldsymbol{w} = (w_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)}$ on $(F, \boldsymbol{\eta})$, we define the corresponding *exponential generating functions* for F and $F_{\boldsymbol{\eta}}$, respectively, by[4]

$$\mathrm{GF}_F(z_1, \dots, z_r) := \sum_{n_1, \dots, n_r \geq 0} \sum_{x \in F[([n_1], \dots, [n_r])]} w_{([n_1], \dots, [n_r])}(x) \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!},$$

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1, \dots, z_r) := \sum_{n_1, \dots, n_r \geq 0} \sum_{x \in F_{\boldsymbol{\eta}}[([n_1], \dots, [n_r])]} w_{([n_1], \dots, [n_r])}(x) \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!},$$

where we suppress the dependence on $\boldsymbol{w}$ in the notation for better readability.

We are now ready to state our first main result, an *exponential principle*, which generalises Part (a) of the main result in [10].

Theorem 1. *Let r be a positive integer, $F : \mathbf{Set}^r \to \mathbf{Set}$ an r-sort species, and let $\boldsymbol{\eta} : F \times F \to F \circ \amalg$ be a natural transformation. If F is decomposable and $\boldsymbol{\eta}$ is a composition operator for F, then the generating functions GF_F and $\mathrm{GF}_{F_{\boldsymbol{\eta}}}$ are connected via the relation*

$$\mathrm{GF}_F(z_1, \dots, z_r) = \exp\big(\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1, \dots, z_r)\big). \tag{2.3}$$

The proof of Theorem 1 is given in Section 4. It requires several preparatory results, which are established in the next section.

In analogy to [10], there is a refinement of Theorem 1 in the spirit of [17, 19], which we explain next. Making use of the map $F_{\boldsymbol{\eta}}$ defined above, we define a sequence of mappings

$$F_{\boldsymbol{\eta}}^{(k)} : \mathrm{Ob}(\mathbf{Set}^r) \to \mathrm{Ob}(\mathbf{Set}), \quad k \geq 0,$$

with the property that $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] \subseteq F[\boldsymbol{\Omega}]$ by induction on k via

$$F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\Omega}] := \begin{cases} F[\boldsymbol{\emptyset}], & \boldsymbol{\Omega} = \boldsymbol{\emptyset} \\ \emptyset, & \boldsymbol{\Omega} \neq \boldsymbol{\emptyset} \end{cases}$$

and

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] := \bigcup_{\substack{\boldsymbol{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r) \\ \boldsymbol{\Omega}_1 \subseteq \boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times F_{\boldsymbol{\eta}}^{(k-1)}[\boldsymbol{\Omega} - \boldsymbol{\Omega}_1]\big), \quad k \geq 1. \tag{2.4}$$

As the definition suggests, one should think of $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ as the subset of objects in $F[\boldsymbol{\Omega}]$ consisting of exactly k "indecomposable" elements (components).

An immediate induction on

$$\|\boldsymbol{\Omega}\| = \|(\Omega^{(1)}, \Omega^{(2)}, \dots, \Omega^{(r)})\| := \sum_{j=1}^{r} |\Omega^{(j)}| \tag{2.5}$$

shows that

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] = \emptyset, \quad k > \|\boldsymbol{\Omega}\|. \tag{2.6}$$

[3] As already remarked in the introduction, it would be easy to generalise our set-up to cover weighted multisort species in full generality, by relaxing the condition that $F[\boldsymbol{\Omega}]$ needs to be finite, and requiring instead that each preimage $w_{\boldsymbol{\Omega}}^{-1}(\lambda)$ is finite and that the ring Λ is *multiplicatively finite*, in the sense that the number of different product representations of $\lambda \in \Lambda$ is always finite.

[4] For a non-negative integer n, we write $[n]$ for the standard set $\{1, 2, \dots, n\}$ of cardinality n.

Again, by definition, $F_{\boldsymbol{\eta}}^{(k)}$ is just a *map* from $\mathrm{Ob}(\mathbf{Set}^r)$ to $\mathrm{Ob}(\mathbf{Set})$. In Lemma 16 in Section 5 we shall show that $F_{\boldsymbol{\eta}}^{(k)}$ is in fact a functor, that is, an r-sort species.

It is not difficult to see that, for any $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, the sets $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ with $k = 0, 1, 2, \dots$ cover all of $F[\boldsymbol{\Omega}]$.

Proposition 2. *For every* $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, *we have*

$$F[\boldsymbol{\Omega}] = \bigcup_{k \geq 0} F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]. \tag{2.7}$$

The proof of Proposition 2 can be found in Section 6.

In [10], a second axiom, (D2), was imposed on the composition operators $\boldsymbol{\eta}$ for obtaining a refined exponential principle that takes into account the filtration given by the sets $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ appearing on the right-hand side of (2.7) (for the case of 1-sort species): it required pairwise disjointness of the sets $F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]$ for $k = 0, 1, 2, \dots$. The proposition below says that, actually, this assertion is a consequence of Axiom (D1) (within the general set-up), and this is even true for multisort species.

Proposition 3. *If* $F : \mathbf{Set}^r \to \mathbf{Set}$ *is an* r*-sort species and* $\boldsymbol{\eta}$ *is a composition operator for* F, *then we have, for*[5] $k, \ell \in \mathbb{N}_0$ *and* $k \neq \ell$,

$$F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}] \cap F_{\boldsymbol{\eta}}^{(\ell)}[\boldsymbol{\Omega}] = \emptyset, \quad \boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r). \tag{2.8}$$

Proposition 3 is also proved in Section 6. Its proof depends crucially on the fact that "$\boldsymbol{\eta}$-bracketings" of F-sets and $F_{\boldsymbol{\eta}}$-sets do not depend on the order of the terms $F[\boldsymbol{\Omega}]$ respectively $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]$ involved, nor on the type of bracketing used; see Lemmas 11 and 15 in Sections 3 and 5, respectively. From a technical point of view, this is the decisive improvement over the results in [10], and it is the reason that the dependence of Axiom (D2) from Axiom (D1) was not observed there.

Given a Λ-weight $\boldsymbol{w}$ on $(F, \boldsymbol{\eta})$, the above propositions allow us to refine the weighting to

$$\widetilde{w}_{\boldsymbol{\Omega}}(x) := y^k w_{\boldsymbol{\Omega}}(x), \qquad x \in F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}].$$

We then can define the refined generating function

$$\begin{aligned}\widetilde{\mathrm{GF}}_F(z_1, \dots, z_r, y) &:= \sum_{n_1, \dots, n_r \geq 0} \sum_{x \in F[([n_1], \dots, [n_r])]} \widetilde{w}_{([n_1], \dots, [n_r])}(x) \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!} \\ &= \sum_{n_1, \dots, n_r \geq 0} \sum_{k \geq 0} \sum_{x \in F_{\boldsymbol{\eta}}^{(k)}[([n_1], \dots, [n_r])]} y^k \, w_{([n_1], \dots, [n_r])}(x) \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1! \cdots n_r!}.\end{aligned}$$

With the above notation, we have the following refinement of Theorem 1, which is our second main result.

Theorem 4. *Under the hypotheses of Theorem* 1, *we have*

$$\widetilde{\mathrm{GF}}_F(z_1, \dots, z_r, y) = \exp\big(y \, \mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1, \dots, z_r)\big), \tag{2.9}$$

as well as

$$\widetilde{\mathrm{GF}}_F(z_1, \dots, z_r, y) = \big(\, \mathrm{GF}_F(z_1, \dots, z_r)\big)^y. \tag{2.10}$$

The proof of Theorem 4 is given in Section 6, as a simple consequence of (the proof of) Proposition 3.

[5]We denote by $\mathbb{N}_0$ the set of non-negative integers.

3. Auxiliary results, I

The purpose of this section is to establish several lemmas, which will be needed in the next section in the proof of Theorem 1. At the same time, they also form the basis for the proofs of the auxiliary results in Section 5, which eventually will lead to proofs of Propositions 2 and 3, and of Theorem 4, in Section 6. In all of this section, we assume that F is a decomposable r-sort species with composition operator $\boldsymbol{\eta}$.

Lemma 5. *We have* $|F[\boldsymbol{\emptyset}]| = 1$.

Proof. By the injectivity of $\eta_{(\boldsymbol{\emptyset},\boldsymbol{\emptyset})} : F[\boldsymbol{\emptyset}] \times F[\boldsymbol{\emptyset}] \to F[\boldsymbol{\emptyset}]$, the set $F[\boldsymbol{\emptyset}]$ is either empty or a 1-set. Suppose that $F[\boldsymbol{\emptyset}] = \emptyset$. Choose $\boldsymbol{\Omega}_1 = (\Omega_1^{(1)}, \dots, \Omega_1^{(r)}) \in \mathrm{Ob}(\mathbf{Set}^r)$ with $F[\boldsymbol{\Omega}_1] \neq \emptyset$, and $\boldsymbol{\Omega}_2 = (\Omega_2^{(1)}, \dots, \Omega_2^{(r)}) \in \mathrm{Ob}(\mathbf{Set}^r)$ such that $\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2 = \boldsymbol{\emptyset}$ and $|\Omega_1^{(i)}| = |\Omega_2^{(i)}|$ for $1 \le i \le r$. Now consider (D1) for the partition $\boldsymbol{\Omega} := \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2$, $\widetilde{\boldsymbol{\Omega}}_i := \boldsymbol{\Omega}_i$ for $i = 1, 2$. By the functoriality of F, we also have $F[\boldsymbol{\Omega}_2] \neq \emptyset$ and, consequently, the left-hand side of (2.2) is non-empty, whereas the right-hand side of (2.2) would be empty in case $F[\boldsymbol{\emptyset}] = \emptyset$, a contradiction. □

Lemma 6 (Commutativity for $(F, \boldsymbol{\eta})$)**.** *For every pair* $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$*, we have*

$$\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) = \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big). \tag{3.1}$$

Proof. Applying Axiom (D1) to the partitions

$$\boldsymbol{\Omega} := \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_1,$$

we find that

$$\begin{aligned} \mathfrak{I} &:= \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big) \\ &= \eta\Big(\eta\big(F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big) \times \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2]\big)\Big). \end{aligned}$$

By Lemma 5 and injectivity of the $\boldsymbol{\eta}$-maps, the map

$$\eta_{(\boldsymbol{\emptyset}, \boldsymbol{\Omega}_1)} : F[\boldsymbol{\emptyset}] \times F[\boldsymbol{\Omega}_1] \to F[\boldsymbol{\Omega}_1]$$

is surjective; that is,

$$\eta\big(F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big) = F[\boldsymbol{\Omega}_1].$$

Similarly, we have

$$\eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_2]\big) = F[\boldsymbol{\Omega}_2].$$

Thus,

$$\mathfrak{I} = \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big).$$

By an analogous application of (D1) and Lemma 5 to the partitions

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2,$$

we find that

$$\mathfrak{I} = \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1]\big),$$

and the proof is complete. □

Lemma 7 (3-Associativity for $(F, \boldsymbol{\eta})$)**.** *For pairwise disjoint* $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3 \in \mathrm{Ob}(\mathbf{Set}^r)$*, we have*

$$\eta\Big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\Big) = \eta\Big(F[\boldsymbol{\Omega}_1] \times \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big)\Big). \tag{3.2}$$

Proof. We shall show that both sides of (3.2) equal the intersection

$$\mathfrak{I} := \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3]\big).$$

Applying (D1) to the partitions

$$\boldsymbol{\Omega} := (\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2) \amalg \boldsymbol{\Omega}_3 = \boldsymbol{\Omega}_1 \amalg (\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3),$$

we find that

$$\mathfrak{I} = \eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times \eta\big(F[\boldsymbol{\Omega}_1 \cap \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_3]\big)\big);$$

and, arguing as in the proof of Lemma 6, this equation simplifies to

$$\mathfrak{I} = \eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big).$$

The same argument, when applied to the partitions

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_1 \amalg (\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3) = (\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2) \amalg \boldsymbol{\Omega}_3,$$

yields

$$\mathfrak{I} = \eta\big(F[\boldsymbol{\Omega}_1] \times \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big)\big),$$

whence (3.2). □

For the sake of convenience, for pairwise disjoint elements $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m \in \mathrm{Ob}(\mathbf{Set}^r)$, let us call expressions formed by applying $\boldsymbol{\eta}$-maps to $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$ (in any order), with each set $F[\boldsymbol{\Omega}_i]$ occurring exactly once, an *$\boldsymbol{\eta}$-bracketing of* $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$. More formally, for any $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, we call the set $F[\boldsymbol{\Omega}]$ an $\boldsymbol{\eta}$-bracketing of $F[\boldsymbol{\Omega}]$; and, given an $\boldsymbol{\eta}$-bracketing B_1 of $F[\boldsymbol{\Omega}_{i_1}]$, …, $F[\boldsymbol{\Omega}_{i_k}]$ and an $\boldsymbol{\eta}$-bracketing B_2 of $F[\boldsymbol{\Omega}_{i_{k+1}}]$, …, $F[\boldsymbol{\Omega}_{i_m}]$, with $1 \le k \le n-1$ and $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, m\}$, the expression $\eta(B_1 \times B_2)$ is, by definition, an $\boldsymbol{\eta}$-bracketing of $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$. For example, the left-hand side and the right-hand side of (3.2) are two possible $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, $F[\boldsymbol{\Omega}_2]$, $F[\boldsymbol{\Omega}_3]$, as are

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_1]\big) \quad \text{and} \quad \eta\big(F[\boldsymbol{\Omega}_3] \times \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big)\big).$$

A simple consequence of Lemma 6 and (the proof of) Lemma 7 is the following fact.

Corollary 8. *All $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, $F[\boldsymbol{\Omega}_2]$, $F[\boldsymbol{\Omega}_3]$ are equal to*

$$\eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_2]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_1]\big). \quad (3.3)$$

Proof. From the proof of Lemma 7, we know that

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big) = \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3]\big).$$

If, in this equation, we interchange $\boldsymbol{\Omega}_1$ and $\boldsymbol{\Omega}_2$ and use Lemma 6 (commutativity), then we obtain

$$\eta\big(\eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2]\big) \times F[\boldsymbol{\Omega}_3]\big) = \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_3]\big).$$

Both equations together, plus another application of Lemma 6, imply our claim. □

Lemma 9 (4-Permutability for $(F, \boldsymbol{\eta})$)**.** *If $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4$ are pairwise disjoint elements of* $\mathrm{Ob}(\mathbf{Set}^r)$*, then all $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, $F[\boldsymbol{\Omega}_2]$, $F[\boldsymbol{\Omega}_3]$, $F[\boldsymbol{\Omega}_4]$ are equal to each other.*

Proof. The possible $\boldsymbol{\eta}$-bracketings of $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$ are

$$\eta\big(\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big), \tag{3.4}$$

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times \eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3]\big)\big)\times F[\mathbf{\Omega}_4]\big), \tag{3.5}$$

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big), \tag{3.6}$$

$$\eta\big(F[\mathbf{\Omega}_1]\times \eta\big(\eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big)\big), \tag{3.7}$$

$$\eta\big(F[\mathbf{\Omega}_1]\times \eta\big(F[\mathbf{\Omega}_2]\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big)\big), \tag{3.8}$$

together with all expressions arising from the above by permuting $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$. By Lemma 7 (3-associativity), the bracketing (3.4) equals the bracketing (3.5), and the bracketing (3.7) equals the bracketing (3.8). It suffices therefore to prove the equality of (3.4), (3.6), (3.8), and all expressions arising from these three by permuting $F[\mathbf{\Omega}_1]$, $F[\mathbf{\Omega}_2]$, $F[\mathbf{\Omega}_3]$, $F[\mathbf{\Omega}_4]$.

Applying (D1) to the partitions

$$\mathbf{\Omega} := (\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2)\amalg(\mathbf{\Omega}_3 \amalg \mathbf{\Omega}_4) = (\mathbf{\Omega}_1 \amalg \mathbf{\Omega}_3)\amalg(\mathbf{\Omega}_2 \amalg \mathbf{\Omega}_4),$$

we find that

$$\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\big)\cap\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_4]\big)$$
$$= \eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big).$$

Using this equation, as well as the one which arises by interchanging $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ and applying Lemma 6 (commutativity) on the resulting right-hand side, we obtain

$$\begin{aligned}\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times \eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big) &= \eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\big)\cap\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_4]\big)\\ &\quad\cap\eta\big(F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_4]\big).\end{aligned} \tag{3.9}$$

On the other hand, by Corollary 8, the expression

$$\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)$$

equals the expression (3.3). Therefore, "bracketing" these two expressions by $\eta(\,.\,\times F[\mathbf{\Omega}_4])$, using the injectivity of η, and applying Lemma 7 (3-associativity) to the expression resulting from (3.3), we arrive at

$$\begin{aligned}\eta\big(\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big) &= \eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big)\cap\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3]\times\eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_4]\big)\big)\\ &\quad\cap\eta\big(F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\times\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_4]\big)\big).\end{aligned} \tag{3.10}$$

Since $\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\subseteq F[\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]$, and similar inclusions hold for other combinations of the $\mathbf{\Omega}_i$'s, we have

$$\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big)\subseteq\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_3\amalg\mathbf{\Omega}_4]\big),$$
$$\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3]\times\eta\big(F[\mathbf{\Omega}_2]\times F[\mathbf{\Omega}_4]\big)\big)\subseteq\eta\big(F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_4]\big),$$
$$\eta\big(F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\times\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_4]\big)\big)\subseteq\eta\big(F[\mathbf{\Omega}_2\amalg\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_1\amalg\mathbf{\Omega}_4]\big).$$

Altogether, these inclusions imply that the right-hand side of (3.10) is contained in the right-hand side of (3.9). We infer that

$$\eta\big(\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times F[\mathbf{\Omega}_3]\big)\times F[\mathbf{\Omega}_4]\big)\subseteq\eta\big(\eta\big(F[\mathbf{\Omega}_1]\times F[\mathbf{\Omega}_2]\big)\times\eta\big(F[\mathbf{\Omega}_3]\times F[\mathbf{\Omega}_4]\big)\big). \tag{3.11}$$

However, by injectivity of the $\boldsymbol{\eta}$-maps, both sides of (3.11) have cardinality

$$|F[\boldsymbol{\Omega}_1]| \cdot |F[\boldsymbol{\Omega}_2]| \cdot |F[\boldsymbol{\Omega}_3]| \cdot |F[\boldsymbol{\Omega}_4]|.$$

Hence, they must be equal, which proves the equality of (3.4) and (3.6).

An analogous argument proves equality of (3.6) and (3.8).

Since, by Lemma 6 (commutativity), the right-hand side of (3.9) is invariant under permutation of $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4$, the proof is complete. □

The (proof of the) above lemma, combined with Lemma 6 and Corollary 8, leads to the following observation.

Corollary 10. *All $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, $F[\boldsymbol{\Omega}_2]$, $F[\boldsymbol{\Omega}_3]$, $F[\boldsymbol{\Omega}_4]$ are equal to*

$$\begin{aligned}\eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_3 \amalg \boldsymbol{\Omega}_4]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_4]\big)&\\ \cap\, \eta\big(F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_4]\big)&\\ \cap\, \eta\big(F[\boldsymbol{\Omega}_1] \times F[\boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3 \amalg \boldsymbol{\Omega}_4]\big) \cap \eta\big(F[\boldsymbol{\Omega}_2] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_3 \amalg \boldsymbol{\Omega}_4]\big)&\\ \cap\, \eta\big(F[\boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_4]\big) \cap \eta\big(F[\boldsymbol{\Omega}_4] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3]\big)&. \qquad (3.12)\end{aligned}$$

Proof. By Corollary 8, we have

$$\begin{aligned}\eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2] \times \eta\big(F[\boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_4]\big)\big)&\\ = \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3] \times F[\boldsymbol{\Omega}_4]\big) \cap \eta\big(F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_4] \times F[\boldsymbol{\Omega}_3]\big)&\\ \cap\, \eta\big(F[\boldsymbol{\Omega}_3 \amalg \boldsymbol{\Omega}_4] \times F[\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2]\big)&.\end{aligned}$$

Analogous identities hold for the other two terms on the right-hand side of (3.10). If we combine this with Lemma 6 (commutativity) and Lemma 9 (4-permutability), then the claim follows immediately. □

Before we state the general permutability result, let us introduce the following shorthand notation, which will be used in its proof. Given a subset I of $[m]$, where $I = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$, we let $\boldsymbol{\Omega}_I$ stand for $\boldsymbol{\Omega}_{i_1} \amalg \cdots \amalg \boldsymbol{\Omega}_{i_k}$.

Lemma 11 (m-Permutability for $(F, \boldsymbol{\eta})$)**.** *If $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ are pairwise disjoint elements of* $\mathrm{Ob}(\mathbf{Set}^r)$*, then all $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$ are equal to each other.*

Proof. We shall prove by induction on m that, for each $m \geq 2$, all $\boldsymbol{\eta}$-bracketings of $F[\boldsymbol{\Omega}_1]$, …, $F[\boldsymbol{\Omega}_m]$ equal

$$\bigcap_{\substack{I,J \subseteq [m] \\ I \cup J = [m],\ I \cap J = \emptyset \\ I \neq \emptyset \neq J}} \eta\big(F[\boldsymbol{\Omega}_I] \times F[\boldsymbol{\Omega}_J]\big). \qquad (3.13)$$

For $m = 2$, the assertion follows from Lemma 6 (commutativity). For $m = 3$, the assertion is equivalent to Corollary 8 (again, modulo Lemma 6), and for $m = 4$, the assertion is equivalent to Corollary 10.

Now let $m \geq 5$, and let us suppose that the assertion is true up to $m - 1$. Consider an $\boldsymbol{\eta}$-bracketing of $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$. There are three possibilities. Either this bracketing has the form

$$\eta\big(\eta(E_1) \times \eta(E_2)\big), \qquad (3.14)$$

where E_1 and E_2 are expressions involving $\boldsymbol{\eta}$-maps and distinct $\mathbf{\Omega}_i$'s, each of them involving at least two $\mathbf{\Omega}_i$'s, or the form

$$\eta\big(F[\mathbf{\Omega}_i] \times \eta\big(E_3\big)\big), \tag{3.15}$$

where E_3 involves $\boldsymbol{\eta}$-maps and $\mathbf{\Omega}_1, \dots, \mathbf{\Omega}_{i-1}, \mathbf{\Omega}_{i+1}, \dots, \mathbf{\Omega}_m$, for some i, or the form

$$\eta\big(\eta\big(E_4\big) \times F[\mathbf{\Omega}_i]\big), \tag{3.16}$$

where E_4 involves $\boldsymbol{\eta}$-maps and $\mathbf{\Omega}_1, \dots, \mathbf{\Omega}_{i-1}, \mathbf{\Omega}_{i+1}, \dots, \mathbf{\Omega}_m$, for some i.

We start by considering (3.14). Let us assume that E_1 involves all $\mathbf{\Omega}_r$'s for $r \in R$, and E_2 involves all $\mathbf{\Omega}_s$'s for $s \in S$, with $R \cup S = [m]$, $R \cap S = \emptyset$, $R \neq \emptyset \neq S$. By the inductive hypothesis, we know that

$$\eta(E_1) = \bigcap_{\substack{I_1, J_1 \subseteq R \\ I_1 \cup J_1 = R,\ I_1 \cap J_1 = \emptyset \\ I_1 \neq \emptyset \neq J_1}} \eta\big(F[\mathbf{\Omega}_{I_1}] \times F[\mathbf{\Omega}_{J_1}]\big) \tag{3.17}$$

and

$$\eta(E_2) = \bigcap_{\substack{I_2, J_2 \subseteq S \\ I_2 \cup J_2 = S,\ I_2 \cap J_2 = \emptyset \\ I_2 \neq \emptyset \neq J_2}} \eta\big(F[\mathbf{\Omega}_{I_2}] \times F[\mathbf{\Omega}_{J_2}]\big). \tag{3.18}$$

If we substitute (3.17) and (3.18) in (3.14) and use injectivity of the $\boldsymbol{\eta}$-maps, then we obtain

$$\bigcap_{\substack{I_1, J_1 \subseteq R \\ I_1 \cup J_1 = R,\ I_1 \cap J_1 = \emptyset \\ I_1 \neq \emptyset \neq J_1}} \bigcap_{\substack{I_2, J_2 \subseteq S \\ I_2 \cup J_2 = S,\ I_2 \cap J_2 = \emptyset \\ I_2 \neq \emptyset \neq J_2}} \eta\Big(\eta\big(F[\mathbf{\Omega}_{I_1}] \times F[\mathbf{\Omega}_{J_1}]\big) \times \eta\big(F[\mathbf{\Omega}_{I_2}] \times F[\mathbf{\Omega}_{J_2}]\big)\Big).$$

We may now apply Corollary 10 to each of the terms on the right-hand side of this equation. It is not difficult to see that, together with Lemma 6 (commutativity), we obtain (3.13).

Next we consider (3.15). By the inductive hypothesis, we know that

$$\eta(E_3) = \bigcap_{\substack{I, J \subseteq [m] - \{i\} \\ I \cup J = [m] - \{i\},\ I \cap J = \emptyset \\ I \neq \emptyset \neq J}} \eta\big(F[\mathbf{\Omega}_I] \times F[\mathbf{\Omega}_J]\big).$$

If we substitute this in (3.15) and use injectivity of the $\boldsymbol{\eta}$-maps, then we obtain

$$\bigcap_{\substack{I, J \subseteq [m] - \{i\} \\ I \cup J = [m] - \{i\},\ I \cap J = \emptyset \\ I \neq \emptyset \neq J}} \eta\Big(F[\mathbf{\Omega}_i] \times \eta\big(F[\mathbf{\Omega}_I] \times F[\mathbf{\Omega}_J]\big)\Big).$$

We may now apply Corollary 8 to each of the terms on the right-hand side of this equation. It is not difficult to see that, together with Lemma 6 (commutativity), we obtain (3.13).

The argument for (3.16) is analogous. This completes the proof of the lemma. □

For $\mathbf{\Omega} = (\Omega^{(1)}, \dots, \Omega^{(r)}) \in \mathrm{Ob}(\mathbf{Set}^r)$ and an integer $\rho \in [r]$, we write $(\omega, \rho) \in \mathbf{\Omega}$ to mean $\omega \in \Omega^{(\rho)}$. This is the concept of *base point* needed in the present context.

Lemma 12. *For non-empty* $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$ *and every choice of base point* $(\omega, \rho) \in \mathbf{\Omega}$, *we have*

$$F[\mathbf{\Omega}] = \bigcup_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r) \\ (\omega,\rho) \in \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big). \tag{3.19}$$

Proof. Let $x \in F[\mathbf{\Omega}]$ be an arbitrary element, and consider the totality of all $\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r)$ such that $(\omega, \rho) \in \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}$ and $x \in \eta\big(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big)$. Such $\mathbf{\Omega}_1$'s do exist; for instance $\mathbf{\Omega}_1 = \mathbf{\Omega}$ has these properties, since the map

$$\eta_{(\mathbf{\Omega},\boldsymbol{\emptyset})} : F[\mathbf{\Omega}] \times F[\boldsymbol{\emptyset}] \hookrightarrow F[\mathbf{\Omega}]$$

is surjective. Among these $\mathbf{\Omega}_1$'s we choose one of minimal norm $\|\mathbf{\Omega}_1\|$ (recall the definition in (2.5)), say $\mathbf{\Omega}_1(x)$. Now suppose that $x \notin \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1(x)] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1(x)]\big)$. Then, by the definition of $F_{\boldsymbol{\eta}}$, the injectivity of η, and the choice of $\mathbf{\Omega}_1(x)$, we must have

$$\begin{aligned} x \in \eta\Big(\big(F[\mathbf{\Omega}_1(x)] - F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1(x)]\big) \times F[\mathbf{\Omega} - \mathbf{\Omega}_1(x)]\Big) \\ = \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r) \\ \boldsymbol{I} \amalg \boldsymbol{J} = \mathbf{\Omega}_1(x) \\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} \eta\Big(\eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big) \times F[\mathbf{\Omega} - \mathbf{\Omega}_1(x)]\Big). \end{aligned}$$

Consequently, there exists $(\boldsymbol{I}_1, \boldsymbol{J}_1) \in \mathrm{Ob}(\mathfrak{D}_r)$ such that $\boldsymbol{I}_1 \amalg \boldsymbol{J}_1 = \mathbf{\Omega}_1(x)$, $\boldsymbol{I}_1 \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}_1$, and

$$x \in \eta\Big(\eta\big(F[\boldsymbol{I}_1] \times F[\boldsymbol{J}_1]\big) \times F[\mathbf{\Omega} - \mathbf{\Omega}_1(x)]\Big).$$

Using Corollary 8 (3-permutability), we see that the latter set is contained in both $\eta\big(F[\boldsymbol{I}_1] \times F[\mathbf{\Omega} - \boldsymbol{I}_1]\big)$ and $\eta\big(F[\boldsymbol{J}_1] \times F[\mathbf{\Omega} - \boldsymbol{J}_1]\big)$. The base point (ω, ρ) is contained in $\boldsymbol{I}_1$ or $\boldsymbol{J}_1$; to fix ideas, say $(\omega, \rho) \in \boldsymbol{I}_1$. Hence, we arrive at the assertion that

$$x \in \eta\big(F[\boldsymbol{I}_1] \times F[\mathbf{\Omega} - \boldsymbol{I}_1]\big),\ (\omega, \rho) \in \boldsymbol{I}_1 \subseteq \mathbf{\Omega},\ \|\boldsymbol{I}_1\| < \|\mathbf{\Omega}_1(x)\|,$$

contradicting the choice of $\mathbf{\Omega}_1(x)$. We conclude that x is indeed contained in

$$\eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1(x)] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1(x)]\big),$$

and (3.19) is proven. □

Lemma 13. *The right-hand side of* (3.19) *is a disjoint union.*

Proof. In the context of Lemma 12, let $\mathbf{\Omega}_1, \mathbf{\Omega}_2 \in \mathrm{Ob}(\mathbf{Set}^r)$ be such that $(\omega, \rho) \in \mathbf{\Omega}_i \subseteq \mathbf{\Omega}$ and $\mathbf{\Omega}_1 \neq \mathbf{\Omega}_2$, say $\mathbf{\Omega}_1 \not\subseteq \mathbf{\Omega}_2$. It is enough to show that

$$\mathfrak{I} := \eta\big(F[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big) \cap \eta\big(F[\mathbf{\Omega}_2] \times F[\mathbf{\Omega} - \mathbf{\Omega}_2]\big)$$

has an empty intersection with $\eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big)$. But, by (D1), we have

$$\mathfrak{I} \subseteq \eta\Big(\eta\big(F[\mathbf{\Omega}_1 \cap \mathbf{\Omega}_2] \times F[\mathbf{\Omega}_1 - \mathbf{\Omega}_2]\big) \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\Big),$$

and, by definition of $F_{\boldsymbol{\eta}}$ and the fact that $(\omega, \rho) \in \mathbf{\Omega}_1 \cap \mathbf{\Omega}_2 \neq \boldsymbol{\emptyset} \neq \mathbf{\Omega}_1 - \mathbf{\Omega}_2$, we have

$$F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \cap \eta\big(F[\mathbf{\Omega}_1 \cap \mathbf{\Omega}_2] \times F[\mathbf{\Omega}_1 - \mathbf{\Omega}_2]\big) = \emptyset.$$

Consequently, by the injectivity of η, we must indeed have

$$\eta\big(F_{\boldsymbol\eta}[\boldsymbol\Omega_1] \times F[\boldsymbol\Omega - \boldsymbol\Omega_1]\big) \cap \mathfrak{I}$$
$$\subseteq \eta\big(F_{\boldsymbol\eta}[\boldsymbol\Omega_1] \times F[\boldsymbol\Omega - \boldsymbol\Omega_1]\big) \cap \eta\Big(\eta\big(F[\boldsymbol\Omega_1 \cap \boldsymbol\Omega_2] \times F[\boldsymbol\Omega_1 - \boldsymbol\Omega_2]\big) \times F[\boldsymbol\Omega - \boldsymbol\Omega_1]\Big) = \emptyset,$$

as required. □

Lemma 14 (Functoriality of $F_{\boldsymbol\eta}$)**.** *Let $\boldsymbol\Omega, \widetilde{\boldsymbol\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, and let $\boldsymbol f : \boldsymbol\Omega \to \widetilde{\boldsymbol\Omega}$ be a morphism. Then*

$$F[\boldsymbol f](F_{\boldsymbol\eta}[\boldsymbol\Omega]) = F_{\boldsymbol\eta}[\widetilde{\boldsymbol\Omega}];$$

that is, setting $F_{\boldsymbol\eta}[\boldsymbol f] := F[\boldsymbol f]|_{F_{\eta}[\boldsymbol\Omega]}$, we get a functor $F_{\boldsymbol\eta} : \mathbf{Set}^r \to \mathbf{Set}$.

Proof. The assertion is obvious if $\boldsymbol\Omega = \boldsymbol\emptyset$, so we may suppose that $\boldsymbol\Omega \neq \boldsymbol\emptyset$. Then, using the naturality of $\boldsymbol\eta$ (that is, the diagram (2.1)), we have

$$\begin{aligned}
F[\boldsymbol f](F_{\boldsymbol\eta}[\boldsymbol\Omega]) &= F[\boldsymbol f]\Bigg(F[\boldsymbol\Omega] - \bigcup_{\substack{(\boldsymbol I_1,\boldsymbol J_1)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_1\amalg\boldsymbol J_1=\boldsymbol\Omega\\ \boldsymbol I_1\neq\boldsymbol\emptyset\neq\boldsymbol J_1}} \eta\big(F[\boldsymbol I_1]\times F[\boldsymbol J_1]\big)\Bigg)\\
&= F[\widetilde{\boldsymbol\Omega}] - \bigcup_{\substack{(\boldsymbol I_1,\boldsymbol J_1)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_1\amalg\boldsymbol J_1=\boldsymbol\Omega\\ \boldsymbol I_1\neq\boldsymbol\emptyset\neq\boldsymbol J_1}} \big(F\circ\amalg\big)\big[(\boldsymbol f|_{\boldsymbol I_1}, \boldsymbol f|_{\boldsymbol J_1})\big]\big(\eta\big(F[\boldsymbol I_1]\times F[\boldsymbol J_1]\big)\big)\\
&= F[\widetilde{\boldsymbol\Omega}] - \bigcup_{\substack{(\boldsymbol I_1,\boldsymbol J_1)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_1\amalg\boldsymbol J_1=\boldsymbol\Omega\\ \boldsymbol I_1\neq\boldsymbol\emptyset\neq\boldsymbol J_1}} \eta\big(F[\boldsymbol f|_{\boldsymbol I_1}](F[\boldsymbol I_1])\times F[\boldsymbol f|_{\boldsymbol J_1}](F[\boldsymbol J_1])\big)\\
&= F[\widetilde{\boldsymbol\Omega}] - \bigcup_{\substack{(\boldsymbol I_2,\boldsymbol J_2)\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol I_2\amalg\boldsymbol J_2=\widetilde{\boldsymbol\Omega}\\ \boldsymbol I_2\neq\boldsymbol\emptyset\neq\boldsymbol J_2}} \eta\big(F[\boldsymbol I_2]\times F[\boldsymbol J_2]\big)\\
&= F[\widetilde{\boldsymbol\Omega}].
\end{aligned}$$

□

4. Proof of Theorem 1

For convenience, let us "extend" the Λ-weight $\boldsymbol w$ to subsets of $F[\boldsymbol\Omega]$, for all $\boldsymbol\Omega \in \mathrm{Ob}(\mathbf{Set}^r)$. To be precise, for $A \subseteq F[\boldsymbol\Omega]$, we define

$$w_{\boldsymbol\Omega}(A) := \sum_{x\in A} w_{\boldsymbol\Omega}(x).$$

Given disjoint $\boldsymbol\Omega_1, \boldsymbol\Omega_2 \in \mathrm{Ob}(\mathbf{Set}^r)$ and subsets $A_1 \subseteq F_{\boldsymbol\eta}[\boldsymbol\Omega_1]$ and $A_2 \subseteq F[\boldsymbol\Omega_2]$, Axiom (W2) says that

$$w_{\boldsymbol\Omega_1\amalg\boldsymbol\Omega_2}(\eta(A_1\times A_2)) = w_{\boldsymbol\Omega_1}(A_1)\cdot w_{\boldsymbol\Omega_2}(A_2). \tag{4.1}$$

In the sequel, we shall suppress the indices of weights w for better readability, the indices always being clear from the context.

As a direct consequence of Lemmas 12 and 13, of the injectivity of the $\boldsymbol{\eta}$-maps, and of (4.1), we have that, for $n_1, n_2, \dots, n_r \in \mathbb{N}_0$ and $n_\rho > 0$,

$$w\big(F[([n_1],\dots,[n_r])]\big) = \sum_{\substack{\Omega_1^{(i)}\subseteq[n_i]\,(1\le i\le r)\\ 1\in\Omega_1^{(\rho)}}} w\big(F_{\boldsymbol{\eta}}[(\Omega_1^{(1)},\dots,\Omega_1^{(r)})]\big)\cdot w\big(F[([n_1]-\Omega_1^{(1)},\dots,[n_r]-\Omega_1^{(r)})]\big). \quad (4.2)$$

Using the functoriality of F and $F_{\boldsymbol{\eta}}$, together with Axiom (W1), each $\boldsymbol{\Omega}_1$ with

$$\boldsymbol{\Omega}_1 = (\Omega_1^{(1)},\dots,\Omega_1^{(r)}) \subseteq ([n_1],\dots,[n_r]),$$

$(1,\rho)\in\boldsymbol{\Omega}_1$, and cardinalities $|\Omega_1^{(i)}| = \mu_i$ $(1\le i\le r)$, is seen to contribute

$$w\big(F_{\boldsymbol{\eta}}[([\mu_1],\dots,[\mu_r])]\big)\cdot w\big(F[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big) \quad (4.3)$$

to the right-hand side of (4.2). We observe that (4.3) does not depend upon $\boldsymbol{\Omega}_1$ itself, but only on the cardinalities $\mu_1,\dots,\mu_r$ of the components $\Omega_1^{(1)},\dots,\Omega_1^{(r)}$. Therefore, the

$$\frac{\mu_\rho}{n_\rho}\prod_{1\le i\le r}\binom{n_i}{\mu_i}$$

elements $\boldsymbol{\Omega}_1\in \mathrm{Ob}(\mathbf{Set}^r)$ with $\boldsymbol{\Omega}_1\subseteq([n_1],\dots,[n_r])$, $(1,\rho)\in\boldsymbol{\Omega}_1$, and such that $|\Omega_1^{(i)}| = \mu_i$ for $1\le i\le r$, contribute

$$\frac{\mu_\rho}{n_\rho}\Bigg(\prod_{1\le i\le r}\binom{n_i}{\mu_i}\Bigg) w\big(F_{\boldsymbol{\eta}}[([\mu_1],\dots,[\mu_r])]\big)\cdot w\big(F[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big)$$

to the right-hand side of (4.2), and we obtain

$$w\big(F[([n_1],\dots,[n_r])]\big) = \sum_{0\le\mu_i\le n_i\,(1\le i\le r)} \frac{\mu_\rho}{n_\rho}\Bigg(\prod_{1\le i\le r}\binom{n_i}{\mu_i}\Bigg) w\big(F_{\boldsymbol{\eta}}[([\mu_1],\dots,[\mu_r])]\big) \cdot w\big(F[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big), \qquad n_\rho>0,$$

or, equivalently,

$$n_\rho\cdot w\big(F[([n_1],\dots,[n_r])]\big) = \sum_{0\le\mu_i\le n_i\,(1\le i\le r)} \mu_\rho\Bigg(\prod_{1\le i\le r}\binom{n_i}{\mu_i}\Bigg) w\big(F_{\boldsymbol{\eta}}[([\mu_1],\dots,[\mu_r])]\big) \cdot w\big(F[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big), \quad (4.4)$$

as long as $n_\rho > 0$. However, Equation (4.4) holds as well for $n_\rho = 0$, with both sides vanishing, so that we are allowed to drop the restriction $n_\rho > 0$.

Fix $\rho\in[r]$, multiply both sides of (4.4) by

$$z_1^{n_1}\cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho-1} z_{\rho+1}^{n_{\rho+1}}\cdots z_r^{n_r}/(n_1!\cdots n_r!),$$

and sum over all tuples $(n_1,\dots,n_r)\in\mathbb{N}_0^r$, to get

$$\sum_{n_1,\dots,n_r\ge 0} w\big(F[([n_1],\dots,[n_r])]\big)\,\frac{z_1^{n_1}\cdots z_{\rho-1}^{n_{\rho-1}}\,z_\rho^{n_\rho-1}\,z_{\rho+1}^{n_{\rho+1}}\cdots z_r^{n_r}}{n_1!\cdots n_{\rho-1}!\,(n_\rho-1)!\,n_{\rho+1}!\cdots n_r!}$$
$$=\sum_{n_1,\dots,n_r\ge 0}\ \sum_{0\le \mu_i\le n_i\,(1\le i\le r)} \frac{w\big(F_{\boldsymbol\eta}[([\mu_1],\dots,[\mu_r])]\big)}{\mu_1!\cdots\mu_{\rho-1}!\,(\mu_\rho-1)!\,\mu_{\rho+1}!\cdots\mu_r!}$$
$$\cdot\frac{w\big(F[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big)}{(n_1-\mu_1)!\cdots(n_r-\mu_r)!}\,z_1^{n_1}\cdots z_{\rho-1}^{n_{\rho-1}}\,z_\rho^{n_\rho-1}\,z_{\rho+1}^{n_{\rho+1}}\cdots z_r^{n_r},\qquad(4.5)$$

where $1/(-1)!$ has to be interpreted as 0. The left-hand side equals

$$\frac{\partial\,\mathrm{GF}_F}{\partial z_\rho},$$

while the right-hand side is identified as

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho}\,\mathrm{GF}_F;$$

whence the equations

$$\frac{\partial\,\mathrm{GF}_F}{\partial z_\rho}=\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho}\,\mathrm{GF}_F,\qquad 1\le\rho\le r.\qquad(4.6)$$

Set

$$Q(z_1,\dots,z_r):=\mathrm{GF}_F(z_1,\dots,z_r)\exp\big(-\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,\dots,z_r)\big).$$

Then, in view of Equations (4.6), the series Q satisfies

$$\frac{\partial Q}{\partial z_\rho}=0,\qquad 1\le\rho\le r.$$

These last equations force Q to be independent of $z_1,z_2,\dots,z_r$. However, since $\mathrm{GF}_{F_{\boldsymbol\eta}}(0,\dots,0)=0$ by definition of $F_{\boldsymbol\eta}$, and since $\mathrm{GF}_F(0,\dots,0)=1$ by Axiom (W0) and Lemma 5, direct inspection shows that

$$Q(0,0,\dots,0)=1,$$

and (2.3) follows. □

5. Auxiliary results, II

In this section, we complement the results of Section 3 by establishing several further results which will be needed in the proofs of Proposition 3 and Theorem 4, to be given in the next section. The first lemma provides the analogue of Lemma 11 for $F_{\boldsymbol\eta}$, namely that arbitrary permutability holds also for $\boldsymbol\eta$-bracketings of $F_{\boldsymbol\eta}$-images (see the subsequent paragraph for the precise definition). All the remaining lemmas concern the maps $F_{\boldsymbol\eta}^{(k)}$. In all of this section, we assume that F is a decomposable r-sort species with composition operator $\boldsymbol\eta$.

In complete analogy with the corresponding definition in Section 3, we define the concept of an *$\boldsymbol\eta$-bracketing of* $F_{\boldsymbol\eta}[\boldsymbol\Omega_1],\dots,F_{\boldsymbol\eta}[\boldsymbol\Omega_m]$ for pairwise disjoint elements $\boldsymbol\Omega_1,\dots,\boldsymbol\Omega_m\in\mathrm{Ob}(\mathbf{Set}^r)$: simply replace F by $F_{\boldsymbol\eta}$ everywhere in the definition just after the proof of Lemma 7.

Lemma 15 (m-Permutability for $(F_{\boldsymbol{\eta}}, \boldsymbol{\eta})$)**.** *If $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ are pairwise disjoint elements of* $\mathrm{Ob}(\mathbf{Set}^r)$*, then all $\boldsymbol{\eta}$-bracketings of $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1]$, ..., $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]$ are equal to each other.*

Proof. Since $F_{\boldsymbol{\eta}}[\boldsymbol{\emptyset}] = \emptyset$ by definition of $F_{\boldsymbol{\eta}}$, our claim holds if at least one of $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ equals $\boldsymbol{\emptyset}$. Hence, we may assume that all of $\boldsymbol{\Omega}_1, \dots, \boldsymbol{\Omega}_m$ are non-empty.

Next we note that, for sets $M_1, \dots, M_m, A_1, \dots, A_m$, we have

$$
\begin{aligned}
&(M_1 - A_1) \times (M_2 - A_2) \times \cdots \times (M_m - A_m) \\
&= M_1 \times M_2 \times \cdots \times M_m - \left(\bigcup_{k=1}^{m} M_1 \times \cdots \times M_{k-1} \times A_k \times M_{k+1} \times \cdots \times M_m \right).
\end{aligned} \tag{5.1}
$$

Now assume that we are given two $\boldsymbol{\eta}$-bracketings of $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]$, say

$$
B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big) \quad \text{and} \quad \bar{B}_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big).
$$

Substituting the definition of $F_{\boldsymbol{\eta}}$ into $B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big)$, and applying Ident– ity (5.1) plus injectivity of $\boldsymbol{\eta}$-maps, we find that

$$
\begin{aligned}
&B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big) = B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big) \\
&\quad - \bigcup_{k=1}^{m} B_{\boldsymbol{\eta}}\left(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r) \\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega}_k \\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m] \right) \\
&= B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big) \\
&\quad - \bigcup_{k=1}^{m} \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J}) \in \mathrm{Ob}(\mathfrak{D}_r) \\ \boldsymbol{I} \amalg \boldsymbol{J} = \boldsymbol{\Omega}_k \\ \boldsymbol{I} \neq \boldsymbol{\emptyset} \neq \boldsymbol{J}}} B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\big).
\end{aligned}
$$

The same argument shows that $\bar{B}_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big)$ equals the last expression where every occurrence of $B_{\boldsymbol{\eta}}$ is replaced by $\bar{B}_{\boldsymbol{\eta}}$. By Lemma 11 ($m$-permutability for $(F, \boldsymbol{\eta})$), we have

$$
B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big) = \bar{B}_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_m]\big)
$$

and

$$
\begin{aligned}
&B_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\big) \\
&\qquad = \bar{B}_{\boldsymbol{\eta}}\big(F[\boldsymbol{\Omega}_1], \dots, F[\boldsymbol{\Omega}_{k-1}], \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big), F[\boldsymbol{\Omega}_{k+1}], \dots, F[\boldsymbol{\Omega}_m]\big),
\end{aligned}
$$

hence

$$
B_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big) = \bar{B}_{\boldsymbol{\eta}}\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1], \dots, F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_m]\big),
$$

which establishes our claim. □

Lemma 16 (Functoriality of $F_{\boldsymbol{\eta}}^{(k)}$)**.** *For each morphism $\boldsymbol{f} : \boldsymbol{\Omega} \to \widetilde{\boldsymbol{\Omega}}$ in $\mathbf{Set}^r$ and every integer $k \geq 0$, we have*

$$
F[\boldsymbol{f}](F_{\boldsymbol{\eta}}^{(k)}[\boldsymbol{\Omega}]) = F_{\boldsymbol{\eta}}^{(k)}[\widetilde{\boldsymbol{\Omega}}].
$$

Proof. We use induction on k, our claim being obvious for $k = 0$. Suppose that the assertion holds for $0 \leq k < K$ with some $K \geq 1$. Then, using the definition of $F_{\boldsymbol{\eta}}^{(k)}$, the functoriality of $F_{\boldsymbol{\eta}}$ already demonstrated in Lemma 14, the inductive hypothesis, as well as the naturality of $\boldsymbol{\eta}$, we find that

$$\begin{aligned}
F[\boldsymbol{f}](F_{\boldsymbol{\eta}}^{(K)}[\boldsymbol{\Omega}]) &= F[\boldsymbol{f}]\left(\bigcup_{\substack{\boldsymbol{\Omega}'\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}'\subseteq\boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}'] \times F_{\boldsymbol{\eta}}^{(K-1)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}']\big)\right)\\
&= \bigcup_{\substack{\boldsymbol{\Omega}'\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}'\subseteq\boldsymbol{\Omega}}} \big(F\circ\amalg\big)\big[(\boldsymbol{f}|_{\boldsymbol{\Omega}'}, \boldsymbol{f}|_{\boldsymbol{\Omega}-\boldsymbol{\Omega}'})\big]\big(\eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}'] \times F_{\boldsymbol{\eta}}^{(K-1)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}']\big)\big)\\
&= \bigcup_{\substack{\boldsymbol{\Omega}'\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}'\subseteq\boldsymbol{\Omega}}} \eta\big(F[\boldsymbol{f}|_{\boldsymbol{\Omega}'}](F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}']) \times F[\boldsymbol{f}|_{\boldsymbol{\Omega}-\boldsymbol{\Omega}'}](F_{\boldsymbol{\eta}}^{(K-1)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}'])\big)\\
&= \bigcup_{\substack{\widetilde{\boldsymbol{\Omega}}'\in\mathrm{Ob}(\mathbf{Set}^r)\\ \widetilde{\boldsymbol{\Omega}}'\subseteq\widetilde{\boldsymbol{\Omega}}}} \eta\big(F_{\boldsymbol{\eta}}[\widetilde{\boldsymbol{\Omega}}'] \times F_{\boldsymbol{\eta}}^{(K-1)}[\widetilde{\boldsymbol{\Omega}}-\widetilde{\boldsymbol{\Omega}}']\big)\\
&= F_{\boldsymbol{\eta}}^{(K)}[\widetilde{\boldsymbol{\Omega}}].
\end{aligned}$$

□

Lemma 17. *The functors $F_{\boldsymbol{\eta}}^{(1)}$ and $F_{\boldsymbol{\eta}}$ coincide.*

Proof. It suffices to show that $F_{\boldsymbol{\eta}}^{(1)}[\boldsymbol{\Omega}] = F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]$ for every $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$. By (2.6), this holds if $\boldsymbol{\Omega} = \boldsymbol{\emptyset}$, so assume that $\boldsymbol{\Omega} \neq \boldsymbol{\emptyset}$. Then, using the definition of $F_{\boldsymbol{\eta}}^{(k)}$, the injectivity of $\boldsymbol{\eta}$-maps, Lemma 7 (3-associativity), and Lemma 5, we have

$$\begin{aligned}
F_{\boldsymbol{\eta}}^{(1)}[\boldsymbol{\Omega}] &= \bigcup_{\substack{\boldsymbol{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\Omega}_1\subseteq\boldsymbol{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_1] \times F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}_1]\big)\\
&= \eta\big(F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}] \times F[\boldsymbol{\emptyset}]\big)\\
&= \eta\left(\left(F[\boldsymbol{\Omega}] - \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J})\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I}\amalg\boldsymbol{J}=\boldsymbol{\Omega}\\ \boldsymbol{I}\neq\boldsymbol{\emptyset}\neq\boldsymbol{J}}} \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big)\right) \times F[\boldsymbol{\emptyset}]\right)\\
&= \eta\big(F[\boldsymbol{\Omega}] \times F[\boldsymbol{\emptyset}]\big) - \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J})\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I}\amalg\boldsymbol{J}=\boldsymbol{\Omega}\\ \boldsymbol{I}\neq\boldsymbol{\emptyset}\neq\boldsymbol{J}}} \eta\Big(\eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big) \times F[\boldsymbol{\emptyset}]\Big)\\
&= F[\boldsymbol{\Omega}] - \bigcup_{\substack{(\boldsymbol{I},\boldsymbol{J})\in\mathrm{Ob}(\mathfrak{D}_r)\\ \boldsymbol{I}\amalg\boldsymbol{J}=\boldsymbol{\Omega}\\ \boldsymbol{I}\neq\boldsymbol{\emptyset}\neq\boldsymbol{J}}} \eta\big(F[\boldsymbol{I}] \times F[\boldsymbol{J}]\big)\\
&= F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}],
\end{aligned}$$

proving our claim. □

In the next lemma, we require again the concept of a base point, which was introduced just before Lemma 12.

Lemma 18. *For every non-empty $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, each choice of base point $(\omega, \rho) \in \mathbf{\Omega}$, and every integer $k \geq 1$, we have*

$$F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}] = \coprod_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r) \\ (\omega,\rho) \in \mathbf{\Omega}_1 \subseteq \mathbf{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F_{\boldsymbol{\eta}}^{(k-1)}[\mathbf{\Omega} - \mathbf{\Omega}_1]\big). \tag{5.2}$$

Proof. The fact that the terms on the right-hand side of (5.2) are pairwise disjoint follows from Lemma 13, since a term $\eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F_{\boldsymbol{\eta}}^{(k-1)}[\mathbf{\Omega} - \mathbf{\Omega}_1]\big)$ is contained in the larger set $\eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big)$.

An immediate induction using the definition of $F_{\boldsymbol{\eta}}^{(m)}$ shows that, for all $m \geq 2$, we have

$$F_{\boldsymbol{\eta}}^{(m)}[\mathbf{\Omega}] = \bigcup_{\substack{\mathbf{\Omega}_1,\dots,\mathbf{\Omega}_m \in \mathrm{Ob}(\mathbf{Set}^r) \\ \mathbf{\Omega}_1 \amalg \cdots \amalg \mathbf{\Omega}_m = \mathbf{\Omega}}} \eta\Big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_2] \times \cdots \times \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_{m-1}] \times F_{\boldsymbol{\eta}}[\mathbf{\Omega}_m]\big) \cdots\Big). \tag{5.3}$$

We first consider the case where $k = 1$. Here, by definition of $F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\emptyset}]$, the only contribution to the union on the right-hand side of (5.2) arises for $\mathbf{\Omega}_1 = \mathbf{\Omega}$. In that situation, we have

$$\eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}] \times F_{\boldsymbol{\eta}}^{(0)}[\boldsymbol{\emptyset}]\big) = \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}] \times F[\boldsymbol{\emptyset}]\big).$$

However, this is also the only contribution on the right-hand side of the definition of $F_{\boldsymbol{\eta}}^{(1)}$ given in (2.4), thus proving (5.2) for $k = 1$.

Now we consider the case where $k \geq 2$. If $k \geq 3$, then, given $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, we substitute the right-hand side of (5.3) with $m = k - 1$ in (5.2). As a result, we obtain

$$\bigcup_{\substack{\mathbf{\Omega}_1,\mathbf{\Omega}_2,\dots,\mathbf{\Omega}_k \in \mathrm{Ob}(\mathbf{Set}^r) \\ \mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2 \amalg \cdots \amalg \mathbf{\Omega}_k = \mathbf{\Omega} \\ (\omega,\rho) \in \mathbf{\Omega}_1}} \eta\Big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_2] \times \cdots \times \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_{k-1}] \times F_{\boldsymbol{\eta}}[\mathbf{\Omega}_k]\big) \cdots\Big) \tag{5.4}$$

for the right-hand side of (5.2). We note that Expression (5.4) also agrees with the right-hand side of (5.2) for $k = 2$ (taking into account the fact that we already know that the union on the right-hand side of (5.2) is a disjoint union).

Expression (5.4) is almost (5.3) with $m = k$, except that $\mathbf{\Omega}_1$ is distinguished by having to contain the given base point (ω, ρ). However, by Lemma 15 (m-permutability for $(F_{\boldsymbol{\eta}}, \boldsymbol{\eta})$), the ordering of $\mathbf{\Omega}_1, \mathbf{\Omega}_2, \dots, \mathbf{\Omega}_k$ in the $\boldsymbol{\eta}$-bracketing in the union on the right-hand side of (5.4) is of no relevance. Thus, the restriction that $(\omega, \rho) \in \mathbf{\Omega}_1$ can be dropped. This shows that the right-hand side of (5.2) equals $F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}]$, as claimed. $\square$

6. Proofs of Propositions 2 and 3, and of Theorem 4

We begin this section with the proof of Proposition 2. With Lemma 18 in hand, we are finally in the position to also establish Proposition 3. Theorem 4 is then a simple consequence of an identity on which the proof of Proposition 3 rests (see (6.1) below).

Proof of Proposition 2. We use induction on $\|\mathbf{\Omega}\|$, where $\|\,.\,\|$ has been defined in (2.5). By (2.6) and the definition of $F_{\boldsymbol{\eta}}^{(0)}$, the statement holds if $\|\mathbf{\Omega}\| = 0$, that is, if $\mathbf{\Omega} = \boldsymbol{\emptyset}$.

Let $\mathbf{\Omega}$ be such that $\|\mathbf{\Omega}\| = N$ for some integer $N > 0$, and suppose that (2.7) holds for all $\mathbf{\Omega}' \in \mathrm{Ob}(\mathbf{Set}^r)$ of norm strictly less than N. Then we have $\mathbf{\Omega} \neq \boldsymbol{\emptyset}$, and therefore

$$\begin{aligned}
\bigcup_{k\ge0} F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}] &= \bigcup_{k\ge1} F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}]\\
&= \bigcup_{k\ge1} \bigcup_{\substack{\mathbf{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \mathbf{\Omega}_1\subseteq\mathbf{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F_{\boldsymbol{\eta}}^{(k-1)}[\mathbf{\Omega}-\mathbf{\Omega}_1]\big)\\
&= \bigcup_{\substack{\mathbf{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \mathbf{\Omega}_1\subseteq\mathbf{\Omega}}} \eta\bigg(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times \Big(\bigcup_{k\ge1} F_{\boldsymbol{\eta}}^{(k-1)}[\mathbf{\Omega}-\mathbf{\Omega}_1]\Big)\bigg)\\
&= \bigcup_{\substack{\mathbf{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \boldsymbol{\emptyset}\neq\mathbf{\Omega}_1\subseteq\mathbf{\Omega}}} \eta\bigg(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times \Big(\bigcup_{k\ge0} F_{\boldsymbol{\eta}}^{(k)}[\mathbf{\Omega}-\mathbf{\Omega}_1]\Big)\bigg)\\
&= \bigcup_{\substack{\mathbf{\Omega}_1\in\mathrm{Ob}(\mathbf{Set}^r)\\ \mathbf{\Omega}_1\subseteq\mathbf{\Omega}}} \eta\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega}-\mathbf{\Omega}_1]\big)\\
&= F[\mathbf{\Omega}].
\end{aligned}$$

Here, we have used Lemma 12 for the last equality, and the inductive hypothesis in the second but last step (here it is important that $\mathbf{\Omega}_1 \neq \boldsymbol{\emptyset}$ in order to guarantee that $\|\mathbf{\Omega}-\mathbf{\Omega}_1\| < \|\mathbf{\Omega}\|$). □

Proof of Proposition 3. For $k \ge 0$, let us define the generating function for $F_{\boldsymbol{\eta}}^{(k)}$ by

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}^{(k)}}(z_1,\dots,z_r) := \sum_{n_1,\dots,n_r\ge0} \sum_{x\in F_{\boldsymbol{\eta}}^{(k)}[([n_1],\dots,[n_r])]} w_{([n_1],\dots,[n_r])}(x)\, \frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1!\cdots n_r!}.$$

Again, in the sequel, we shall suppress the indices to weights w for better readability, the indices always being clear from the context.

The first step consists in showing that

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}^{(k)}}(z_1,\dots,z_r) = \frac{1}{k!}\big(\,\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1,\dots,z_r)\big)^k. \tag{6.1}$$

By definition of $F_{\boldsymbol{\eta}}^{(0)}$, the left-hand side of (6.1) equals 1, so that (6.1) holds for $k = 0$. Therefore, we may in the sequel assume that $k \ge 1$.

We now proceed in a manner similar to the proof of Theorem 1 given in Section 4. Here, however, we use Lemma 18 instead of Lemmas 12 and 13, and we also need the functoriality of $F_{\boldsymbol{\eta}}^{(m)}$ for $m = 0, 1, 2, \dots$ established in Lemma 16. In this way, we obtain from (5.2) the identity

$$\begin{aligned}
n_\rho \cdot w\big(F_{\boldsymbol{\eta}}^{(k)}[([n_1],\dots,[n_r])]\big) = \sum_{0\le\mu_i\le n_i\,(1\le i\le r)} \mu_\rho\bigg(\prod_{1\le i\le r}\binom{n_i}{\mu_i}\bigg) w\big(F_{\boldsymbol{\eta}}[([\mu_1],\dots,[\mu_r])]\big)&\\
\cdot\, w\big(F_{\boldsymbol{\eta}}^{(k-1)}[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big).&
\end{aligned} \tag{6.2}$$

Fixing $\rho \in [r]$, multiplying both sides of (6.2) by

$$z_1^{n_1}\cdots z_{\rho-1}^{n_{\rho-1}} z_\rho^{n_\rho-1} z_{\rho+1}^{n_{\rho+1}}\cdots z_r^{n_r}/(n_1!\cdots n_r!),$$

and summing over all tuples $(n_1,\dots,n_r)\in\mathbb{N}_0^r$, gives

$$\sum_{n_1,\dots,n_r\ge 0} w\big(F_{\boldsymbol\eta}^{(k)}[([n_1],\dots,[n_r])]\big)\,\frac{z_1^{n_1}\cdots z_{\rho-1}^{n_{\rho-1}}\,z_\rho^{n_\rho-1}\,z_{\rho+1}^{n_{\rho+1}}\cdots z_r^{n_r}}{n_1!\cdots n_{\rho-1}!\,(n_\rho-1)!\,n_{\rho+1}!\cdots n_r!}$$
$$=\sum_{n_1,\dots,n_r\ge 0}\ \sum_{0\le\mu_i\le n_i\,(1\le i\le r)} \frac{w\big(F_{\boldsymbol\eta}[([\mu_1],\dots,[\mu_r])]\big)}{\mu_1!\cdots\mu_{\rho-1}!\,(\mu_\rho-1)!\,\mu_{\rho+1}!\cdots\mu_r!}$$
$$\cdot\frac{w\big(F_{\boldsymbol\eta}^{(k-1)}[([n_1-\mu_1],\dots,[n_r-\mu_r])]\big)}{(n_1-\mu_1)!\cdots(n_r-\mu_r)!}\,z_1^{n_1}\cdots z_{\rho-1}^{n_{\rho-1}}\,z_\rho^{n_\rho-1}\,z_{\rho+1}^{n_{\rho+1}}\cdots z_r^{n_r}, \quad (6.3)$$

where, again, $1/(-1)!$ has to be interpreted as 0. The left-hand side equals

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}}{\partial z_\rho},$$

while the right-hand side is identified as

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho}\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k-1)}},$$

whence the equations

$$\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}}{\partial z_\rho}=\frac{\partial\,\mathrm{GF}_{F_{\boldsymbol\eta}}}{\partial z_\rho}\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k-1)}},\qquad 1\le\rho\le r. \quad (6.4)$$

Assuming inductively that

$$\mathrm{GF}_{F_{\boldsymbol\eta}^{(k-1)}}=\frac{1}{(k-1)!}\big(\,\mathrm{GF}_{F_{\boldsymbol\eta}}\,\big)^{k-1},$$

we infer from (6.4) that

$$\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}=\frac{1}{k!}\big(\,\mathrm{GF}_{F_{\boldsymbol\eta}}\,\big)^{k}+C,$$

where C is independent of $z_1,z_2,\dots,z_r$. Making use of the facts that $\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}(0,\dots,0)=0$ (since $k\ge1$) and that $\mathrm{GF}_{F_{\boldsymbol\eta}}(0,\dots,0)=0$, we see that $C=0$, which proves (6.1).

On the other hand, by Theorem 1, we know that

$$\mathrm{GF}_F(z_1,z_2,\dots,z_r)=\exp\big(\,\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,z_2,\dots,z_r)\big),$$

or, equivalently,

$$\mathrm{GF}_F(z_1,z_2,\dots,z_r)=\sum_{k\ge0}\frac{1}{k!}\big(\,\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,z_2,\dots,z_r)\big)^k. \quad (6.5)$$

If there were a non-empty intersection between $F_{\boldsymbol\eta}^{(k_1)}[\boldsymbol\Omega]$ and $F_{\boldsymbol\eta}^{(k_2)}[\boldsymbol\Omega]$, for some k_1,k_2 with $k_1<k_2$ and some $\boldsymbol\Omega\in\mathrm{Ob}(\mathbf{Set}^r)$, then Proposition 2 would contradict (6.5) and (6.1). This proves the assertion of the proposition. ☐

Proof of Theorem 4. By definition of $\widetilde{\mathrm{GF}}_F(z_1,\dots,z_r,y)$, we have

$$\widetilde{\mathrm{GF}}_F(z_1,\dots,z_r,y)=\sum_{k\ge0}y^k\,\mathrm{GF}_{F_{\boldsymbol\eta}^{(k)}}(z_1,\dots,z_r).$$

If we now substitute (6.1), then we immediately obtain (2.9). Identity (2.10) results from using Theorem 1 to express $\mathrm{GF}_{F_{\boldsymbol\eta}}(z_1,\dots,z_r)$ in terms of $\mathrm{GF}_F(z_1,\dots,z_r)$ and substituting the result in (2.9). ☐

7. Illustrations, I: Three examples

We give here three illustrations for the application of our theory. In the first and second example below, bipartite graphs are considered. Example 1 is, in some sense, "standard," since it addresses the case where the composition operator $\boldsymbol{\eta}$ consists in "putting objects together," so that the combinatorial objects in our (in this case, 2-sort) species are sets of indecomposable objects, a situation which is well covered by classical species theory. In Example 2, however, the composition operator $\boldsymbol{\eta}$ is different, "non-standard," so that classical species theory does not apply, but our (extension of species) theory does. On the other hand, we shall see in Section 10 that this composition operator is pointwise associative and commutative (for the precise definition see Section 10), and that this family of composition operators is closely related to the classical operation of "putting objects together." (See Theorem 24 for the precise statement.) Our last example in this section, Example 3, presents an example of a composition operator that is neither pointwise associative nor pointwise commutative. A particular aspect demonstrated by Examples 1 and 2 that we want to highlight is that composition operators need not be unique.

Example 1 (Bipartite graphs I). Let the 2-sort species $F : \mathbf{Set}^2 \to \mathbf{Set}$ be defined by

$$F[\boldsymbol{\Omega}] = F[(\Omega^{(1)}, \Omega^{(2)})] := 2^{\Omega^{(1)}\times\Omega^{(2)}}, \quad \boldsymbol{\Omega} = (\Omega^{(1)}, \Omega^{(2)}) \in \mathrm{Ob}(\mathbf{Set}^2).$$

Thus, $F[(\Omega^{(1)}, \Omega^{(2)})]$ can be considered as set of all bipartite graphs, where the set of "white" vertices is $\Omega^{(1)}$ and the set of "black" vertices is $\Omega^{(2)}$. For $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_2)$, $b_1 \in F[\boldsymbol{\Omega}_1]$ and $b_2 \in F[\boldsymbol{\Omega}_2]$, put

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((b_1, b_2)\big) := b_1 \amalg b_2.$$

This means that $\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}$ merely forms the disjoint union of the bipartite graphs b_1 and b_2. Then it is not difficult to see that $\boldsymbol{\eta}$ is a natural transformation satisfying (D1). Moreover, $F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]$ consists of the *connected* bipartite graphs with bipartition $\boldsymbol{\Omega} = (\Omega^{(1)}, \Omega^{(2)})$.

For a weight, we choose $\Lambda = \mathbb{Z}[t]$ and

$$w_{\boldsymbol{\Omega}}(b) := t^{|b|}, \quad b \in F[\boldsymbol{\Omega}].$$

Again, it is not difficult to see that $\boldsymbol{w}$ satisfies Axioms (W0)–(W2); that is, $\boldsymbol{w}$ is a Λ-weight on $(F, \boldsymbol{\eta})$.

Theorem 1 then says that

$$\mathrm{GF}_F(z_1, z_2) = \exp\big(\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1, z_2)\big), \tag{7.1}$$

where

$$\mathrm{GF}_F(z_1, z_2) = \sum_{n_1,n_2\ge 0} \sum_{b\in F[([n_1],[n_2])]} t^{|b|} \frac{z_1^{n_1} z_2^{n_2}}{n_1!\, n_2!}$$

and

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1, z_2) = \sum_{n_1,n_2\ge 0} \sum_{b\in F_{\boldsymbol{\eta}}[([n_1],[n_2])]} t^{|b|} \frac{z_1^{n_1} z_2^{n_2}}{n_1!\, n_2!}.$$

However, by straightforward counting, one sees that

$$\mathrm{GF}_F(z_1, z_2) = \sum_{n_1,n_2\ge 0} (1+t)^{n_1 n_2} \frac{z_1^{n_1} z_2^{n_2}}{n_1!\, n_2!}.$$

From (7.1), it then follows that the generating function for connected bipartite graphs is given by

$$\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z_1,z_2)=\log\left(\sum_{n_1,n_2\ge0}(1+t)^{n_1n_2}\frac{z_1^{n_1}\,z_2^{n_2}}{n_1!\,n_2!}\right),$$

while (2.10) implies that

$$\begin{aligned}\widetilde{\mathrm{GF}}_F(z_1,z_2)&:=\sum_{n_1,n_2\ge0}\ \sum_{b\in F[([n_1],[n_2])]}t^{|b|}y^{\#(\text{connected components of }b)}\frac{z_1^{n_1}\,z_2^{n_2}}{n_1!\,n_2!}\\&=\left(\sum_{n_1,n_2\ge0}(1+t)^{n_1n_2}\frac{z_1^{n_1}\,z_2^{n_2}}{n_1!\,n_2!}\right)^y.\end{aligned}\tag{7.2}$$

This example can be considered as a 2-dimensional analogue of the example in [10, Sec. 3] (with the first of the two composition operators considered there). The knowledgeable reader will recognise (7.2) as the exponential generating function for the Tutte polynomials of complete bipartite graphs (cf. e.g. [19, Eq. (3.10)]).

Example 2 (Bipartite graphs II). Let $F:\mathbf{Set}^2\to\mathbf{Set}$ be as in Example 1. Here, for $(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)\in\mathrm{Ob}(\mathfrak{D}_2)$, where $\boldsymbol{\Omega}_i=(\Omega_i^{(1)},\Omega_i^{(2)})$, $i=1,2$, for $b_1\in F[\boldsymbol{\Omega}_1]$ and $b_2\in F[\boldsymbol{\Omega}_2]$, we put

$$\eta'_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((b_1,b_2)\big):=b_1\amalg b_2\amalg\left(\Omega_1^{(1)}\times\Omega_2^{(2)}\right)\amalg\left(\Omega_1^{(2)}\times\Omega_2^{(1)}\right).$$

The graph $\eta'_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((b_1,b_2)\big)$ can be considered as a kind of bipartite completion of the disjoint union of b_1 and b_2. Again, it is not difficult to see that $\boldsymbol{\eta}'$ is a natural transformation satisfying (D1). Moreover, $F_{\boldsymbol{\eta}'}(\boldsymbol{\Omega})$ consists of the *complements* of connected bipartite graphs with bipartition $\boldsymbol{\Omega}=(\Omega^{(1)},\Omega^{(2)})$, where the complement b^c of a bipartite graph $b\in F[\boldsymbol{\Omega}]$ is defined as $b^c:=\big(\Omega^{(1)}\times\Omega^{(2)}\big)-b$.

If we now were to choose the weight of Example 1, then Axiom (W2) would be violated. Instead, with $\Lambda=\mathbb{Z}[t]$, we set

$$w'_{\boldsymbol{\Omega}}(b):=t^{|\Omega^{(1)}|\cdot|\Omega^{(2)}|-|b|},\quad b\in F[\boldsymbol{\Omega}]=F\big[(\Omega^{(1)},\Omega^{(2)})\big].$$

Then it is not difficult to see that $\boldsymbol{w}'$ does satisfy Axioms (W0)–(W2); that is, $\boldsymbol{w}'$ is a Λ-weight on $(F,\boldsymbol{\eta}')$.

Theorem 1 then says that

$$\mathrm{GF}_F(z_1,z_2)=\exp\big(\mathrm{GF}_{F_{\boldsymbol{\eta}'}}(z_1,z_2)\big),\tag{7.3}$$

where

$$\mathrm{GF}_F(z_1,z_2)=\sum_{n_1,n_2\ge0}\ \sum_{b\in F[([n_1],[n_2])]}t^{n_1n_2-|b|}\frac{z_1^{n_1}\,z_2^{n_2}}{n_1!\,n_2!}$$

and

$$\mathrm{GF}_{F_{\boldsymbol{\eta}'}}(z_1,z_2)=\sum_{n_1,n_2\ge0}\ \sum_{b\in F_{\boldsymbol{\eta}'}([n_1],[n_2])}t^{n_1n_2-|b|}\frac{z_1^{n_1}\,z_2^{n_2}}{n_1!\,n_2!}.$$

Again, by straightforward counting, one sees that

$$\mathrm{GF}_F(z_1,z_2)=\sum_{n_1,n_2\ge0}(1+t)^{n_1n_2}\frac{z_1^{n_1}\,z_2^{n_2}}{n_1!\,n_2!},$$

and we obtain the formulae

$$\mathrm{GF}_{F_{\boldsymbol{\eta}'}}(z_1,z_2)=\log\left(\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\right)$$

and

$$\begin{aligned}\widetilde{\mathrm{GF}}_F(z_1,z_2)&:=\sum_{n_1,n_2\ge 0}\ \sum_{b\in F[([n_1],[n_2])]} t^{n_1n_2-|b|}y^{\#(\text{connected components of }b^c)}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\\&=\left(\sum_{n_1,n_2\ge 0}(1+t)^{n_1n_2}\frac{z_1^{n_1}z_2^{n_2}}{n_1!\,n_2!}\right)^y.\end{aligned}\tag{7.4}$$

This example can be viewed as a 2-dimensional analogue of the example in [10, Sec. 3] (with the second of the two composition operators considered there).

The alert reader will have noticed that the $\boldsymbol{\eta}'$-maps could have been alternatively defined by

$$\eta'_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((b_1,b_2)\big):=\big(b_1^c\amalg b_2^c\big)^c,\tag{7.5}$$

where the complements have to be taken in the appropriate complete bipartite graphs. This construction will be generalised in Section 10.

Example 3 (Binary functions). Let the (1-sort) species $F:\mathbf{Set}\to\mathbf{Set}$ be defined by

$$F[\boldsymbol{\Omega}]:=\{0,1\}^{\boldsymbol{\Omega}},\quad \boldsymbol{\Omega}\in\mathrm{Ob}(\mathbf{Set}).$$

For $(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)\in\mathrm{Ob}(\mathfrak{D}_1)$, $f_1\in F[\boldsymbol{\Omega}_1]$, and $f_2\in F[\boldsymbol{\Omega}_2]$, put

$$\big(\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((f_1,f_2)\big)\big)(\omega):=\begin{cases}f_1(\omega), & \text{if }\omega\in\boldsymbol{\Omega}_1,\\ 1-f_2(\omega), & \text{if }\omega\in\boldsymbol{\Omega}_2.\end{cases}$$

Then it is easy to see that $\boldsymbol{\eta}$ is a natural transformation satisfying (D1). Moreover,

$$F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}]=\begin{cases}\{0_{\boldsymbol{\Omega}},1_{\boldsymbol{\Omega}}\}, & \text{if }|\boldsymbol{\Omega}|=1,\\ \{\}, & \text{otherwise,}\end{cases}$$

where $0_{\boldsymbol{\Omega}}$ and $1_{\boldsymbol{\Omega}}$ are the constant functions on $\boldsymbol{\Omega}$ taking the value 0 and 1, respectively. We note that, in contrast to Examples 1 and 2, the $\boldsymbol{\eta}$-maps of the present example are *pointwise non-associative and non-commutative* (cf. Section 10); to be precise, in general we have

$$\eta_{(\boldsymbol{\Omega}_1\amalg\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_3)}\Big(\big(\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((f_1,f_2)\big),f_3\big)\Big)\ne\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2\amalg\boldsymbol{\Omega}_3)}\Big(\big(f_1,\eta_{(\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_3)}\big((f_2,f_3)\big)\big)\Big)$$

and

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((f_1,f_2)\big)\ne\eta_{(\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_1)}\big((f_2,f_1)\big).$$

For the sake of completeness, we remark that, choosing the trivial weighting

$$w_{\boldsymbol{\Omega}}(f):=1,\quad f\in F[\boldsymbol{\Omega}],$$

Theorem 1 yields the trivial identity

$$\mathrm{GF}_F(z)=\sum_{n\ge 0}2^n\frac{z^n}{n!}=\exp\big(\mathrm{GF}_{F_{\boldsymbol{\eta}}}(z)\big)=\exp(2z).$$

The construction of this example can also be generalised to produce many more (multisort) species with pointwise non-associative and non-commutative composition operator, see Theorem 25 in Section 10.

8. Illustrations, II: Magic squares

The purpose of this section is to illustrate the increased flexibility of our present multivariate setting. We show that a number of generating function identities for combinatorial matrices found scattered throughout the literature can be uniformly explained, and generalised, in the context of our theory.

By a *combinatorial matrix* on $\mathbf{\Omega} = (\Omega^{(1)}, \Omega^{(2)}) \in \mathrm{Ob}(\mathbf{Set}^2)$ we shall mean any map

$$m : \Omega^{(1)} \times \Omega^{(2)} \to \mathbb{N}_0.$$

The pair of sets $\mathbf{\Omega}$ is called the *support* of m. Let m_1, m_2 be two combinatorial matrices with supports $\mathbf{\Omega}_1 = (\Omega_1^{(1)}, \Omega_1^{(2)})$ and $\mathbf{\Omega}_2 = (\Omega_2^{(1)}, \Omega_2^{(2)})$, respectively, and suppose that $\mathbf{\Omega}_1 \cap \mathbf{\Omega}_2 = \boldsymbol{\emptyset}$. Then we define their *direct sum* $m = m_1 \oplus m_2$ to be the combinatorial matrix with support $\mathbf{\Omega} := \mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2$ given by

$$m(\omega_1, \omega_2) := \begin{cases} m_1(\omega_1, \omega_2), & (\omega_1, \omega_2) \in \Omega_1^{(1)} \times \Omega_1^{(2)}, \\ m_2(\omega_1, \omega_2), & (\omega_1, \omega_2) \in \Omega_2^{(1)} \times \Omega_2^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$$

A combinatorial matrix m on $\mathbf{\Omega}$ is termed *s-magic*,[6] s a positive integer, if

$$\sum_{\omega_2 \in \Omega^{(2)}} m(\omega_1, \omega_2) = s, \quad \omega_1 \in \Omega^{(1)},$$

and

$$\sum_{\omega_1 \in \Omega^{(1)}} m(\omega_1, \omega_2) = s, \quad \omega_2 \in \Omega^{(2)}.$$

Computing the sum of entries, we find that an s-magic matrix is necessarily square, $|\Omega^{(1)}| = |\Omega^{(2)}|$. The enumeration of s-magic squares has a long history, going back to MacMahon [18, §404–419]. A good account of the enumerative theory of magic squares can be found in [24, Sec. 4.6], with many pointers to further literature. For more recent work, see for instance [4, 9].

For $s \in \mathbb{N}$ and $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^2)$, denote by $F_s(\mathbf{\Omega})$ the set of all s-magic matrices on $\mathbf{\Omega}$, and by $\bar{F}_s(\mathbf{\Omega})$ the set of those s-magic matrices on $\mathbf{\Omega}$ which do not contain s as an entry. We thus have mappings

$$F_s, \bar{F}_s : \mathrm{Ob}(\mathbf{Set}^2) \to \mathrm{Ob}(\mathbf{Set}),$$

which we turn into functors $F_s, \bar{F}_s : \mathbf{Set}^2 \to \mathbf{Set}$ by assigning to a morphism

$$\boldsymbol{f} = (f_1, f_2) : \mathbf{\Omega} \to \widetilde{\mathbf{\Omega}}$$

in $\mathbf{Set}^2$ the map (denoted $F_s[\boldsymbol{f}]$ respectively $\bar{F}_s[\boldsymbol{f}]$) sending a combinatorial matrix m in the respective domain to $m \circ (f_1^{-1} \times f_2^{-1})$. Moreover, given s and a finite set Ω, let $F_s^*(\Omega)$ be the set of *symmetric* s-magic matrices on $\mathbf{\Omega} = (\Omega, \Omega)$; that is, combinatorial matrices satisfying

$$m(\omega_1, \omega_2) = m(\omega_2, \omega_1), \quad (\omega_1, \omega_2) \in \Omega^2;$$

and denote by $\bar{F}_s^*(\Omega)$ the subset of $F_s^*(\Omega)$ consisting of those matrices with no entry equal to s. Just as above, the maps

$$F_s^*, \bar{F}_s^* : \mathrm{Ob}(\mathbf{Set}) \to \mathrm{Ob}(\mathbf{Set})$$

[6] Strictly speaking, the correct term here would be "s-semi-magic," since we do not require diagonals to sum up to s as well, see e.g. [4]. However, we prefer the term "s-magic" for the sake of brevity.

become functors $F_s^*, \bar{F}_s^* : \mathbf{Set} \to \mathbf{Set}$ by assigning to a morphism $f : \Omega \to \widetilde{\Omega}$ in **Set** the map sending a combinatorial matrix m in the respective domain to $m \circ (f^{-1} \times f^{-1})$.

Next, given $s \in \mathbb{N}$, a pair $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_2)$, and a pair $(\Omega_1, \Omega_2) \in \mathrm{Ob}(\mathfrak{D}_1)$, the direct sum construction provides us with injective maps

$$\begin{aligned}(\eta_s)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)} &: F_s(\boldsymbol{\Omega}_1) \times F_s(\boldsymbol{\Omega}_2) \to F_s(\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2),\\ (\bar{\eta}_s)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)} &: \bar{F}_s(\boldsymbol{\Omega}_1) \times \bar{F}_s(\boldsymbol{\Omega}_2) \to \bar{F}_s(\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2),\\ (\eta_s^*)_{(\Omega_1,\Omega_2)} &: F_s^*(\Omega_1) \times F_s^*(\Omega_2) \to F_s^*(\Omega_1 \amalg \Omega_2),\\ (\bar{\eta}_s^*)_{(\Omega_1,\Omega_2)} &: \bar{F}_s^*(\Omega_1) \times \bar{F}_s^*(\Omega_2) \to \bar{F}_s^*(\Omega_1 \amalg \Omega_2).\end{aligned}$$

A certain amount of checking is required in order to convince oneself that these definitions fit into the framework of Theorems 1 and 4. The next lemma states the corresponding result. We leave its proof, which essentially amounts to a routine verification, to the reader.

Lemma 19. (i) *For each $s \in \mathbb{N}$, the collection of maps*

$$\boldsymbol{\eta}_s = \big((\eta_s)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)\in \mathrm{Ob}(\mathfrak{D}_2)}$$

is a natural transformation from the functor $F_s \times F_s$ to the functor $F_s \circ \amalg$. Analogous statements hold for the functors $\bar{F}_s, F_s^*, \bar{F}_s^*$, and the families of maps

$$\begin{aligned}\bar{\boldsymbol{\eta}}_s &= \big((\bar{\eta}_s)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)\in \mathrm{Ob}(\mathfrak{D}_2)},\\ \boldsymbol{\eta}_s^* &= \big((\eta_s^*)_{(\Omega_1,\Omega_2)}\big)_{(\Omega_1,\Omega_2)\in \mathrm{Ob}(\mathfrak{D}_1)},\\ \bar{\boldsymbol{\eta}}_s^* &= \big((\bar{\eta}_s^*)_{(\Omega_1,\Omega_2)}\big)_{(\Omega_1,\Omega_2)\in \mathrm{Ob}(\mathfrak{D}_1)}.\end{aligned}$$

(ii) *For each s, the pair $(F_s, \boldsymbol{\eta}_s)$ satisfies Axiom* (D1), *an analogous statement holding for each of the other pairs $(\bar{F}_s, \bar{\boldsymbol{\eta}}_s)$, $(F_s^*, \boldsymbol{\eta}_s^*)$, and $(\bar{F}_s^*, \bar{\boldsymbol{\eta}}_s^*)$.*

It follows from Lemma 19 and Theorem 4, that Equations (2.9) and (2.10) hold for each of the pairs $(F_s, \boldsymbol{\eta}_s)$, $(\bar{F}_s, \bar{\boldsymbol{\eta}}_s)$, $(F_s^*, \boldsymbol{\eta}_s^*)$, and $(\bar{F}^*, \bar{\boldsymbol{\eta}}_s^*)$; in particular, we find that

$$\widetilde{\mathrm{GF}}_{F_s}(z_1, z_2, y) = \exp\big(y\, \mathrm{GF}_{(F_s)_{\boldsymbol{\eta}_s}}(z_1, z_2)\big), \tag{8.1}$$

$$\widetilde{\mathrm{GF}}_{\bar{F}_s}(z_1, z_2, y) = \exp\big(y\, \mathrm{GF}_{(\bar{F}_s)_{\bar{\boldsymbol{\eta}}_s}}(z_1, z_2)\big), \tag{8.2}$$

$$\widetilde{\mathrm{GF}}_{F_s^*}(z, y) = \exp\big(y\, \mathrm{GF}_{(F_s^*)_{\boldsymbol{\eta}_s^*}}(z)\big), \tag{8.3}$$

$$\widetilde{\mathrm{GF}}_{\bar{F}_s^*}(z, y) = \exp\big(y\, \mathrm{GF}_{(\bar{F}_s^*)_{\bar{\boldsymbol{\eta}}_s^*}}(z)\big). \tag{8.4}$$

Note that in these identities the variable y keeps track of the number of indecomposable matrices into which the matrices which are counted by the respective generating functions on the left-hand sides can be decomposed. Clearly, the generating functions occurring in (8.1) and (8.2) can be viewed as formal power series in z_1z_2 and y; that is, z_1z_2 could be replaced by a single variable. However, we prefer to keep z_1 and z_2 separate, since this is more in line with our general theory.

We note certain dependencies among the series $\widetilde{\mathrm{GF}}_{F_s}$, $\widetilde{\mathrm{GF}}_{\bar{F}_s}$, $\widetilde{\mathrm{GF}}_{F_s^*}$, $\widetilde{\mathrm{GF}}_{\bar{F}_s^*}$; for instance, we observe that an indecomposable s-magic matrix on $([n_1], [n_2])$ cannot contain an entry equal to s, unless $n_1 = n_2 = 1$. It follows that

$$\big|(F_s)_{\boldsymbol{\eta}_s}([n_1], [n_2])\big| = \begin{cases} 1 + \big|(\bar{F}_s)_{\bar{\boldsymbol{\eta}}_s}([1], [1])\big|, & n_1 = n_2 = 1,\\ \big|(\bar{F}_s)_{\bar{\boldsymbol{\eta}}_s}([n_1], [n_2])\big|, & \text{otherwise,}\end{cases}$$

and hence, by Equations (8.1) and (8.2),

$$\widetilde{\mathrm{GF}}_{\bar{F}_s}(z_1, z_2, y) = e^{-z_1 z_2 y}\, \widetilde{\mathrm{GF}}_{F_s}(z_1, z_2, y). \tag{8.5}$$

Similarly, we have

$$\widetilde{\mathrm{GF}}_{\bar{F}^*_s}(z, y) = e^{-y(z+z^2/2)} \widetilde{\mathrm{GF}}_{F^*_s}(z, y). \tag{8.6}$$

Indeed, for $n = 1, 2$, there exist indecomposable symmetric s-magic matrices on $([n], [n])$ containing an entry s:

$$(s) \text{ and } \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}.$$

Now let m be a symmetric s-magic matrix on $([n], [n])$ with $n \geq 3$, and suppose that m contains an entry equal to s in position (i, j). Then, if $i = j$, we have $m = (s) \oplus m'$, where m' has support $([n] - \{i\}, [n] - \{i\})$. If, on the other hand, $i \neq j$, then, by symmetry, m also contains s in position (j, i), and we find that m splits as

$$m = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \oplus m',$$

where m' has support $([n] - \{i, j\}, [n] - \{i, j\})$, and is non-empty since $n \geq 3$. Thus, in both cases, m is in fact decomposable. Hence,

$$\left|(F^*_s)_{\boldsymbol{\eta}^*_s}([n])\right| = \begin{cases} 1 + \left|(\bar{F}^*_s)_{\bar{\boldsymbol{\eta}}^*_s}([n])\right|, & n = 1, 2, \\ \left|(\bar{F}^*_s)_{\bar{\boldsymbol{\eta}}^*_s}([n])\right|, & \text{otherwise}, \end{cases}$$

and (8.6) follows from Equations (8.3) and (8.4).

The enumeration can be done exactly if $s = 2$. For, according to Birkhoff's Theorem (cf. [6] or [2, Corollary 8.40]), a 2-magic matrix m is the sum of two permutation matrices, say p_1 and p_2. If m is indecomposable, then the pair $\{p_1, p_2\}$ is uniquely determined. Premultiplying by p_1^{-1}, we obtain a situation where p_1 is the identity; indecomposability forces p_2 to be the permutation matrix corresponding to a cyclic permutation. So there are $n!(n-1)!$ choices for (p_1, p_2), and half this many choices for m (assuming, as we may, that $n > 1$). Note that this formula gives half the correct number for $n = 1$. So we have

$$\left|(F_2)_{\boldsymbol{\eta}_2}([n_1], [n_2])\right| = \begin{cases} 1, & n_1 = n_2 = n = 1, \\ \frac{n!(n-1)!}{2}, & n_1 = n_2 = n > 1, \\ 0, & n_1 \neq n_2, \end{cases}$$

that is,

$$\mathrm{GF}_{(F_2)_{\boldsymbol{\eta}_2}}(z_1, z_2) = \frac{1}{2} z_1 z_2 - \frac{1}{2} \log(1 - z_1 z_2),$$

and therefore

$$\widetilde{\mathrm{GF}}_{F_2}(z_1, z_2, y) = (1 - z_1 z_2)^{-y/2}\, e^{z_1 z_2 y/2} \tag{8.7}$$

by Equation (8.1). Also,

$$\widetilde{\mathrm{GF}}_{\bar{F}_2}(z_1, z_2, y) = (1 - z_1 z_2)^{-y/2}\, e^{-z_1 z_2 y/2}, \tag{8.8}$$

making use of Equation (8.5) and the last result. Special cases of Identities (8.7) and (8.8) appear in [3, Sections 8.1 and 8.3] (see also [23, Eqs. (23) and (24) in Example 6.11]). For $s > 2$, enumeration is more difficult; see Stanley's paper [22] and Comtet [8, pp. 124–125] for comments in this direction; also Goulden and Jackson [14, Sections 3.4 and 3.5] for some variations on this counting problem.[7]

[7] Note however, that the formula given in [8] for $s = 3$ is erroneous.

For symmetric matrices, it is again possible to count the indecomposables with $s = 2$. For $n > 2$, such a matrix can be represented as a graph in which every vertex has degree 2; loops are permitted, but contribute only one to the degree of a vertex. Indecomposability of the matrix is reflected in connectedness of the graph. So the graphs we must consider are paths (with a loop at each end) and cycles; and, for $n > 2$, their number is $\frac{1}{2}(n! + (n-1)!)$. Including the cases where $n \le 2$, we obtain

$$\mathrm{GF}_{(F_2^*)_{\eta_2^*}}(z) = \frac{z^2}{4} + \frac{z}{2(1-z)} - \frac{1}{2}\log(1-z),$$

and, hence

$$\widetilde{\mathrm{GF}}_{F_2^*}(z,y) = (1-z)^{-y/2} \exp\Big(\frac{yz^2}{4} + \frac{yz}{2(1-z)}\Big), \tag{8.9}$$

as well as

$$\widetilde{\mathrm{GF}}_{\bar{F}_2^*}(z,y) = (1-z)^{-y/2} \exp\Big(-\frac{yz^2}{4} - yz + \frac{yz}{2(1-z)}\Big). \tag{8.10}$$

Identity (8.9) generalises [15, Eq. (6.3)], whereas (8.10) generalises [15, Eq. (6.4)].

9. Illustrations, III: Magic cubes

Our theory applies as well to higher-dimensional analogues of (semi-)magic matrices, but the situation becomes more complicated. We look briefly at two instances.

Three-dimensional arrays with line sums s: These are maps $m : \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{N}_0$ satisfying

$$\sum_{\omega_3 \in \Omega_3} m(\omega_1, \omega_2, \omega_3) = s$$

for all $\omega_1 \in \Omega_1$ and $\Omega_2 \in \Omega_2$, as well as two similar equations. We assume that $s > 0$. This generalisation of the concept of magic squares to higher dimensions appears already in the literature, see e.g. [1, 4].

The sum of all the entries in the array is

$$s \cdot |\Omega_1| \cdot |\Omega_2| = s \cdot |\Omega_1| \cdot |\Omega_3| = s \cdot |\Omega_2| \cdot |\Omega_3|;$$

so $|\Omega_1| = |\Omega_2| = |\Omega_3|$.

For $s = 1$, these objects are equivalent to Latin squares counted up to isotopy: the sets of rows, columns, and symbols of the corresponding Latin square are Ω_1, Ω_2, and Ω_3 respectively, and the entry in position (ω_1, ω_2) of the Latin square is ω_3 if and only if $m(\omega_1, \omega_2, \omega_3) = 1$.

Unfortunately, the functor corresponding to this construction has no composition operator in the sense of our theory. For suppose that both $\boldsymbol{\Omega}' = (\Omega_1', \Omega_2', \Omega_3')$ and $\boldsymbol{\Omega} - \boldsymbol{\Omega}' = (\Omega_1 - \Omega_1', \Omega_2 - \Omega_2', \Omega_3 - \Omega_3')$ have line sums s, where Ω_i' is a proper subset of Ω_i for $i = 1, 2, 3$. Then we have

$$m(\omega_1, \omega_2, \omega_3) = 0 \quad \text{for all } \omega_1 \in \Omega_1',\ \omega_2 \in \Omega_2',\ \omega_3 \in \Omega_3 - \Omega_3', \tag{9.1}$$

and further equalities of the same type arising by permuting $1, 2, 3$. This, in its turn, entails that

$$\sum_{\omega_2 \in \Omega_2 - \Omega_2'} m(\omega_1, \omega_2, \omega_3) = s \quad \text{for all } \omega_1 \in \Omega_1',\ \text{and } \omega_3 \in \Omega_3 - \Omega_3',$$

whence

$$\sum_{\omega_1\in\Omega_1',\,\omega_2\in\Omega_2-\Omega_2',\,\omega_3\in\Omega_3-\Omega_3'} m(\omega_1,\omega_2,\omega_3) = s\cdot|\Omega_1'|\cdot(|\Omega_3|-|\Omega_3'|), \tag{9.2}$$

and two further equations arising by permuting $1,2,3$.

Let us write $N=|\Omega_1|=|\Omega_2|=|\Omega_3|$ and $N'=|\Omega_1'|=|\Omega_2'|=|\Omega_3'|$. Then, using (9.1) and (9.2), we infer

$$\begin{aligned}\sum_{\omega_1\in\Omega_1,\,\omega_2\in\Omega_2,\,\omega_3\in\Omega_3} m(\omega_1,\omega_2,\omega_3) &= s\cdot N^2\\ &= s\cdot(N')^2+s\cdot(N-N')^2+3s\cdot N'(N-N'),\end{aligned}$$

and hence $N'N-(N')^2=0$, or, equivalently, $N'=0$ or $N'=N$, both of which are absurd.

Higher-dimensional arrays with hyperplane sums s: Consider a variant of the preceding discussion, namely maps $m:\Omega_1\times\Omega_2\times\Omega_3\to\mathbb{N}_0$ satisfying

$$\sum_{\omega_2\in\Omega_2,\,\omega_3\in\Omega_3} m(\omega_1,\omega_2,\omega_3)=s$$

for all $\omega_1\in\Omega_1$, as well as two similar equations. We assume again that $s>0$. Strangely enough, we have not been able to locate this generalisation of magic squares to higher dimensions in the literature.

This time, the sum of all the entries is

$$s\cdot|\Omega_1|=s\cdot|\Omega_2|=s\cdot|\Omega_3|;$$

so again $|\Omega_1|=|\Omega_2|=|\Omega_3|$.

Here, the obvious transformation $\boldsymbol{\eta}$ (given by the direct sum construction suitably generalised) is indeed a composition operator for the corresponding functor on $\mathbf{Set}^3$ (that is, it satisfies Axiom (D1)). In fact, Lemma 19 extends to d-dimensional arrays with hyperplane sums s, for any $d>2$. We now consider this case further.

Counting higher-dimensional magic matrices is made more difficult by the fact that the analogue of Birkhoff's Theorem fails for them. For example, the 3-dimensional 2-magic matrix with ones in positions $(1,1,1)$, $(1,2,3)$, $(2,1,2)$, $(2,2,1)$, $(3,3,2)$ and $(3,3,3)$ is not the sum of two 1-magic matrices. However, it is possible to count the 2-magic matrices of any dimension. Below we count the indecomposable matrices; the exponential principle then gives us the total number.

An indecomposable 2-magic matrix with an entry 2 has size 1. So it is enough to consider zero-one matrices.

Proposition 20. *The number $u_n(d)$ of indecomposable d-dimensional zero-one matrices of size n with all hyperplane sums equal to 2 satisfies $u_1(d)=0$ and $u_n(d)=(n!)^{d-1}v_n(d)/2^n$ for $n>1$, where $v_1(d)=1$ and*

$$\sum_{k=1}^{n}\binom{n-1}{k-1}((2n-2k-1)!!)^{d-1}v_k(d)=((2n-1)!!)^{d-1},\quad n>1.$$

Here, $(2n-1)!!=1\cdot3\cdot5\cdots(2n-1)$ is the product of the first n odd positive integers for $n>0$, and, by convention, $(-1)!!=1$.

Proof. Recall that $(2n-1)!!$ is the number of fixed-point-free involutions on a set of size $2n$. We take t_1 to be the permutation

$$t_1=(1,2)(3,4)\cdots(2n-1,2n), \tag{9.3}$$

and let $v_n(d)$ be the number of choices of $d-1$ fixed-point-free involutions $t_2, \ldots, t_d$ on $\{1, \ldots, 2n\}$ such that the group $G = \langle t_1, t_2, \ldots, t_d \rangle$ generated by $t_1, t_2, \ldots, t_d$ is transitive. Then the number of choices of involutions $t_1, t_2, \ldots, t_d$, where t_1 is as in (9.3), such that the G-orbit containing 1 has size $2k$ is

$$\binom{n-1}{k-1}((2n-2k-1)!!)^{d-1}\, v_k(d),$$

since we can choose in order

(i) $k-1$ of the $n-1$ cycles of t_1 other than $(1,2)$ such that the elements not fixed by all of these $k-1$ transpositions together with $\{1,2\}$ form the desired orbit, O say;
(ii) $d-1$ fixed-point-free involutions on O which, together with the restriction of t_1 to O, generate a transitive group;
(iii) $d-1$ arbitrary fixed-point-free involutions on the complement of O.

Summing these values shows that the numbers $v_n(d)$ satisfy the desired recurrence.

Next, we establish the relation between the numbers $u_n(d)$ and $v_n(d)$. Let m be an indecomposable d-dimensional 2-magic matrix of size n, where $n > 1$. Then m is a zero-one matrix, and it contains $2n$ entries equal to 1, the rest being zero. We assume that $\Omega_i = \{1, \ldots, n\}$ for $i = 1, \ldots, d$. Number the positions of the 1's in m from 1 to $2n$ in such a way that the positions with first coordinate j are numbers $2j-1$ and $2j$ for $j = 1, \ldots, n$. (There are 2^n ways to do this.) Then, for $i = 1, \ldots, d$, let t_i be the fixed-point-free involution whose cycles are the pairs of numbers in $\{1, \ldots, 2n\}$ indexing positions of 1's with the same i-th coordinate. Note that t_1 is the permutation defined earlier. It is straightforward to show that indecomposability of m implies transitivity of the group generated by these involutions. So each matrix gives rise to 2^n such d-tuples of involutions. Thus, the number of pairs consisting of a matrix and a corresponding sequence of permutations is $2^n\, u_n(d)$.

For instance, the example of a matrix failing the analogue of Birkhoff's Theorem given above, with the entries numbered in the order given, produces the three permutations $(1,2)(3,4)(5,6)$, $(1,3)(2,4)(5,6)$ and $(1,4)(2,6)(3,5)$.

Conversely, let $t_1, \ldots, t_d$ be fixed-point-free involutions on the set $\{1, \ldots, 2n\}$ which generate a transitive group, where t_1 is the standard involution defined above. Number the cycles of each t_i from 1 to n such that the cycle $(2j-1, 2j)$ of t_1 has number j. (There are $(n!)^{d-1}$ such numberings.) Now construct a d-dimensional matrix m as follows: for $k = 1, \ldots, 2n$, if k lies in cycle number p_i of t_i, then $m(p_1, p_2, \ldots, p_d) = 1$; all other entries are zero. Then m is 2-magic. Consequently, each sequence of permutations gives rise to $(n!)^{d-1}$ matrices; and the number of pairs consisting of a matrix and a corresponding sequence of permutations equals $(n!)^{d-1}\, v_n(d)$.

Comparing these two expressions, we obtain $u_n(d) = (n!)^{d-1}\, v_n(d)/2^n$, as required. □

In the case $d = 2$, we have already seen that $u_n(2) = n!(n-1)!/2$, so that

$$v_n(2) = 2^{n-1}\,(n-1)! = (2n-2)!!,$$

where $(2n-2)!!$ is the product of the even integers up to $2n-2$ (with $0!! = 1$ by convention). Hence, we have proved the somewhat curious looking identity

$$\sum_{k=1}^{n} \binom{n-1}{k-1} (2n-2k-1)!!\,(2k-2)!! = (2n-1)!!$$

d	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
2	1	1	6	72	1440	43200
	0	1	6	90	2040	67950
	1	3	21	282	6210	202410
3	1	8	900	359424	370828800	820150272000
	0	8	900	366336	378028800	833156928000
	1	12	1152	431424	427723200	920031955200

Table 1

for $n > 1$. It is, however, just an instance of the Chu–Vandermonde identity, which becomes obvious when the left-hand side is written in standard hypergeometric notation (cf. e.g. [20, (1.7.7), Appendix (III.4)]) as

$$2^{n-1}\,(1/2)_{n-1}\cdot {}_2F_1\left[\begin{matrix}-n+1,1\\ -n+\frac{1}{2}\end{matrix};1\right].$$

For $d > 2$, we have not been able to solve the recurrence explicitly. However, it is easy to calculate terms in the sequences, and we can describe their asymptotics.

Table 1 gives counts of all indecomposable matrices, all zero-one matrices, and all non-negative integer matrices, with dimension d and hyperplane sums 2. The sequences for $d = 2$ are numbers A010796, A001499, and A000681 in the On-Line Encyclopedia of Integer Sequences [21]. For $d = 3$, they are A112578, A112579 and A112580.

Asymptotically, for $d > 2$, the following result holds.

Proposition 21. *Let $d \geq 3$ be fixed, let w_n be the number of d-dimensional zero-one matrices of order n with all hyperplane sums 2, and let u_n be the number of those which are indecomposable. Then*

$$w_n \sim u_n \sim ((2n-1)!!)^{d-1}(n!)^{d-1}/2^n.$$

Proof. We first derive the asymptotics for u_n. We have $u_n = (n!)^{d-1}v_n/2^n$, with v_n as above, so we have to show that

$$v_n \sim ((2n-1)!!)^{d-1}.$$

We will use the estimates

$$\sqrt{2(n+1)} \leq \frac{2^n\,n!}{(2n-1)!!} \leq 2\sqrt{n}$$

for $n \geq 1$. With $c_n = 2^n n!/(2n-1)!!$, we have $c_{n+1}/c_n = (2n+2)/(2n+1)$, and both inequalities are easily proved by induction. From these estimates, we obtain the inequality

$$\frac{(2n-1)!!}{(2k-1)!!\,(2n-2k-1)!!} \geq \binom{n}{k}\left(\frac{(k+1)(n-k+1)}{n}\right)^{1/2}. \tag{9.4}$$

To simplify our formulae, we denote the left-hand side of this inequality by $\left(\!\binom{n}{k}\!\right)$.

Now, v_n satisfies the recurrence

$$v_n = ((2n-1)!!)^{d-1} - \sum_{k=1}^{n-1}\binom{n-1}{k-1}((2n-2k-1)!!)^{d-1}v_k.$$

Clearly $v_n \le ((2n-1)!!)^{d-1}$. We show that $v_n \ge ((2n-1)!!)^{d-1}(1-O(1/n))$, an estimate which, in view of the above recurrence, would follow from

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(\!\!\binom{n}{k}\!\!\right)^{-(d-1)} = O\left(\frac{1}{n}\right).$$

Using (9.4), the quantity L on the left satisfies

$$\begin{aligned} L &\le \frac{n}{(2n-1)^{d-1}} + \sum_{k=2}^{n-2} \binom{n-1}{k-1} \binom{n}{k}^{-(d-1)} \left(\frac{n}{(k+1)(n-k+1)}\right)^{(d-1)/2} \\ &\le \frac{n}{(2n-1)^{d-1}} + \sum_{k=2}^{n-2} \frac{k}{n} \binom{n}{k}^{-(d-2)} \left(\frac{n}{(k+1)(n-k+1)}\right)^{(d-1)/2}. \end{aligned}$$

Since $k/n < 1$, $n/(k+1)(n-k+1) < 1/2$, and $\binom{n}{k} \ge \binom{n}{2}$, and there are fewer than $n-1$ terms in the sum, the second term is at most

$$n^{-(d-2)}(n-1)^{-(d-3)} \cdot 2^{d-2} \cdot 2^{-(d-1)/2} \le \frac{1}{n},$$

as required.

For the estimate of w_n, the exponential principle (in the form of Equation (4.4), with $r = d$, $n_1 = \ldots = n_r = n$, and $\mu_1 = \ldots = \mu_r = k$) gives

$$w_n = u_n + \sum_{k=1}^{n-1} \frac{k}{n} \binom{n}{k}^{d} u_k w_{n-k}.$$

Putting $w_n = u_n + x_n$, we have to show that $x_n = o(u_n)$. We assume inductively that $x_n \le ((2n-1)!!)^{d-1}(n!)^{d-1}/2^n$; the induction starts since we have $x_1 = x_2 = 0$.

Now, using the inductive hypothesis with the recurrence relation, we have

$$\begin{aligned} \frac{x_n 2^n}{((2n-1)!!)^{d-1}(n!)^{d-1}} &\le 2 \sum_{k=1}^{n-1} \frac{k}{n} \binom{n}{k}^{d} \left(\!\!\binom{n}{k}\!\!\right)^{-(d-1)} \binom{n}{k}^{-(d-1)} \\ &\le 2 \sum_{k=1}^{n-1} \frac{k}{n} \binom{n}{k}^{-(d-2)} \left(\frac{n}{(k+1)(n-k+1)}\right)^{(d-1)/2} \\ &\le (2^{1/2} n)^{-(d-3)}, \end{aligned}$$

which establishes the result if $d > 3$. For $d = 3$, this inequality gives the inductive step (that is, that the left-hand side is at most 1); the fact that it is $o(1)$ for large n is proved by an argument like that in the first part of the proof. □

Remark. The proof of Proposition 21 depends on the fact that v_n grows sufficiently rapidly (roughly like $(n!)^{(d-1)}$). Indeed, the result is false for $d = 2$.

10. Characterisation of pointwise associative and commutative composition operators

In this final section, we address the question how far the composition operator $\boldsymbol{\eta}$ of our theory can be from the standard operation for the species of sets of combinatorial structures given by forming the disjoint union, of which Example 1 in Section 7 is a prototypical example. While we do not have an answer in general, we are able to completely resolve the case when $\boldsymbol{\eta}$ is pointwise associative and commutative. Here, given an r-sort species $F : \mathbf{Set}^r \to \mathbf{Set}$ with composition operator $\boldsymbol{\eta}$, we say that

$\boldsymbol{\eta}$ is *pointwise associative* if, for all pairwise disjoint $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3 \in \mathrm{Ob}(\mathbf{Set}^r)$, and all elements $x_1 \in F[\boldsymbol{\Omega}_1]$, $x_2 \in F[\boldsymbol{\Omega}_2]$, $x_3 \in F[\boldsymbol{\Omega}_3]$, we have

$$\eta_{(\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3)}\Big(\big(\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((x_1,x_2)\big), x_3\big)\Big) = \eta_{(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \amalg \boldsymbol{\Omega}_3)}\Big(\big(x_1, \eta_{(\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_3)}\big((x_2,x_3)\big)\big)\Big),$$

and we say that $\boldsymbol{\eta}$ is *pointwise commutative* if, for all $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, and all elements $x_1 \in F[\boldsymbol{\Omega}_1]$, $x_2 \in F[\boldsymbol{\Omega}_2]$, we have

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((x_1,x_2)\big) = \eta_{(\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_1)}\big((x_2,x_1)\big).$$

The $\boldsymbol{\eta}$-maps in Examples 1 and 2 are instances of pointwise associative and commutative composition operators, while the composition operator in Example 3 is neither pointwise associative nor pointwise commutative. The notion of pointwise commutativity and associativity as defined above should not be confused with the commutativity and the 3-associativity proved in Lemmas 6 and 7, respectively, which are (in general) strictly weaker assertions.

Before we are able to make the above statement precise, we need two preparatory results. Recall that a *species isomorphism* between two r-sort species F_1 and F_2 is a collection of maps $\boldsymbol{\varphi} = (\varphi_{\boldsymbol{\Omega}})_{\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)}$, where, for each $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$,

$$\varphi_{\boldsymbol{\Omega}} : F_1[\boldsymbol{\Omega}] \to F_2[\boldsymbol{\Omega}]$$

is a bijection, with the property that, for every morphism $\boldsymbol{f} : \boldsymbol{\Omega} \to \widetilde{\boldsymbol{\Omega}}$ in the category $\mathbf{Set}^r$, the diagram

$$\begin{array}{ccc} F_1[\boldsymbol{\Omega}] & \xrightarrow{F_1[\boldsymbol{f}]} & F_1[\widetilde{\boldsymbol{\Omega}}] \\ {\scriptstyle\varphi_{\boldsymbol{\Omega}}}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_{\widetilde{\boldsymbol{\Omega}}}} \\ F_2[\boldsymbol{\Omega}] & \xrightarrow[F_2[\boldsymbol{f}]]{} & F_2[\widetilde{\boldsymbol{\Omega}}] \end{array} \tag{10.1}$$

commutes. If F_1 carries a weak Λ_1-weight $\boldsymbol{w}_1$ and F_2 carries a weak Λ_2-weight $\boldsymbol{w}_2$, where, by "weak," we mean that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ satisfy Axioms (W0) and (W1), but not necessarily (W2) (cf. Section 2), then an isomorphism $\boldsymbol{\varphi} : F_1 \to F_2$ is called *weight-preserving*, if there exists a ring homomorphism $\lambda : \Lambda_1 \to \Lambda_2$ such that the diagram

$$\begin{array}{ccc} F_1[\boldsymbol{\Omega}] & \xrightarrow{(w_1)_{\boldsymbol{\Omega}}} & \Lambda_1 \\ {\scriptstyle\varphi_{\boldsymbol{\Omega}}}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ F_2[\boldsymbol{\Omega}] & \xrightarrow[(w_2)_{\boldsymbol{\Omega}}]{} & \Lambda_2 \end{array} \tag{10.2}$$

commutes.

The lemma below tells us that, if F_1 and F_2 are two isomorphic r-sort species, where F_1 is decomposable with composition operator $\boldsymbol{\eta}_1$, then $\boldsymbol{\eta}_1$ can be lifted to a composition operator for F_2, demonstrating that F_2 is decomposable as well.

Lemma 22. *Let F_1 and F_2 be two isomorphic r-sort species, where F_1 is decomposable with composition operator $\boldsymbol{\eta}_1$. Furthermore, let $\boldsymbol{\varphi}$ be an isomorphism between F_1 and F_2. Then F_2 is decomposable, and the family of maps $\boldsymbol{\eta}_2 = ((\eta_2)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)})_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)}$ defined by*

$$(\eta_2)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}(x_1,x_2) := \varphi_{\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2}\Big((\eta_1)_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}\big((\varphi^{-1}_{\boldsymbol{\Omega}_1}(x_1), \varphi^{-1}_{\boldsymbol{\Omega}_2}(x_2))\big)\Big),$$
$$x_1 \in F_2[\boldsymbol{\Omega}_1],\ x_2 \in F_2[\boldsymbol{\Omega}_2],$$

is a composition operator for F_2.

Proof. We have to show that $\boldsymbol{\eta}_2$ is a natural transformation from $F_2 \times F_2$ to $F_2 \circ \amalg$, and that the pair $(F_2, \boldsymbol{\eta}_2)$ satisfies Axiom (D1). The former follows immediately from the corresponding property for $(F_1, \boldsymbol{\eta}_1)$ and the naturality condition (10.1). In order to verify (D1), we start with the left-hand side of (2.2) for the pair $(F_2, \boldsymbol{\eta}_2)$, suppressing the indices of η_1, η_2, φ for better readability:

$$\begin{aligned}
&\eta_2\big(F_2[\boldsymbol{\Omega}_1] \times F_2[\boldsymbol{\Omega}_2]\big) \cap \eta_2\big(F_2[\widetilde{\boldsymbol{\Omega}}_1] \times F_2[\widetilde{\boldsymbol{\Omega}}_2]\big)\\
&\quad = \varphi\Big(\eta_1\big(\varphi^{-1}(F_2[\boldsymbol{\Omega}_1]) \times \varphi^{-1}(F_2[\boldsymbol{\Omega}_2])\big)\Big) \cap \varphi\Big(\eta_1\big(\varphi^{-1}(F_2[\widetilde{\boldsymbol{\Omega}}_1]) \times \varphi^{-1}(F_2[\widetilde{\boldsymbol{\Omega}}_2])\big)\Big)\\
&\quad = \varphi\Big(\eta_1\big(F_1[\boldsymbol{\Omega}_1] \times F_1[\boldsymbol{\Omega}_2]\big)\Big) \cap \varphi\Big(\eta_1\big(F_1[\widetilde{\boldsymbol{\Omega}}_1] \times F_1[\widetilde{\boldsymbol{\Omega}}_2]\big)\Big)\\
&\quad = \varphi\Big(\eta_1\big(F_1[\boldsymbol{\Omega}_1] \times F_1[\boldsymbol{\Omega}_2]\big) \cap \eta_1\big(F_1[\widetilde{\boldsymbol{\Omega}}_1] \times F_1[\widetilde{\boldsymbol{\Omega}}_2]\big)\Big).
\end{aligned}$$

Here we have used the injectivity of φ to obtain the last line. Now we substitute the right-hand side of (2.2) for the pair $(F_1, \boldsymbol{\eta}_1)$, to obtain

$$\begin{aligned}
&\eta_2\big(F_2[\boldsymbol{\Omega}_1] \times F_2[\boldsymbol{\Omega}_2]\big) \cap \eta_2\big(F_2[\widetilde{\boldsymbol{\Omega}}_1] \times F_2[\widetilde{\boldsymbol{\Omega}}_2]\big)\\
&\qquad\qquad = \varphi\Big(\eta_1\big(\eta_1\big(F_1[\boldsymbol{\Omega}_{11}] \times F_1[\boldsymbol{\Omega}_{12}]\big) \times \eta_1\big(F_1[\boldsymbol{\Omega}_{21}] \times F_1[\boldsymbol{\Omega}_{22}]\big)\big)\Big),
\end{aligned}$$

where $\boldsymbol{\Omega}_{ij} := \boldsymbol{\Omega}_i \cap \widetilde{\boldsymbol{\Omega}}_j$ for $i, j \in \{1, 2\}$. Using $F_1[\boldsymbol{\Omega}_{ij}] = \varphi^{-1}(F_2[\boldsymbol{\Omega}_{ij}])$ at each possible place, and inserting $\mathrm{id} = \varphi^{-1} \circ \varphi$ at two places, we arrive at

$$\begin{aligned}
&\eta_2\big(F_2[\boldsymbol{\Omega}_1] \times F_2[\boldsymbol{\Omega}_2]\big) \cap \eta_2\big(F_2[\widetilde{\boldsymbol{\Omega}}_1] \times F_2[\widetilde{\boldsymbol{\Omega}}_2]\big)\\
&\qquad = \varphi\Big(\eta_1\big(\varphi^{-1}\big(\varphi\big(\eta_1\big(\varphi^{-1}(F_2[\boldsymbol{\Omega}_{11}]) \times \varphi^{-1}(F_2[\boldsymbol{\Omega}_{12}])\big)\big)\big)\\
&\qquad\qquad \times \varphi^{-1}\big(\varphi\big(\eta_1\big(\varphi^{-1}(F_2[\boldsymbol{\Omega}_{21}]) \times \varphi^{-1}(F_2[\boldsymbol{\Omega}_{22}])\big)\big)\big)\big)\Big)\\
&\qquad = \eta_2\big(\eta_2\big(F_2[\boldsymbol{\Omega}_{11}] \times F_2[\boldsymbol{\Omega}_{12}]\big) \times \eta_2\big(F_2[\boldsymbol{\Omega}_{21}] \times F_2[\boldsymbol{\Omega}_{22}]\big)\big),
\end{aligned}$$

which is exactly (2.2) for the pair $(F_2, \boldsymbol{\eta}_2)$. □

The second preparatory result, Proposition 23 below, states that, given a decomposable r-sort species F with pointwise associative and commutative composition operator $\boldsymbol{\eta}$, F is isomorphic to $E(F_{\boldsymbol{\eta}})$, where $E(F_{\boldsymbol{\eta}})$ denotes the species of sets of $F_{\boldsymbol{\eta}}$-structures (cf. [7, p. 8] for the definition of the species of sets, E, and [7, p. 41] for the definition of composition of species). In rigorous terms, for $\boldsymbol{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$, the set $E(F_{\boldsymbol{\eta}})[\boldsymbol{\Omega}]$ can be defined by

$$\begin{aligned}
E(F_{\boldsymbol{\eta}})[\boldsymbol{\Omega}] := \Big\{&\big\{(x_1, \boldsymbol{\Omega}_1), \dots, (x_k, \boldsymbol{\Omega}_k)\big\} : x_i \in F_{\boldsymbol{\eta}}[\boldsymbol{\Omega}_i],\ i = 1, \dots, k,\\
&\text{for some } k \in \mathbb{N}_0 \text{ and } \boldsymbol{\Omega}_1 \amalg \cdots \amalg \boldsymbol{\Omega}_k = \boldsymbol{\Omega},\ \text{all } \boldsymbol{\Omega}_i\text{'s being non-empty}\Big\},
\end{aligned}$$

with the obvious notion of induced morphisms. If F carries a weak Λ-weight $\boldsymbol{w}$, then $\boldsymbol{w}$ can be lifted to a weak Λ-weight of $E(F_{\boldsymbol{\eta}})$ by setting

$$w_{\boldsymbol{\Omega}}\Big(\big\{(x_1, \boldsymbol{\Omega}_1), \dots, (x_k, \boldsymbol{\Omega}_k)\big\}\Big) := w_{\boldsymbol{\Omega}_1}(x_1) \cdots w_{\boldsymbol{\Omega}_k}(x_k).$$

Proposition 23. *Let F be a decomposable weighted r-sort species with composition operator $\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is pointwise associative and commutative. Then there exists a weight-preserving isomorphism between F and $E(F_{\boldsymbol{\eta}})$.*

Proof. The starting point is the combination of Lemmas 12 and 13. It says that, for each non-empty $\mathbf{\Omega} \in \mathrm{Ob}(\mathbf{Set}^r)$ and every choice of base point $(\omega, \rho) \in \mathbf{\Omega}$, we have

$$F[\mathbf{\Omega}] = \coprod_{\substack{\mathbf{\Omega}_1 \in \mathrm{Ob}(\mathbf{Set}^r) \\ (\omega,\rho)\in\mathbf{\Omega}_1\subseteq\mathbf{\Omega}}} \eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}-\mathbf{\Omega}_1)}\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big). \tag{10.3}$$

We are now going to construct bijective maps $\psi_{\mathbf{\Omega}} : F[\mathbf{\Omega}] \to E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]$ by induction on $\|\mathbf{\Omega}\|$, where $\|\,.\,\|$ has been defined in (2.5). For $\mathbf{\Omega} = \boldsymbol{\emptyset}$, we have $|F[\mathbf{\Omega}]| = |E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]| = 1$ by Lemma 5 respectively the definition of $E(F_{\boldsymbol{\eta}})$, whence the construction of $\psi_{\boldsymbol{\emptyset}}$ is trivial. Henceforth, we shall suppose that $\|\mathbf{\Omega}\| \geq 1$, and we assume that we have constructed maps $\psi_{\widetilde{\mathbf{\Omega}}}$ for all $\widetilde{\mathbf{\Omega}} \in \mathrm{Ob}(\mathbf{Set}^r)$ with $\|\widetilde{\mathbf{\Omega}}\| < N$.

Now let $\|\mathbf{\Omega}\| = N$. Choose a base point $(\omega, \rho) \in \mathbf{\Omega}$, and let $x \in F[\mathbf{\Omega}]$. By (10.3), there is a unique $\mathbf{\Omega}_1$ such that $x \in \eta_{(\mathbf{\Omega}_1,\mathbf{\Omega}-\mathbf{\Omega}_1)}\big(F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1] \times F[\mathbf{\Omega} - \mathbf{\Omega}_1]\big)$ and $(\omega, \rho) \in \mathbf{\Omega}_1$. Let (y_1, x_1) be the uniquely determined pair with $y_1 \in F_{\boldsymbol{\eta}}[\mathbf{\Omega}_1]$ and $x_1 \in F[\mathbf{\Omega} - \mathbf{\Omega}_1]$, such that

$$(y_1, x_1) := \eta^{-1}_{(\mathbf{\Omega}_1,\mathbf{\Omega}-\mathbf{\Omega}_1)}(x). \tag{10.4}$$

By the inductive hypothesis, there exist uniquely determined elements $y_2, \dots, y_k$, for some $k \in \mathbb{N}$ and $y_i \in F_{\boldsymbol{\eta}}[\mathbf{\Omega}_i]$, $i = 2, \dots, k$, with $\mathbf{\Omega}_2 \amalg \cdots \amalg \mathbf{\Omega}_k = \mathbf{\Omega} - \mathbf{\Omega}_1$, such that

$$\psi_{\mathbf{\Omega}-\mathbf{\Omega}_1}(x_1) = \big\{(y_2, \mathbf{\Omega}_2), \dots, (y_k, \mathbf{\Omega}_k)\big\}. \tag{10.5}$$

Define

$$\psi_{\mathbf{\Omega}}(x) := \big\{(y_1, \mathbf{\Omega}_1), (y_2, \mathbf{\Omega}_2), \dots, (y_k, \mathbf{\Omega}_k)\big\}.$$

We claim that this yields a well-defined bijection $\psi_{\mathbf{\Omega}} : F[\mathbf{\Omega}] \to E(F_{\boldsymbol{\eta}})[\mathbf{\Omega}]$. What needs to be checked here first of all is that different choices of base points would always lead to the same result. So, let us suppose, that, by choosing a different base point, we would have obtained

$$\bar{\psi}_{\mathbf{\Omega}}(x) := \big\{(\bar{y}_1, \bar{\mathbf{\Omega}}_1), (\bar{y}_2, \bar{\mathbf{\Omega}}_2), \dots, (\bar{y}_l, \bar{\mathbf{\Omega}}_l)\big\},$$

for some l, instead. Since we must have

$$\mathbf{\Omega} = \mathbf{\Omega}_1 \amalg \mathbf{\Omega}_2 \amalg \cdots \mathbf{\Omega}_k = \bar{\mathbf{\Omega}}_1 \amalg \bar{\mathbf{\Omega}}_2 \amalg \cdots \bar{\mathbf{\Omega}}_l,$$

there is a j such that $(w, \rho) \in \bar{\mathbf{\Omega}}_j$. By our inductive construction via (10.4) and (10.5), we have

$$x \in \eta\Big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_1] \times \eta\big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_2] \times \cdots \times \big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{l-1}] \times F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_l]\big) \cdots\big)\Big).$$

By Lemma 15 (m-permutability for $(F_{\boldsymbol{\eta}}, \boldsymbol{\eta})$), this is equivalent to saying that

$$x \in \eta\Big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_j] \times \eta\big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{\sigma(2)}] \times \cdots \times \big(F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{\sigma(l-1)}] \times F_{\boldsymbol{\eta}}[\bar{\mathbf{\Omega}}_{\sigma(l)}]\big) \cdots\big)\Big), \tag{10.6}$$

where $\sigma(2), \dots, \sigma(l-1), \sigma(l)$ is some permutation of $\{1, \dots, j-1, j+1, \dots, l\}$. If $\bar{\mathbf{\Omega}}_j \neq \mathbf{\Omega}_1$, then (10.4) and (10.6) would contradict the disjointness in (10.3). Hence, we must have $\bar{\mathbf{\Omega}}_j = \mathbf{\Omega}_1$, and, by our assumption that $\boldsymbol{\eta}$ be pointwise associative and commutative, we even must have $\bar{y}_j = y_1$. The inductive hypothesis applied to $\mathbf{\Omega} - \mathbf{\Omega}_1$ then guarantees that, moreover,

$$\{(y_2, \mathbf{\Omega}_1), \dots, (y_k, \mathbf{\Omega}_k)\} = \{(\bar{y}_1, \mathbf{\Omega}_1), \dots, (\bar{y}_{j-1}, \bar{\mathbf{\Omega}}_{j-1}), (\bar{y}_{j+1}, \bar{\mathbf{\Omega}}_{j+1}), \dots, (\bar{y}_l, \mathbf{\Omega}_l)\}.$$

This proves that $\psi_{\mathbf{\Omega}}$ is indeed well-defined.

The facts that each map $\psi_{\mathbf{\Omega}}$ is a bijection, and that the family $\boldsymbol{\psi} = (\psi_{\mathbf{\Omega}})_{\mathbf{\Omega}\in\mathrm{Ob}(\mathbf{Set}^r)}$ is an isomorphism between F and $E(F_{\boldsymbol{\eta}})$, are not hard to verify. The fact that $\boldsymbol{\psi}$ is weight-preserving is obvious from the definition of $\boldsymbol{\psi}$ and Axiom (W2) for $(F, \boldsymbol{\eta}, \boldsymbol{w})$. This completes the proof of the proposition. ☐

If we combine Lemma 22 and Proposition 23, then we can say exactly how a decomposable r-sort species F with pointwise associative and commutative composition operator $\boldsymbol{\eta}$ arises from the composition of the species of sets with the species of $F_{\boldsymbol{\eta}}$-structures ("components"). We should point out here that, clearly, a natural composition operator for $E(F_{\boldsymbol{\eta}})$ is given by

$$(y_1, y_2) \mapsto y_1 \amalg y_2, \quad y_1 \in E(F_{\boldsymbol{\eta}})[\boldsymbol{\Omega}_1],\ y_2 \in E(F_{\boldsymbol{\eta}})[\boldsymbol{\Omega}_2],\ \boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}. \tag{10.7}$$

Theorem 24. *Let F be a decomposable r-sort species with composition operator $\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is pointwise associative and commutative. Then, for all $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, the composition operator $\boldsymbol{\eta}$ can be expressed as follows:*

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}(x_1, x_2) = \psi^{-1}_{\boldsymbol{\Omega}_1 \amalg \boldsymbol{\Omega}_2}\Big(\psi_{\boldsymbol{\Omega}_1}(x_1) \amalg \psi_{\boldsymbol{\Omega}_2}(x_2)\Big), \quad x_1 \in F[\boldsymbol{\Omega}_1],\ x_2 \in F[\boldsymbol{\Omega}_2], \tag{10.8}$$

where $\boldsymbol{\psi}$ is the isomorphism between F and $E(F_{\boldsymbol{\eta}})$ constructed in the proof of Proposition 23.

Proof. One combines Lemma 22 with Proposition 23, where the role of the isomorphism $\boldsymbol{\varphi}$ in Lemma 22 is played by the family of maps $\boldsymbol{\psi}^{-1}$ constructed in the proof of Proposition 23. □

In summary, all decomposable r-sort species with pointwise associative and commutative composition operator can be constructed from $E(G)$, for some species G, equipped with the natural composition operator as given in (10.7) (in the case where $G = F_{\boldsymbol{\eta}}$), by applying a lift in the sense of Lemma 22 via a species isomorphism. In order to see how Example 2 in Section 7 fits into the setting of Theorem 24, recall that the isomorphism between F and $E(F_{\boldsymbol{\eta}})$ in that example can be defined by mapping the bipartite graph $b \in F[\boldsymbol{\Omega}]$ to its complement b^c, identifying the connected components (in the classical sense of graph theory) of b^c, and forming the set of complements of these connected components (restricted to the set of vertices which a component involves). If this isomorphism is inserted in (10.8), the result is (7.5).

We conclude our paper by pointing out that the construction in Theorem 24 can be "twisted" to produce pointwise non-associative and non-commutative composition operators as well.

Theorem 25. *Let G be a weighted r-sort species, and let $\boldsymbol{g} : G \to G$ be a weight-preserving isomorphism. We extend $\boldsymbol{g}$ to $E(G)$ by setting*

$$g_{\boldsymbol{\Omega}\amalg\cdots\amalg\boldsymbol{\Omega}_k}\Big(\big\{(y_1, \boldsymbol{\Omega}_1), \dots, (y_k, \boldsymbol{\Omega}_k)\big\}\Big) = \big\{(g_{\boldsymbol{\Omega}_1}(y_1), \boldsymbol{\Omega}_1), \dots, (g_{\boldsymbol{\Omega}_k}(y_k), \boldsymbol{\Omega}_k)\big\}.$$

Then the family $\boldsymbol{\eta} = (\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)})_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)\in \mathrm{Ob}(\mathfrak{D}_r)}$ of maps defined by

$$\eta_{(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2)}(x_1, x_2) = x_1 \amalg g_{\boldsymbol{\Omega}_2}(x_2), \quad x_1 \in E(G)[\boldsymbol{\Omega}_1],\ x_2 \in E(G)[\boldsymbol{\Omega}_2],$$

where $(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2) \in \mathrm{Ob}(\mathfrak{D}_r)$, is a composition operator for the weighted species $E(G)$.

It is obvious from the definition that the composition operator $\boldsymbol{\eta}$ of Theorem 25 will, in general, be neither pointwise associative nor pointwise commutative. Example 3 in Section 7 provides a typical example of the above construction, with G given by

$$G[\boldsymbol{\Omega}] = \begin{cases} \{0_{\boldsymbol{\Omega}}, 1_{\boldsymbol{\Omega}}\}, & \text{if } |\boldsymbol{\Omega}| = 1, \\ \{\}, & \text{otherwise,} \end{cases}$$

where $0_{\boldsymbol{\Omega}}$ and $1_{\boldsymbol{\Omega}}$ are the constant functions on $\boldsymbol{\Omega}$ taking the value 0 and 1, respectively, and where the isomorphism $\boldsymbol{g}$ is given by $g_{\boldsymbol{\Omega}}(0_{\boldsymbol{\Omega}}) = 1_{\boldsymbol{\Omega}}$ and $g_{\boldsymbol{\Omega}}(1_{\boldsymbol{\Omega}}) = 0_{\boldsymbol{\Omega}}$ for $|\boldsymbol{\Omega}| = 1$.

However, we expect that there are many composition operators $\boldsymbol{\eta}$ not obtainable in this way.

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