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Decomposable Functors and the Exponential Principle

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We develop a combinatorial theory for the operation of forming the logarithm of generating functions associated with covariant functors on the category of finite sets and bijective maps. This leads to a rather general and flexible exponential principle for labeled combinatorial structures. We also introduce the concept of a topos-like category, which turns out to be a convenient framework for the discussion of a wide range of applications a few of which are studied in detail. © 1997 Academic Press

0. INTRODUCTION

The main purpose of the present paper is the construction of a combinatorial theory for a large class of enumeration problems leading to an identity of the form

$$1+\sum_{n>0} a_n Z^n/n! = \exp\left(\sum_{n>0} b_n Z^n/n!\right) \tag{0.1}$$

or, more generally

$$1+\sum_{n>0}\sum_{k>0} a_{nk} Z^n Y^k/n! = \exp\left(Y\sum_{n>0} b_n Z^n/n!\right). \tag{0.2}$$

It has often been noted that the identity (0.1) arises in connection with the problem of counting labeled combinatorial structures, which are composed of a finite number of indecomposable substructures, and several combinatorial models for (0.1) or its refinement (0.2) have been proposed on the basis of this observation, cf. for example [BG, Section 3; F, Chap. IV; or W1, Chap. 3]. It appears, however, that no satisfactory definition has been offered so far for the most fundamental notion of such a theory, the concept of decomposition of a labeled combinatorial structure. In [BG], a composition of combinatorial structures is introduced axiomatically. The strength of Bender and Goldman's prefab theory lies largely in the fact that it can deal with completely labeled and completely

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unlabeled structures in a uniform way, an advantage, which at the same time causes their list of axioms to be rather long and involved. Indeed, their approach seems much too complicated and abstract if one is mainly interested in the identities (0.1) and (0.2) and their interpretation in terms of labeled combinatorial structures. On the other hand, both [F, W1] introduce explicit set-theoretic constructions, thus only implicitly defining the particular kind of decomposition underlying their respective construction and automatically ensuring decomposability, and then relate the enumeration of the constructed objects to the enumeration of the set of objects they started from.

Denote by $\widetilde{\mathrm{Ens}}$ the category of finite sets and injective maps and by Ens the subcategory consisting of finite sets and bijective maps[1]. It is natural to think of labeled combinatorial structures as covariant functors on Ens. Given such a functor F: Ens → Ens, the basic idea of our approach is to study natural transformations η from the functor

$$F \times F: \mathrm{Ens}^2 \xrightarrow{F^2} \mathrm{Ens}^2 \xrightarrow{\times} \mathrm{Ens} \hookrightarrow \widetilde{\mathrm{Ens}}$$

to the functor

$$F \circ \dot{\cup}: \mathrm{Ens}^2 \xrightarrow{\dot{\cup}} \mathrm{Ens} \xrightarrow{F} \mathrm{Ens} \hookrightarrow \widetilde{\mathrm{Ens}},$$

where $\times$ and $\dot{\cup}$ denote the cartesian product and disjoint union of sets. Among such transformations η, a subclass is singled out by means of a certain rather natural reconstruction axiom, (D1), and any η in this subclass is called a weak decomposition of F. The functor F is termed weakly decomposable, if $F \neq \emptyset$, i.e., if $F(\Omega) \neq \emptyset$ for some finite set Ω and if a weak decomposition η of F exists. The class of weak decompositions of F is further reduced by imposing a second condition (D2), thus leading to the concept of a decomposable functor F and its decomposition(s). Our main result associates a general exponential formula of type (0.1) or (0.2) with each such weakly decomposable or decomposable functor F, respectively. This result and its conceptual framework are explained in more detail in the first section. We also comment there on Wilf's exponential formula associated with exponential families which arises as a special case of our result; and we demonstrate by means of an example that a decomposition of a functor F, if it exists, is in general not uniquely determined by F. The next section contains the proof of our main result. In Section 3, we concentrate on three specific applications of somewhat algebraic flavor: (i) the connection between group actions and finite-index subgroups; (ii) the equation $X^\alpha = X^\beta$ in symmetric semigroups; and (iii) cyclic sets. Finally, in

[1] Such functors are often called species; cf., for example, [J].

Section 4, we introduce the concept of topos-like categories. These are categories endowed with a canonical forgetful functor into the category of sets, which generalize certain important aspects of the category of G-sets for a given group G. Our theorem allows us to relate the problem of counting objects in such a category, which lie over a given finite set and are the coproduct of a given finite number of indecomposable objects, to the enumeration of the indecomposable objects over a finite set (Proposition 4). In fact, these categories whose axioms are also discussed in the theory of toposes (or, rather, topoi), turn out to be a convenient framework for the discussion of a wide range of applications of our theorem. In particular, all instances of a relation between two generating functions of the type described in (0.1) and (0.2) which are discussed in the present paper, turn out to be interpretable in the context of topos-like categories, which, however, invites many further applications.

1. THE EXPONENTIAL PRINCIPLE

The crucial concept for our approach to the exponential principle is that of a decomposition of a (combinatorial) functor F. This concept and the exponential formulas it leads to are explained in the first subsection. In Subsection 2 we introduce Wilf's concept of an exponential family and show how his exponential formula associated with such a family arises as a special case of our result. Moreover, in Subsection 3 we demonstrate by means of an example that the decomposition of a functor F, if it exists, is in general not uniquely determined by F.

1. *Decomposition of Functors*

Denote by $\widetilde{\text{Ens}}$ the category of finite sets and injective mappings and by Ens the subcategory consisting of finite sets and bijective maps. For a given (covariant) functor $F\colon \text{Ens} \to \text{Ens}$ we want to study natural transformations η from the functor

$$F \times F\colon \text{Ens}^2 \xrightarrow{F^2} \text{Ens}^2 \xrightarrow{\times} \text{Ens} \xrightarrow{\iota} \widetilde{\text{Ens}}$$

to the functor

$$F \circ \dot{\cup}\colon \text{Ens}^2 \xrightarrow{\dot{\cup}} \text{Ens} \xrightarrow{F} \text{Ens} \xrightarrow{\iota} \widetilde{\text{Ens}}.$$

Here $\times$ and $\dot{\cup}$ denote the natural product and coproduct in the category of sets, i.e., the Cartesian product and disjoint union, respectively; and $\iota\colon \text{Ens} \to \widetilde{\text{Ens}}$ is the inclusion functor. Given F and such a transformation

η we define a map F_η: $|\mathrm{Ens}| \to |\mathrm{Ens}|$ on the class $|\mathrm{Ens}|$ of all finite sets as follows:

$$F_\eta(\Omega) := \begin{cases} F(\Omega) - \bigcup\limits_{\substack{I \cup J = \Omega \\ I \neq \varnothing \neq J}} \eta(F(I) \times F(J)), & \Omega \neq \varnothing \\ \varnothing, & \Omega = \varnothing \end{cases} \qquad (\Omega \in |\mathrm{Ens}|)$$

$(\eta(F(I) \times F(J)))$, of course, denotes the image of the injective map $\eta_{(I,J)}$: $F(I) \times F(J) \hookrightarrow F(\Omega)$ for $\Omega = I \cup J$). Furthermore, using this map F_η we define a sequence of mappings $F_\eta^{(k)}$: $|\mathrm{Ens}| \to |\mathrm{Ens}|$ $(k = 0, 1, ...)$ with the property that $F_\eta^{(k)}(\Omega) \subseteq F(\Omega)$ by induction on k:

$$F_\eta^{(0)}(\Omega) := \begin{cases} F(\varnothing), & \Omega = \varnothing \\ \varnothing, & \Omega \neq \varnothing; \end{cases}$$

$$F_\eta^{(k)}(\Omega) := \bigcup_{\Omega_1 \subseteq \Omega} \eta(F_\eta(\Omega_1) \times F_\eta^{(k-1)}(\Omega - \Omega_1)), \qquad k \geqslant 1.$$

An immediate induction on $\#\Omega$ shows that

$$F_\eta^{(k)}(\Omega) = \varnothing, \qquad k > \#\Omega. \tag{1.1}$$

Denoting by $\mathbb{Z}_+$ the set of nonnegative integers, we associate with such a pair (F, η) three arithmetic functions φ_F^η: $\mathbb{Z}_+ \to \mathbb{Z}_+$, ψ_F^η: $\mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$, and ψ_F: $\mathbb{Z}_+ \to \mathbb{Z}_+$ as follows:

$$\varphi_F^\eta(n) := \# F_\eta(\{1, ..., n\})$$
$$\psi_F^\eta(n, k) := \# F_\eta^{(k)}(\{1, ..., n\})$$
$$\psi_F(n) := \# F(\{1, ..., n\}).$$

By (1.1) we have that

$$\psi_F^\eta(n, k) = 0, \qquad k > n.$$

The functions φ_F^η, ψ_F^η, and ψ_F in turn give rise to (formal) generating functions:

$$\Phi(Z) = \Phi_F^\eta(Z) := \sum_{n \geqslant 0} \varphi_F^\eta(n)\, Z^n/n!$$

$$\Psi(Z, Y) = \Psi_F^\eta(Z, Y) := \sum_{n \geqslant 0} \sum_{k \geqslant 0} \psi_F^\eta(n, k)\, Z^n Y^k/n!$$

$$\Psi(Z) = \Psi_F(Z) := \sum_{n \geqslant 0} \psi_F(n)\, Z^n/n!.$$

Given F, we call a natural transformation $\eta: F \times F \to F \circ \cup$ a *weak decomposition* of F if the following condition (D1) holds.

(D1) For each finite set Ω and any two partitions $\Omega = \Omega_1 \cup \Omega_2 = \Omega'_1 \cup \Omega'_2$ of Ω into disjoint parts we have that

$$\eta(F(\Omega_1) \times F(\Omega_2)) \cap \eta(F(\Omega'_1) \times F(\Omega'_2)) = \eta(\eta(F(\Omega_{11}) \times F(\Omega_{12})) \times \eta(F(\Omega_{21}) \times F(\Omega_{22}))), \quad (1.2)$$

where $\Omega_{ij} := \Omega_i \cap \Omega'_j$ for $i, j \in \{1, 2\}$.

A weak decomposition η of F is termed a *decomposition* of F if it satisfies in addition the following condition.

(D2) For each finite set Ω the sets $F_\eta^{(0)}(\Omega), F_\eta^{(1)}(\Omega), F_\eta^{(2)}(\Omega), \ldots$ are pairwise disjoint.

A functor F will be called *weakly decomposable* if $F \neq \emptyset$, i.e., $F(\Omega) \neq \emptyset$ for some finite set Ω, and if F admits some such weak decomposition η. Similarly, F will be called *decomposable* if $F \neq \emptyset$ and if F admits a decomposition η. We shall see that for a weakly decomposable functor F and a weak decomposition η of F in particular the following is true:

$$\# F(\emptyset) = 1. \quad (1.3)$$

$$F_\eta^{(1)} = F_\eta. \quad (1.4)$$

For each finite set Ω *we have* $F(\Omega) = \bigcup_{k \geq 0} F_\eta^{(k)}(\Omega)$. (1.5)

Hence, for such a pair (F, η)

$$\psi_F^\eta(0, 0) = \# F_\eta^{(0)}(\emptyset) = \# F(\emptyset) = 1$$

and

$$\psi_F^\eta(n, 1) = \varphi_F^\eta(n), \qquad n \geq 0.$$

If in addition to (D1) η also satisfies condition (D2), i.e., if F is decomposable and η a decomposition of F then for each finite set Ω the set $F(\Omega)$ is the disjoint union of the family $\{F_\eta^{(k)}(\Omega)\}_{k=0}^{\infty}$.

The main objective of the present paper is to establish the following *exponential principle.*

THEOREM. *Let* F *be a covariant functor on the category* Ens *and* $\eta: F \times F \to F \circ \cup$ *a natural transformation.*

(a) *If* F *is weakly decomposable and* η *a weak decomposition of* F, *then the generating functions* $\Phi(Z)$ *and* $\Psi(Z)$ *are connected by the transformation*

$$\Psi(Z) = \exp(\Phi(Z)). \quad (1.6)$$

(b) *If F is decomposable and η a decomposition of F, then*

$$\begin{aligned} &\text{(i)} \quad \Psi(Z, Y) = \exp(Y\Phi(Z)) \\ &\text{(ii)} \quad \Psi(Z, 1) = \Psi(Z). \end{aligned} \tag{1.7}$$

It is often convenient to view Eq. (1.7)(i) in a slightly different (but equivalent) way. For a set $T \subseteq \mathbb{Z}_+$ of nonnegative integers put

$$\psi_F^\eta(n, T) := \sum_{k \in T} \psi_F^\eta(n, k)$$

and

$$\Psi^T(Z) = \Psi^T_{(F,\eta)}(Z) := \sum_{n \geqslant 0} \psi_F^\eta(n, T)\, Z^n/n!.$$

Moreover, for a series $f(Z) = \sum_{n \geqslant 0} a_n Z^n$ and a set $S \subseteq \mathbb{Z}_+$ we denote by $f(Z)_S = \sum_{n \in S} a_n Z^n$ the truncation of $f(Z)$ corresponding to S. Given a functor F: Ens $\to$ Ens and a natural transformation $\eta: F \times F \to F \circ \cup$ Eq. (1.7)(i) is then equivalent to the following assertion:

For every set $T \subseteq \mathbb{Z}_+$ of nonnegative integers we have $\Psi^T(Z) = e_T(\Phi(Z))$, where $e_T(Z) := \exp(Z)_T$. (1.8)

Indeed, define for fixed $k \geqslant 0$

$$\Psi_k(Z) := \sum_{n \geqslant 0} \psi_F^\eta(n, k)\, Z^n/n!.$$

Assuming Eq. (1.7)(i) we have

$$\Psi(Z, Y) = \sum_{k \geqslant 0} \Psi_k(Z)\, Y^k = \sum_{k \geqslant 0} \frac{1}{k!}\, Y^k(\Phi(Z))^k.$$

Comparing coefficients we find that

$$\Psi_k(Z) = \frac{1}{k!}\,(\Phi(Z))^k, \qquad k \geqslant 0, \tag{1.9}$$

from which (1.8) follows by summing over the set T. Conversely, using (1.8) with $T = \{k\}$ gives (1.9) from which Eq. (1.7)(i) results upon multiplying by Y^k and summing over all $k \geqslant 0$. We shall freely use this equivalence whenever it is convenient.

Remarks. 1. Condition (D1) can be rephrased in terms of pullback diagrams. Recall that pullbacks always exist in the category $\widetilde{\text{Ens}}$ and are constructed in an obvious fashion. Denoting by P the right-hand side of (1.2) it is then easy to see that Eq. (1.2) is equivalent to the statement that

(i) $P \subseteq \eta(F(\Omega_1) \times F(\Omega_2)) \cap \eta(F(\Omega'_1) \times F(\Omega'_2))$ and

(ii) the diagram

$$\begin{array}{ccc} P & \xrightarrow{\eta^{-1}_{(\Omega_1,\,\Omega_2)}} & F(\Omega_1) \times F(\Omega_2) \\ \downarrow{\scriptstyle \eta^{-1}_{(\Omega'_1,\,\Omega'_2)}} & & \downarrow{\scriptstyle \eta_{(\Omega_1,\,\Omega_2)}} \\ F(\Omega'_1) \times F(\Omega'_2) & \xrightarrow{\eta_{(\Omega'_1,\,\Omega'_2)}} & F(\Omega) \end{array}$$

is a pullback diagram in the category $\widetilde{\text{Ens}}$.

2. Given a functor F: Ens $\rightarrow$ Ens with associated generating function $\Psi_F(Z) = \sum_{n \geqslant 0} \#F(\underline{n})\, Z^n/n!$ our theorem supplies a (not necessarily canonical) combinatorial interpretation of the function $\log \Psi_F$, i.e., a functor F_η: Ens $\rightarrow$ Ens such that

$$\sum_{n \geqslant 0} \#F_\eta(\underline{n})\, Z^n/n! = \log(\Psi_F(Z)),$$

provided that F is weakly decomposable. Obviously, the functor F can be reconstructed from F_η and the defining weak decomposition η of F. More general, given an arbitrary functor H: Ens $\rightarrow$ Ens (rather than F_η), consider a functor $G = G_H$: Ens $\rightarrow$ Ens with

$$G(\Omega) = \bigoplus_{\pi \in Eq(\Omega)} \prod_{X \in \Omega/\pi} H(X), \qquad \Omega \in |\text{Ens}|.$$

Then an immediate calculation shows that

$$\sum_{n \geqslant 0} \#G(\underline{n})\, Z^n/n! = \exp(\Psi_H(Z) - \#H(\underline{0}));$$

i.e., one always has a (canonical) combinatorial interpretation of the image of $\sum_{n \geqslant 1} \#H(\underline{n})\, Z^n/n!$ under the exponential map.

2. *Exponential Families*

In his recent book [W1] on generating functions, Wilf introduces the concept of an exponential family and associates an exponential formula with each such family. Unfortunately, Wilf's proof of his exponential principle is not entirely correct; cf. [M4]. However, the result itself is valid and is in fact a special case of our theorem. We briefly review the definition of an exponential family and state Wilf's exponential formula before deducing it from our result.

Given a set $\mathscr{P}$ the elements of which are called "pictures" Wilf defines a *card* $\mathscr{C}=\mathscr{C}(S, p)$ as a pair consisting of a finite set S of positive integers (the label set) and a picture $p\in\mathscr{P}$. The *weight* of a card $\mathscr{C}(S, p)$ is $\#S$ and a card of weight n is *standard* if its label set is $\{1, ..., n\}$. By hand H he means a set of cards whose label sets form a partition of $\{1, ..., n\}$ for some n; i.e., these label sets are nonempty, pairwise disjoint and their union is $\{1, ..., n\}$, where n is the sum of the weights of the cards in the hand. This n is called the weight of H. A *relabeling* of a card $\mathscr{C}=\mathscr{C}(S, p)$ with a set S' is defined if $\#S=\#S'$, in which case it is the card $\mathscr{C}'=\mathscr{C}(S', p)$. If $S'=\{1, ..., \#S\}$ then $\mathscr{C}'$ is the *standard relabeling* of $\mathscr{C}$. A *deck* $\mathscr{D}$ is a finite set of standard cards whose weights are all the same. This common weight of the cards in $\mathscr{D}$ is called the weight of $\mathscr{D}$. Finally, an *exponential family* $\mathscr{F}$ is a sequence of decks $\mathscr{D}_1, \mathscr{D}_2, ...,$ where for each $n=1, 2, \ldots$ the deck $\mathscr{D}_n$ is of weight n. Given an arbitrary exponential family $\mathscr{F}$ put $d_n := \#\mathscr{D}_n$ and define $h(n, k)$ to be the number of hands of weight n consisting of precisely k cards, each of these cards being a relabeling of some card in some deck of $\mathscr{F}$. Moreover, introduce the generating functions $\mathscr{D}(x)=\sum_{n\geqslant 1} d_n x^n/n!$ and $\mathscr{H}(x, y)=\sum_{n, k\geqslant 0} h(n, k)\, x^n y^k/n!$. Then Wilf's exponential principle [W1, Theorem 3.4.1] asserts that

$$\mathscr{H}(x, y)=e^{y\mathscr{D}(x)}. \tag{1.10}$$

In order to reveal (1.10) as a special case of our theorem we have to slightly generalize Wilf's concepts so as to allow arbitrary finite sets as label sets. Hence, a card in our sense is of the form $\mathscr{C}=\mathscr{C}(\Omega, p)$ with an arbitrary finite set Ω (the label set). The weight $w(\mathscr{C})$ of $\mathscr{C}=\mathscr{C}(\Omega, p)$ is $\#\Omega$. A hand $H(\Omega)$ on a (finite) set Ω is a set of cards whose label sets form a partition of Ω. Note that there exists precisely one hand on the empty set, the empty hand $H(\varnothing)=\varnothing$. The weight of a hand $H(\Omega)$ on Ω is $\sum_{\mathscr{C}\in H(\Omega)} w(\mathscr{C})=\#\Omega$. Finally, a relabeling of a card $\mathscr{C}(\Omega, p)$ is a card $\mathscr{C}(\Omega', p)$ with $\#\Omega=\#\Omega'$. Call a hand *admissible* if each of its cards is a relabeling of some card in $\bigcup_{n\geqslant 1}\mathscr{D}_n$. For a finite set Ω denote by $F(\Omega)=F_{\mathscr{F}}(\Omega)$ the set of all admissible hands on Ω. Since by definition of an exponential family each deck $\mathscr{D}_n$ is finite $F_{\mathscr{F}}(\Omega)$ is again a finite set. Hence, with the obvious definition of induced maps, $F=F_{\mathscr{F}}$ is a functor on the category Ens and the disjoint union of hands defines a natural transformation $\eta=\eta_{\mathscr{F}}: F_{\mathscr{F}}\times F_{\mathscr{F}}\to F_{\mathscr{F}}\circ\dot\cup$. The fact that $F_{\mathscr{F}}(\varnothing)=\{H(\varnothing)\}\neq\varnothing$ shows that $F_{\mathscr{F}}$ is not the empty functor. Let

$$\begin{aligned} H(\Omega)=\{\mathscr{C}_1(\Delta_1, p_1), ..., \mathscr{C}_k(\Delta_k, p_k)\} &\in \eta_{\mathscr{F}}(F_{\mathscr{F}}(\Omega_1)\times F_{\mathscr{F}}(\Omega_2)) \\ &\cap \eta_{\mathscr{F}}(F_{\mathscr{F}}(\Omega_1')\times F_{\mathscr{F}}(\Omega_2')) \end{aligned}$$

be an element of the left-hand side of (1.2). Then we can partition the index set $\{1, ..., k\}$ in two ways, $\{1, ..., k\} = I_1 \cup I_2 = I'_1 \cup I'_2$, such that

$$\Omega_i = \bigcup_{\mu \in I_i} \Delta_\mu \quad (i=1,2), \qquad \Omega'_j = \bigcup_{\mu \in I'_j} \Delta_\mu \quad (j=1,2).$$

For $i, j \in \{1, 2\}$ put $I_{ij} := I_i \cap I'_j$. By definition we have

$$\bigcup_{\mu \in I_{ij}} \Delta_\mu \subseteq \Omega_{ij} \qquad (i, j \in \{1, 2\})$$

and since $\bigcup_{\mu=1}^{k} \Delta_\mu = \Omega$ we must in fact have equality. This shows that $H(\Omega)$ is contained in the right-hand side of (1.2), and since the other inclusion is clear, we have verified condition (D1) for the pair $(F_{\mathscr{F}}, \eta_{\mathscr{F}})$. By definition $F_\eta(\Omega)$ is the set of all admissible hands on Ω consisting solely of one card, and for $\Omega \neq \emptyset$ there is an obvious bijection between $F_\eta(\Omega)$ and $\mathscr{D}_{\#\Omega}$. Moreover, by an immediate induction on k, $F_\eta^{(k)}(\Omega)$ is the set of all admissible hands on Ω consisting of precisely k cards; in particular, condition (D2) holds. Hence, $F_{\mathscr{F}}$ is decomposable and $\eta_{\mathscr{F}}$ is a decomposition of $F_{\mathscr{F}}$. Wilf's result (1.10) now follows from our theorem since $\varphi_F^\eta(n) = \#F_\eta(\{1, ..., n\}) = d_n \ (n \geqslant 1)$ and $\psi_F^\eta(n, k) = \#F_\eta^{(k)}(\{1, ..., n\}) = h(n, k) \ (n, k \geqslant 0)$.

3. *An Example*

Given a decomposable functor F: Ens $\to$ Ens, its decomposition η will, in general, not be uniquely determined by F. As an example consider the functor F: Ens $\to$ Ens given by

$$F(\Omega) := 2^{\binom{\Omega}{2}}, \qquad \Omega \in |\mathrm{Ens}|,$$

which maps a finite set Ω onto the set of all subsets of $\binom{\Omega}{2} = \{\{\omega, \omega'\} : \omega, \omega' \in \Omega, \ \omega \neq \omega'\}$. For $\Omega_1, \Omega_2 \in |\mathrm{Ens}|$ put

$$\eta_{(\Omega_1, \Omega_2)}(E_1, E_2) := E_1 \cup E_2$$

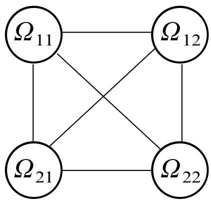
and

$$\eta'_{(\Omega_1, \Omega_2)}(E_1, E_2) := E_1 \cup E_2 \cup \left(\binom{\Omega_1 \cup \Omega_2}{2} - \binom{\Omega_1}{2} - \binom{\Omega_2}{2}\right),$$

where $E_i \subseteq \binom{\Omega_i}{2}$. $F(\Omega)$ can, of course, be interpreted as the set of all undirected combinatorial graphs with vertex set Ω. In this language $\eta_{(\Omega_1, \Omega_2)}(E_1, E_2)$ is the disjoint sum and $\eta'_{(\Omega_1, \Omega_2)}(E_1, E_2)$ the bipartite completion of the disjoint sum of the graphs corresponding to E_1 and E_2. Clearly, η is a decomposition of F with $F_\eta^{(k)}(\Omega)$ consisting of those graphs on Ω which have exactly k connected components. Our theorem then tells us that, for every set $T \subseteq \mathbb{Z}_+$, the exponential generating function $\Psi^T(Z)$ of the number $\psi_F^\eta(n, T)$ of all those graphs on n labeled vertices with number of connected components in T is given by

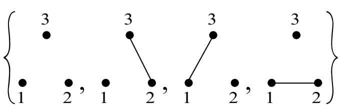
$$\Psi^T(Z) = e_T\left(\log\left(1 + \sum_{n \geqslant 1} 2^{\binom{n}{2}} Z^n/n!\right)\right).$$

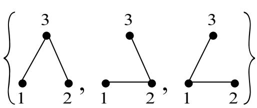
A moment's thought will also convince you that the transformation η' satisfies condition (D1). Essentially, one only has to observe that in the notation of (1.2) a graph corresponding to a set $E \in \eta'(F(\Omega_1) \times F(\Omega_2)) \cap \eta'(F(\Omega_1') \times F(\Omega_2'))$ is in fact of the form

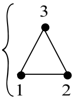


(a straight line denotes the bipartite completion of the two disjoint graphs it connects). By definition, a graph E on the set $\Omega \neq \varnothing$ is indecomposable with respect to η' (i.e., belongs to $F_{\eta'}(\Omega)$) if and only if for every partition $\Omega = \Omega_1 \mathbin{\dot\cup} \Omega_2$ of Ω into two nonempty disjoint parts the graph E does not contain all $\#\Omega_1 \cdot \#\Omega_2$ possible diagonal edges between Ω_1 and Ω_2. But this is equivalent to saying that the complement $\binom{\Omega}{2} - E$ is a connected graph on Ω. By induction on k, $F_{\eta'}^{(k)}(\Omega)$ is then identified as the set of all those graphs on Ω whose complement has exactly k connected components, whence (D2) for the transformation η'.

How far then is the decomposition of a decomposable functor F determined? If η and η' are two decompositions of some functor F: Ens $\to$ Ens then by our remark above for every set $\Omega \in |\mathrm{Ens}|$ the set $F(\Omega)$ is the disjoint union of the family $F_\eta^{(k)}(\Omega)$ as well as of the family $F_{\eta'}^{(k)}(\Omega)$ and by our theorem we must have that $\#F_\eta^{(k)}(\Omega) = \#F_{\eta'}^{(k)}(\Omega)$ for every $k \geqslant 0$. However, these two partitions of $F(\Omega)$ may still look very different from each other. Returning to our example, if we take for instance $\Omega = \{1, 2, 3\}$, we find that

$F_{\eta'}^{(1)}(\Omega) =$ 

$F_{\eta'}^{(2)}(\Omega) =$ 

$F^{(3)}_{\eta'}(\Omega) = \Big\{$  $\Big\}$

$$F^{(k)}_{\eta'}(\Omega) = \varnothing \qquad \text{if} \quad k = 0 \qquad \text{or} \qquad k \geqslant 4.$$

2. PROOF OF THE THEOREM

We are going to analyze the situation in a sequence of 12 steps. Our general assumption will be that F is weakly decomposable and that η is a weak decomposition of F (although condition (D1) is not always needed in each of these steps). Axiom (D2) will only come into play in steps (x) and (xii).

(i) *Functoriality of* F_η. Let Ω and Ω' be finite sets and $f\colon \Omega \to \Omega'$ a bijective map. Then

$$F(f)(F_\eta(\Omega)) = F_\eta(\Omega'); \tag{2.1}$$

i.e., putting $F_\eta(f) := F(f)|_{F_\eta(\Omega)}$ we get a functor $F_\eta\colon \text{Ens} \to \text{Ens}$.

Proof. This follows straightforward from the naturality of the transformation η, the functoriality of F and the fact that the map f is bijective. ▮

(ii) *Functoriality of* $F^{(k)}_\eta$, $k \geqslant 0$. For each bijection $f\colon \Omega \to \Omega'$ between finite sets Ω and Ω' and every integer $k \geqslant 0$ we have

$$F(f)(F^{(k)}_\eta(\Omega)) = F^{(k)}_\eta(\Omega'). \tag{2.2}$$

Proof. This follows by induction on k, using the functoriality of F_η already proved. ▮

(iii) $\#F(\varnothing) = 1$.

Proof. By the injectivity of $\eta_{(\varnothing,\varnothing)}\colon F(\varnothing) \times F(\varnothing) \to F(\varnothing)$, $F(\varnothing)$ is either empty or a 1-set. Suppose that $F(\varnothing) = \varnothing$. Choose a set Ω_1 with $F(\Omega_1) \neq \varnothing$ and a set Ω_2 such that $\Omega_1 \cap \Omega_2 = \varnothing$ and $\#\Omega_1 = \#\Omega_2$, and consider (D1) for the partition $\Omega := \Omega_1 \cup \Omega_2$, $\Omega'_i = \Omega_i$. By the functoriality of F we also have $F(\Omega_2) \neq \varnothing$ and, consequently, the left-hand side of (1.2) is nonempty, whereas the right-hand side of (1.2) would be empty in case $F(\varnothing) = \varnothing$—a contradiction. ▮

(iv) *Commutativity of* η. Given any two disjoint finite sets Ω_1 and Ω_2 we have

$$\eta(F(\Omega_1)\times F(\Omega_2))=\eta(F(\Omega_2)\times F(\Omega_1)). \tag{2.3}$$

Proof. Applying condition (D1) to the decompositions $\Omega:=\Omega_1\cup\Omega_2=\Omega_2\cup\Omega_1$ we find that

$$\begin{aligned}\mathcal{I}&:=\eta(F(\Omega_1)\times F(\Omega_2))\cap\eta(F(\Omega_2)\times F(\Omega_1))\\&=\eta(\eta(F(\Omega_1\cap\Omega_2)\times F(\Omega_1))\times\eta(F(\Omega_2)\times F(\Omega_1\cap\Omega_2)));\end{aligned}$$

(1.3) ensures that the map

$$\eta_{(\varnothing,\Omega_1)}\colon F(\varnothing)\times F(\Omega_1)\to F(\Omega_1)$$

is surjective, i.e., $\eta(F(\Omega_1\cap\Omega_2)\times F(\Omega_1))=F(\Omega_1)$, and, similarly, $\eta(F(\Omega_2)\times F(\Omega_1\cap\Omega_2))=F(\Omega_2)$. Thus we have

$$\mathcal{I}=\eta(F(\Omega_1)\times F(\Omega_2)).$$

By an analogous application of (D1) and (1.3) to the decompositions $\Omega=\Omega_2\cup\Omega_1=\Omega_1\cup\Omega_2$ we find that

$$\mathcal{I}=\eta(F(\Omega_2)\times F(\Omega_1)),$$

and (2.3) is proved. ∎

(v) *Associativity of* η. For any three pairwise disjoint finite sets Ω_1, Ω_2, and Ω_3 we have

$$\eta(\eta(F(\Omega_1)\times F(\Omega_2))\times F(\Omega_3))=\eta(F(\Omega_1)\times\eta(F(\Omega_2)\times F(\Omega_3))). \tag{2.4}$$

Proof. We show that both sides of (2.4) equal the intersection

$$\mathcal{I}:=\eta(F(\Omega_1\cup\Omega_2)\times F(\Omega_3))\cap\eta(F(\Omega_1)\times F(\Omega_2\cup\Omega_3)).$$

By applying (D1) to the decompositions $\Omega:=(\Omega_1\cup\Omega_2)\cup\Omega_3=\Omega_1\cup(\Omega_2\cup\Omega_3)$ we obtain that

$$\mathcal{I}=\eta(\eta(F(\Omega_1)\times F(\Omega_2))\times\eta(F(\Omega_1\cap\Omega_3)\times F(\Omega_3)))$$

and as in the proof of (iv) we find that $\eta(F(\Omega_1\cap\Omega_3)\times F(\Omega_3))=F(\Omega_3)$, i.e.,

$$\mathcal{I}=\eta(\eta(F(\Omega_1)\times F(\Omega_2))\times F(\Omega_3)).$$

The same argument when applied to the decompositions $\Omega = \Omega_1 \cup (\Omega_2 \cup \Omega_3) = (\Omega_1 \cup \Omega_2) \cup \Omega_3$ yields

$$\mathscr{I} = \eta(F(\Omega_1) \times \eta(F(\Omega_2) \times F(\Omega_3))),$$

whence the claim (2.4). ▮

(vi) The functors $F_\eta^{(1)}$ and F_η coincide.

Proof. It suffices to show that

$$F_\eta^{(1)}(\Omega) = F_\eta(\Omega)$$

holds for every finite set Ω. By (1.1) this is true if $\Omega = \varnothing$, so assume that $\Omega \neq \varnothing$. Then, using the injectivity and associativity of η, together with the observation (1.3), we have

$$\begin{aligned}
F_\eta^{(1)}(\Omega) &= \bigcup_{\Omega_1 \subseteq \Omega} \eta(F_\eta(\Omega_1) \times F_\eta^{(0)}(\Omega - \Omega_1)) \\
&= \eta(F_\eta(\Omega) \times F(\varnothing)) \\
&= \eta\left(\left(F(\Omega) - \bigcup_{\substack{I \dot\cup J = \Omega \\ I \neq \varnothing \neq J}} \eta(F(I) \times F(J))\right) \times F(\varnothing)\right) \\
&= \eta(F(\Omega) \times F(\varnothing)) - \bigcup_{\substack{I \dot\cup J = \Omega \\ I \neq \varnothing \neq J}} \eta(\eta(F(I) \times F(J)) \times F(\varnothing)) \\
&= F(\Omega) - \bigcup_{\substack{I \dot\cup J = \Omega \\ I \neq \varnothing \neq J}} \eta(F(I) \times F(J)) = F_\eta(\Omega). \quad ▮
\end{aligned}$$

(vii) For a finite set Ω and any fixed element $\omega \in \Omega$ we have

$$F(\Omega) = \bigcup_{\omega \in \Omega_1 \subseteq \Omega} \eta(F_\eta(\Omega_1) \times F(\Omega - \Omega_1)). \tag{2.5}$$

Proof. Let $x \in F(\Omega)$ be an element and consider all sets Ω_1 such that $\omega \in \Omega_1 \subseteq \Omega$ and $x \in \eta(F(\Omega_1) \times F(\Omega - \Omega_1))$. Such sets exist, for example $\Omega_1 = \Omega$ has these properties since the map $\eta_{(\Omega, \varnothing)}: F(\Omega) \times F(\varnothing) \hookrightarrow F(\Omega)$ is surjective. Of these sets we whose one of minimal cardinality, say, $\Omega_1(x)$. Now suppose that $x \notin \eta(F_\eta(\Omega_1(x)) \times F(\Omega - \Omega_1(x)))$. Then by the injectivity of η and the choice of $\Omega_1(x)$ we must have that

$$\begin{aligned}
x &\in \eta((F(\Omega_1(x)) - F_\eta(\Omega_1(x))) \times F(\Omega - \Omega_1(x))) \\
&= \bigcup_{\substack{I \dot\cup J = \Omega_1(x) \\ I \neq \varnothing \neq J}} \eta(\eta(F(I) \times F(J)) \times F(\Omega - \Omega_1(x))).
\end{aligned}$$

Consequently, there exists a decomposition $I_1 \cup J_1 = \Omega_1(x)$, $I_1 \neq \emptyset \neq J_1$, such that $x \in \eta(\eta(F(I_1) \times F(J_1)) \times F(\Omega - \Omega_1(x)))$. Using (iv) and (v) we see that $\eta(\eta(F(I_1) \times F(J_1)) \times F(\Omega - \Omega_1(x)))$ is contained in both $\eta(F(I_1) \times F(\Omega - I_1))$ and $\eta(F(J_1) \times F(\Omega - J_1))$. The element ω is contained in I_1 or J_1, to fix ideas, say, $\omega \in I_1$. Hence we arrive at the assertion that

$$x \in \eta(F(I_1) \times F(\Omega - I_1)), \qquad \# I_1 < \# \Omega_1(x),$$

contradicting the choice of $\Omega_1(x)$. We conclude that x is indeed contained in $\eta(F_\eta(\Omega_1(x)) \times F(\Omega - \Omega_1(x)))$ and (2.5) is proved. ▮

(viii) The right-hand side of (2.5) is a disjoint union.

Proof. In the context of (vii) let Ω_1 and Ω_2 be two finite sets with $\omega \in \Omega_i \subseteq \Omega$ and $\Omega_1 \neq \Omega_2$, say, $\Omega_1 \cap \Omega_2 \neq \Omega_1$. It is enough to show that

$$\mathscr{I} := \eta(F(\Omega_1) \times F(\Omega - \Omega_1)) \cap \eta(F(\Omega_2) \times F(\Omega - \Omega_2))$$

has an empty intersection with $\eta(F_\eta(\Omega_1) \times F(\Omega - \Omega_1))$. But, by (D1), we have that

$$\mathscr{I} \subseteq \eta(\eta(F(\Omega_1 \cap \Omega_2) \times F(\Omega_1 - \Omega_2)) \times F(\Omega - \Omega_1))$$

and, by definition of F_η and the fact that $\omega \in \Omega_1 \cap \Omega_2 \neq \emptyset \neq \Omega_1 - \Omega_2$, we have

$$F_\eta(\Omega_1) \cap \eta(F(\Omega_1 \cap \Omega_2) \times F(\Omega_1 - \Omega_2)) = \emptyset.$$

So, by the injectivity of η we must indeed have

$$\eta(F_\eta(\Omega_1) \times F(\Omega - \Omega_1)) \cap \eta(F_\eta(\Omega_2) \times F(\Omega - \Omega_2)) = \emptyset,$$

as claimed. ▮

(ix) Given any finite set Ω we have

$$F(\Omega) = \bigcup_{k \geqslant 0} F_\eta^{(k)}(\Omega). \tag{2.6}$$

Proof. We use induction on $\#\Omega$. By (1.1) and the definition of $F_\eta^{(0)}$ the statement holds if $\Omega = \emptyset$. So let $\Omega \neq \emptyset$ and suppose that (2.6) holds for all sets of cardinality less than $\#\Omega$. Then

$$\begin{aligned}
\bigcup_{k\geqslant 0} F_\eta^{(k)}(\Omega) &= \bigcup_{k\geqslant 1} F_\eta^{(k)}(\Omega)\\
&= \bigcup_{k\geqslant 1}\ \bigcup_{\Omega_1\subseteq\Omega} \eta(F_\eta(\Omega_1)\times F_\eta^{(k-1)}(\Omega-\Omega_1))\\
&= \bigcup_{\Omega_1\subseteq\Omega} \eta\left(F_\eta(\Omega_1)\times \bigcup_{k\geqslant 1} F_\eta^{(k-1)}(\Omega-\Omega_1)\right)\\
&= \bigcup_{\varnothing\neq\Omega_1\subseteq\Omega} \eta\left(F_\eta(\Omega_1)\times \bigcup_{k\geqslant 0} F_\eta^{(k)}(\Omega-\Omega_1)\right)\\
&= \bigcup_{\Omega_1\subseteq\Omega} \eta(F_\eta(\Omega_1)\times F(\Omega-\Omega_1))\\
&= F(\Omega),
\end{aligned}$$

where we have used (vii) for the last equality. ∎

For our next step we have to assume that F is decomposable and that η is a decomposition of F.

(x) For a finite set Ω, any fixed element $\omega\in\Omega$ and an integer $k\geqslant 1$ we have

$$F_\eta^{(k)}(\Omega) = \dot{\bigcup_{\omega\in\Omega_1\subseteq\Omega}} \eta(F_\eta(\Omega_1)\times F_\eta^{(k-1)}(\Omega-\Omega_1)). \tag{2.7}$$

Proof. The fact that the terms on the right-hand side of (2.7) are pairwise disjoint follows from (viii) since a term $\eta(F_\eta(\Omega_1)\times F_\eta^{(k-1)}(\Omega-\Omega_1))$ is contained in the larger set $\eta(F_\eta(\Omega_1)\times F(\Omega-\Omega_1))$. Denote the union on the right-hand side of (2.7) by $\tilde{F}_\eta^{(k)}(\Omega)$. It remains to show that $F_\eta^{(k)}(\Omega)=\tilde{F}_\eta^{(k)}(\Omega)$. By definition of $F_\eta^{(k)}$ we have that $\tilde{F}_\eta^{(k)}(\Omega)\subseteq F_\eta^{(k)}(\Omega)$. Moreover, using (ix) and (vii) we see that

$$\begin{aligned}
\bigcup_{k\geqslant 1} \tilde{F}_\eta^{(k)}(\Omega) &= \bigcup_{k\geqslant 1}\ \bigcup_{\omega\in\Omega_1\subseteq\Omega} \eta(F_\eta(\Omega_1)\times F_\eta^{(k-1)}(\Omega-\Omega_1))\\
&= \bigcup_{\omega\in\Omega_1\subseteq\Omega} \eta\left(F_\eta(\Omega_1)\times \bigcup_{k\geqslant 0} F_\eta^{(k)}(\Omega-\Omega_1)\right)\\
&= \bigcup_{\omega\in\Omega_1\subseteq\Omega} \eta(F_\eta(\Omega_1)\times F(\Omega-\Omega_1))\\
&= F(\Omega).
\end{aligned}$$

In the presence of axiom (D2), i.e., the disjointness of the sets $F_\eta^{(k)}(\Omega)$ for different k, this implies $\tilde{F}_\eta^{(k)}(\Omega)=F_\eta^{(k)}(\Omega)$ for every $k\geqslant 1$, whence (x). ∎

(xi) *Proof of* (a). By (vii), (viii) and the injectivity of η we have that

$$\psi_F(n) = \#F(\{1, ..., n\})$$
$$= \sum_{1 \in \Omega_1 \subseteq \{1, ..., n\}} \#(F_\eta(\Omega_1) \times F(\{1, ..., n\} - \Omega_1)). \tag{2.8}$$

Using the functoriality of F and F_η, each subset Ω_1 with $1 \in \Omega_1 \subseteq \{1, ..., n\}$ and of cardinality $\#\Omega_1 = \mu$ contributes $\varphi_F^\eta(\mu)\, \psi_F(n-\mu)$ to the right-hand side of (2.8). Observe that this number does not depend upon Ω_1 itself but only on the cardinality μ of Ω_1. Therefore the $\binom{n-1}{\mu-1}$ μ-subsets of $\{1, ..., n\}$ containing the element 1 contribute $\binom{n-1}{\mu-1}\, \varphi_F^\eta(\mu)\, \psi_F(n-\mu)$ to this sum and we obtain

$$\psi_F(n) = \sum_{\mu=1}^{n} \binom{n-1}{\mu-1} \varphi_F^\eta(\mu)\, \psi_F(n-\mu), \qquad n \geqslant 1. \tag{2.9}$$

Multiplying both sides of (2.9) by $Z^{n-1}/(n-1)!$ and summing over $n \geqslant 1$ gives

$$\Psi'(Z) = \Phi'(Z) \cdot \Psi(Z),$$

from which (1.6) follows in view of (1.3). ∎

(xii) *Proof of* (b). By (ix) and (D2) we have

$$\psi_F(n) = \sum_{k \geqslant 0} \psi_F^\eta(n, k), \qquad n \geqslant 0 \tag{2.10}$$

and, hence,

$$\Psi(Z, 1) = \sum_{n \geqslant 0} \sum_{k \geqslant 0} \psi_F^\eta(n, k)\, Z^n/n! = \sum_{n \geqslant 0} \psi_F(n)\, Z^n/n! = \Psi(Z),$$

whence (1.7)(ii). In order to prove (1.7)(i) we use (x) instead of (vii) and (viii). Arguing as in the last step except that, instead of the functoriality of F, one has to use (ii), we obtain

$$\psi_F^\eta(n, k) = \sum_{\mu=1}^{n} \binom{n-1}{\mu-1} \varphi_F^\eta(\mu)\, \psi_F^\eta(n-\mu, k-1) \qquad (n, k \geqslant 1). \tag{2.11}$$

Multiplying both sides of (2.11) by $Z^{n-1} Y^k/(n-1)!$ and summing over $n \geqslant 1$ and $k \geqslant 1$ gives

$$\sum_{n \geqslant 1} \sum_{k \geqslant 1} \psi_F^\eta(n, k)\, Z^{n-1} Y^k/(n-1)!$$
$$= \sum_{n \geqslant 1} \sum_{k \geqslant 1} \sum_{\mu=1}^{n} \binom{n-1}{\mu-1} \varphi_F^\eta(\mu)\, \psi_F^\eta(n-\mu, k-1)\, Z^{n-1} Y^k/(n-1)!. \tag{2.12}$$

In view of the definition of $F_\eta^{(0)}$ the left-hand side of (2.12) equals $(\partial/\partial Z)\,\Psi(Z, Y)$. The right-hand side can be rewritten as

$$\sum_{k\geqslant 1} Y^k \sum_{n\geqslant 1} \frac{Z^{n-1}}{(n-1)!}\left\langle \frac{Z^{n-1}}{(n-1)!}, \Phi'(Z)\,\Psi_{k-1}(Z)\right\rangle = \sum_{k\geqslant 1} Y^k\Phi'(Z)\,\Psi_{k-1}(Z)$$
$$= Y\Phi'(Z)\sum_{k\geqslant 0} Y^k\Psi_k(Z)$$
$$= Y\Phi'(Z)\,\Psi(Z, Y).$$

The resulting relation,

$$\frac{\partial}{\partial Z}\Psi(Z, Y) = Y\Phi'(Z)\,\Psi(Z, Y),$$

in view of (1.3) now obviously implies Eq. (1.7)(i) and the proof of the theorem is complete. ∎

3. SOME APPLICATIONS

As we saw in Section 1, Wilf's exponential formula for exponential families is a special case of our theorem. We therefore feel freed from the usual obligation to demonstrate the existence of a multitude of interesting applications for our method, since Chapter 3 of [W1] provides an excellent such collection with respect to Wilf's formula and, hence, mutatis mutandis, also for our result. Instead, we concentrate here on three topics of somewhat algebraic flavor, namely (i) the connection between group actions and finite-index subgroups, (ii) equations in one variable in symmetric semigroups, and (iii) cyclic sets.

1. *Group Actions and Subgroups of Finite Index*

Let G be a group, $\Sigma\subseteq G$ a normal subset with $1\notin\Sigma$, and $S\subseteq\mathbb{N}$ a set of positive integers. For a set Ω we denote by $\mathrm{Hom}_\Sigma^S(G, S(\Omega))$ the set of all G-actions τ on Ω with the following two properties:

(i) τ induces a fixed-point-free action of (the elements of) Σ on Ω.

(ii) The lengths of the orbits into which Ω decomposes under τ are contained in the set S.

The elements of $\mathrm{Hom}_\Sigma^S(G, S(\Omega))$ will be called (Σ, S)-*admissible* G-actions on Ω. A triple (G, Σ, S) is termed *admissible* if $h(n) = h_{(G,\Sigma,S)}(n) := \#\mathrm{Hom}_\Sigma^S(G, S_n) < \infty$ for all $n\geqslant 0$. If, for instance, the group G is finitely generated or of finite Prüfer rank, then the triple (G, Σ, S) is admissible for each normal subset $\Sigma \subseteq G \backslash \{1\}$ and every set S of positive integers. A subgroup Γ of index n in G induces a G-action by left multiplication on the n-set G/Γ of left coset which, after suitable renaming, becomes a G-action on $\{1, ..., n\}$ with the property that stab $(1) = \Gamma$. Thus we have an injective mapping from the set of all subgroups of index n in G into $\mathrm{Hom}(G, S_n)$, and if $n = [G : \Gamma] \in S$ and Γ is such that $\Gamma \cap \Sigma = \emptyset$ then the image of Γ will be contained in the subset $\mathrm{Hom}_{\Sigma}^{S}(G, S_n)$. Hence, admissibility of (G, Σ, S) implies that the number

$$s_G^{\Sigma}(n) := \#\{\Gamma : \Gamma \leqslant G, [G : \Gamma] = n, \Gamma \cap \Sigma = \emptyset\}$$

is finite for all $n \in S$.

Let (G, Σ, S) be an admissible triple. For a finite set Ω put $F(\Omega) = F_{(G, \Sigma, S)}(\Omega) := \mathrm{Hom}_{\Sigma}^{S}(G, S(\Omega))$. With the obvious definition of induced maps F is a functor on Ens and the disjoint sum of G-actions defines a natural transformation $\eta: F \times F \to F \circ \dot{\cup}$. It is clear that $F \neq \emptyset$ and that the pair (F, η) satisfies (D1). By definition we have for a finite set Ω that

$$F_\eta(\Omega) = \begin{cases} \text{set of all transitive } \Sigma\text{-free } G\text{-actions on } \Omega, & \#\Omega \in S, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and the set $F_\eta^{(k)}(\Omega)$ consists of all those (Σ, S)-admissible G-actions on Ω which have exactly k orbits. This implies in particular that the pair (F, η) also satisfies (D2); hence, F is decomposable and η is a decomposition of F. Suppose that $n \in S$. The stabilizer of the letter 1 in S_n acts freely by conjugation on $F_\eta(\{1, ..., n\})$, decomposing this set into orbits $\bar{\tau}$ in 1–1 correspondence with the subgroups of index n in G avoiding Σ via $\bar{\tau} \mapsto \mathrm{stab}_\tau(1)$. It follows that

$$\varphi_F^\eta(n) = \#F_\eta(\{1, ..., n\}) = \begin{cases} (n-1)!\, s_G^{\Sigma}(n), & n \in S, \\ 0, & \text{otherwise,} \end{cases}$$

and, hence,

$$\Phi(Z) = \Phi_{(G, \Sigma, S)}(Z) = \sum_{n \in S} \frac{s_G^{\Sigma}(n)}{n} Z^n.$$

Denoting by $h(n, T) = h_{(G, \Sigma, S)}(n, T)$ the number of those (Σ, S)-admissible G-actions on an n-set whose number of orbits is in a given set $T \subseteq \mathbb{Z}_+$, our theorem yields the following.

PROPOSITION 1. *Let (G, Σ, S) be an admissible triple and $T \subseteq \mathbb{Z}_+$ a set of nonnegative integers. Then we have*

$$\sum_{n \geqslant 0} h(n, T)\, Z^n/n! = e_T\left(\sum_{n \in S} \frac{s_G^{\Sigma}(n)}{n} Z^n\right). \tag{3.1}$$

The special case

$$\sum_{n=0}^{\infty} \#\mathrm{Hom}_{\Sigma}(G, S_n)\, Z^n/n! = \exp\left(\sum_{n=1}^{\infty} \frac{s_G^{\Sigma}(n)}{n} Z^n\right) \tag{3.2}$$

of (3.1), where $S = \mathbb{N}$ and $T = \mathbb{Z}_+$ appears to cover every result in the literature relating subgroup numbers to the enumeration of permutation representations or vice versa. Here $\mathrm{Hom}_{\Sigma}(G, S_n) = \mathrm{Hom}_{\Sigma}^{\mathbb{N}}(G, S_n)$ denotes the set of Σ-free G-actions on $\{1, ..., n\}$. For example, putting in addition $\Sigma = \emptyset$ in (3.2) yields Wohlfahrt's result connecting the number $s_G(n) = s_G^{\emptyset}(n)$ of all subgroups of index n in G with the enumeration of all G-actions on an n-set; cf. [Wo]. Wohlfahrt's formula in turn contains in particular Hall's recursion formula [H, Theorem 5.2] for the number of subgroups of index n in a finitely generated free group or Dey's corresponding formula [D, Theorem 6.10] for a free product G. If we take G to be a finite group of order $\#G = m$ then we find from Wohlfahrt's result that

$$\sum_{n=0}^{\infty} \#\mathrm{Hom}(G, S_n)\, Z^n/n! = \exp\left(\sum_{d \mid m} \frac{s_G(d)}{d} Z^d\right). \tag{3.3}$$

This formula, which exhibits the exponential generating function of the sequence $h(n) = \#\mathrm{Hom}(G, S_n)$ for a finite group G as a rather simple entire function, was a starting point in [M3] for the asymptotic enumeration of finite group actions; cf. [M3, Theorem 5]. Remarks concerning the history of the latter problem can be found in [W2] and the introduction of [M3]. For $G = C_m$ a cyclic group formula (3.3) was already proved in [CHS]. If we take G to be a finitely generated virtually free group and $\Sigma = \mathrm{tor}\, G \backslash \{1\}$ as the set of torsion elements in G apart from the identity then after substituting $Z = \bar{Z}^{1/m_G}$ in (3.2), taking log and differentiating we recover the identity [M1, (3)] relating the number of free subgroups in G of given finite index to the enumeration of torsion-free G-actions, i.e., G-actions which are free when restricted to finite subgroups (m_G denotes the least common multiple of the orders of the finite subgroups in G). This relation in turn was the starting point for a detailed analysis of the growth behavior and the asymptotics of the number of free subgroups of given finite index in a virtually free group G; cf. [M1, M2].

We conclude our discussion of Proposition 1 with the following two remarks:

(i) When studying the arithmetic of finite group actions it is sometimes necessary to allow restrictions on the number and size of orbits. Applying (3.1) with $\Sigma = \varnothing$ and S, T arbitrary to a finite group G of order $\# G = m$ we obtain the following refinement of (3.3):

The exponential generating function of the number $h(n, T) = h_{(G, S)}(n, T)$ of G-actions on an n-set with orbits lengths in S and total number of orbits in T is given by

$$\sum_{n \geqslant 0} h(n, T)\, Z^n/n! = e_T\left(\sum_{\substack{d \mid m \\ d \in S}} \frac{s_G(d)}{d}\, Z^d\right), \qquad m = \# G. \tag{3.4}$$

(ii) An equivalent way of expressing Proposition 1 is as follows:

Let (G, Σ, S) be an admissible triple. Then the number $h(n, k)$ of (Σ, S)-admissible G-actions on an n-set with exactly k orbits can be expressed in terms of subgroups numbers as

$$h(n, k) = \frac{n!}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_j \in S}} \frac{s_G^{\Sigma}(n_1) \cdots s_G^{\Sigma}(n_k)}{n_1 \cdots n_k} \qquad (n, k \geqslant 0). \tag{3.5}$$

As an illustration let us calculate the number N of all fixed-point-free $SL_2(\mathbb{Z})$-actions on a 10-set with exactly four orbits. We have $G = SL_2(\mathbb{Z})$, $\Sigma = \varnothing$, $n = 10$, $k = 4$, and we can take $S = \{2, 3, 4\}$. We need to know the subgroup numbers $s_G(n)$ for $n \leqslant 4$. Using the presentation $G \cong \langle A, B \mid A^4 = B^6 = 1, A^2 = B^3 \rangle$ one finds that

$$\begin{aligned} \# \mathrm{Hom}(G, S_1) &= 1 \\ \# \mathrm{Hom}(G, S_2) &= 2 \\ \# \mathrm{Hom}(G, S_3) &= 12 \\ \# \mathrm{Hom}(G, S_4) &= 96 \end{aligned}$$

(only the last case requires a moment's thought). Plugging this information into formula (3.2) with $\Sigma = \varnothing$ and taking the log gives

$$\sum_{n=1}^{\infty} \frac{s_G(n)}{n}\, Z^n = \sum_{\mu=1}^{\infty} (-1)^{\mu - 1} \frac{1}{\mu} (Z + Z^2 + 2Z^3 + 4Z^4 + \cdots)^{\mu},$$

from which we read off that $s_G(1)=s_G(2)=1$, $s_G(3)=4$, and $s_G(4)=9$. Using these values in (3.5) gives

$$N = h_{(SL_2(\mathbb{Z}),\, \varnothing,\, \{2,3,4\})}(10, 4) = 573300.$$

2. *The Equation $X^\alpha = X^\beta$ in Symmetric Semigroups*

For a set Ω denote by $T(\Omega)$ the symmetric semigroup on Ω and put $T(\{1, ..., n\}) =: T_n$. In their paper [HS] Harris and Schoenfeld study the number U_n of idempotent elements, i.e. the solutions of the equation $X^2 = X$ in T_n. They find the exponential generating function

$$\sum_{n \geqslant 0} U_n Z^n/n! = \exp(Ze^Z), \tag{3.6}$$

from which they derive, among other things, some divisibility properties for the U_n and an asymptotic formula. In his survey [S] on generating functions Stanley after rederiving (3.6) poses the problem to find the number $s(n)$ of solutions in T_n of the general equation

$$X^\alpha = X^\beta, \qquad 0 \leqslant \alpha < \beta, \tag{3.7}$$

in one variable. To our knowledge, this function $s(n)$ was first determined by Goulden and Jackson; cf. [GJ, Section 3.3.15, Ex. 3.3.31]. Using our functorial approach we shall derive a refined version of their result.

Given a finite set Ω and a map $f\colon \Omega \to \Omega$, Ω decomposes into nonempty, f-invariant subsets Ω_j, $\Omega = \Omega_1 \,\dot\cup\, \cdots \,\dot\cup\, \Omega_k$, such that each Ω_j is indecomposable, i.e., Ω_j itself is not the union of two disjoint nonempty f-invariant subsets. This decomposition of Ω is uniquely determined by these requirements and is nothing but the partition of Ω given by vertex sets of the connected components of f viewed as a directed graph on Ω. The equivalence relation inducing this decomposition of Ω is given by

$$\omega_1 \sim \omega_2 :\Leftrightarrow \text{there exist integers } k, l \geqslant 0 \text{ such that } f^k(\omega_1) = f^l(\omega_2).$$

We call $\Omega_1, ..., \Omega_k$ the *components* of f, the numbers $\#\Omega_1, ..., \#\Omega_k$ are the *decomposition numbers* and k is the *decomposition length* of f. Given a set $S \subseteq \mathbb{N}$ of positive integers, a map $f\colon \Omega \to \Omega$ is termed *S-admissible* if the decomposition numbers of f are in S. For integers α and β, $0 \leqslant \alpha < \beta$, and a set $S \subseteq \mathbb{N}$ introduce a functor F: Ens $\to$ Ens with

$$F(\Omega) = F_{(\alpha, \beta, S)}(\Omega) := \text{set of } S\text{-admissible solutions of (3.7) in } T(\Omega)$$

and obvious definition of induced maps. The disjoint sum of maps defines a natural transformation $\eta: F \times F \to F \circ \cup$ and it is immediately verified that $F \neq \varnothing$ and that the pair (F, η) satisfies (D1). By definition

$$F_\eta(\Omega) = \begin{cases} \text{set of connected solutions of (3.7) in } T(\Omega), & \#\Omega \in S, \\ \varnothing, & \text{otherwise,} \end{cases}$$

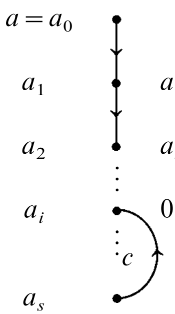
and $F_\eta^{(k)}(\Omega)$ is the set of those S-admissible solutions of Eq. (3.7) in $T(\Omega)$ which have decomposition length exactly k; in particular the pair (F, η) satisfies (D2). Denoting by $s(n, S, T) = s_{\alpha, \beta}(n, S, T)$ the number of those solutions of Eq. (3.7) in T_n whose decomposition length is in a given set $T \subseteq \mathbb{Z}_+$ and whose decomposition numbers lie in the set S, and by $c(n) = c_{\alpha, \beta}(n)$ the number of connected solutions of (3.7) in T_n, our theorem yields

$$\sum_{n \geq 0} s(n, S, T)\, Z^n/n! = e_T\left(\sum_{n \in S} c(n)\, Z^n/n!\right). \tag{3.8}$$

It remains to evaluate the sum $\sum_{n \in S} c(n)\, Z^n/n!$. To this end we first describe the graph-theoretic structure of a connected solution $f: \{1, ..., n\} \to \{1, ..., n\}$, $n \geq 1$, of (3.7):

> f consists of an oriented cycle c of length dividing $\beta - \alpha$ with (pairwise disjoint) trees of height at most α (possibly 0) growing out of the vertices of c.

To see this we apply f to an arbitrary point $a = a_0 \in \{1, ..., n\}$ until we hit (for the first time) a point already produced



$a_0 = a$, $a_1 = f(a_0)$, $a_2 = f(a_1) = f^2(a_0)$, ...,

$a_s = f^s(a_0)$, $f(a_s) = a_i$ $(0 \leq i \leq s)$ and $a_j \neq a_k$ for

$0 \leq j$, $k \leq s$ and $j \neq k$.

Since f is a solution of (3.7) the length of the cycle c must divide $\beta - \alpha$; for the same reason also $i \leq \alpha$, for otherwise we would have $f^\alpha(a) = a_\alpha \neq f^\beta(a)$. Hence, a reaches the cycle c after at most α (possibly 0) steps. In particular, since $n > 0$, there exists such a cycle c, and taking into account all points in $\{1, ..., n\}$ which eventually run into c, we obtain a graph of the form described above. Since f is indecomposable the complement of this graph in f is empty; i.e., we have described the graph of f. Conversely, every map $f: \{1, ..., n\} \to \{1, ..., n\}$ of the structure described clearly is an indecomposable solution of Eq. (3.7). Using this description we now determine the function $c(n)$.

We have

$$c(n) = \sum_{\mu \mid \beta - \alpha} \binom{n}{\mu} (\mu - 1)! \sum_{\substack{n_1 + \cdots + n_\mu = n - \mu \\ n_j \geqslant 0}} \binom{n-\mu}{n_1, ..., n_\mu} T_\alpha(n_1) \cdots T_\alpha(n_\mu), \quad n \geqslant 1, \tag{3.9}$$

where $T_\alpha(n)$ denotes the number of labeled trees on $n+1$ vertices and of height at most α growing out of a given root.

Proof. Suppose we are given (i) a number $\mu \mid \beta - \alpha$, (ii) a μ-cycle c in S_n, (iii) a partition $n - \mu = n_1 + \cdots + n_\mu$ of $n - \mu$ into summands $n_j \geqslant 0$, and (iv) a partition of the remaining $n - \mu$ points of $\{1, ..., n\}$ outside c into parts of length $n_1, n_2, ..., n_\mu$. The n_j points above the jth cycle vertex v_j, together with v_j, can be organized into a tree at height of most α growing out of v_j in exactly $T_\alpha(n_j)$ different ways; hence, there are a total of

$$T_\alpha(n_1)\, T_\alpha(n_2) \cdots T_\alpha(n_\mu)$$

possible ways to complete the given situation into a connected solution of (3.7). This number does not depend on the actual distribution of the $n - \mu$ points outside c onto the vertices of c but only upon the partition $n - \mu = n_1 + \cdots + n_\mu$ of $n - \mu$. Hence, if we start from μ, a given μ-cycle c and a number-theoretic partition $n - \mu = n_1 + \cdots + n_\mu$, there are precisely

$$\binom{n-\mu}{n_1, ..., n_\mu} T_\alpha(n_1) \cdots T_\alpha(n_\mu)$$

possible ways of correctly completing the situation. Summing this number over the (ordered) partitions of $n - \mu$ into μ parts we find that there are exactly

$$\sum_{\substack{n_1 + \cdots + n_\mu = n - \mu \\ n_j \geqslant 0}} \binom{n-\mu}{n_1, ..., n_\mu} T_\alpha(n_1) \cdots T_\alpha(n_\mu)$$

indecomposable solutions of (3.7) which contain the μ-cycle c. Again, this number does not depend on the cycle c but only upon its cardinality μ. Hence, there are precisely

$$\binom{n}{\mu} (\mu - 1)! \sum_{\substack{n_1 + \cdots + n_\mu = n - \mu \\ n_j \geqslant 0}} \binom{n-\mu}{n_1, ..., n_\mu} T_\alpha(n_1) \cdots T_\alpha(n_\mu)$$

indecomposable solutions of (3.7) containing a μ-cycle and summing this last term over the divisors μ of $\beta-\alpha$ we obtain the number $c(n)$. ∎

It remains to determine the tree numbers $T_\alpha(n)$.
We have

$$T_\alpha(n)=\sum_{\substack{n_1+\cdots+n_\alpha=n\\ n_j\geqslant 0}}\binom{n}{n_1,\ldots,n_\alpha} n_1^{n_2}n_2^{n_3}\cdots n_{\alpha-1}^{n_\alpha}\qquad (n\geqslant 0,\ \alpha\geqslant 1).\quad (3.10)$$

Proof. We determine the number $t_h(n)$ of such trees which are precisely of height h. Classifying the n nonroots by height and counting yields a partition $n=n_1+\cdots+n_h$ with $n_j\geqslant 1$. Given such a partition of n and a corresponding set-theoretic partition of the nonroots there are precisely

$$n_1^{n_2}n_2^{n_3}\cdots n_{h-1}^{n_h}$$

ways of drawing the edges from each level downward to the next lower level. As this number does not depend on the set-theoretic partition of the nonroots, there are exactly

$$\binom{n}{n_1,\ldots,n_h} n_1^{n_2}n_2^{n_3}\cdots n_{h-1}^{n_h}$$

such trees of height h realizing the same partition $n=n_1+\cdots+n_h$. Hence,

$$t_h(n)=\sum_{n_1+\cdots+n_h=n}\binom{n}{n_1,\ldots,n_h} n_1^{n_2}n_2^{n_3}\cdots n_{h-1}^{n_h}$$

and

$$\begin{aligned}T_\alpha(n)&=\sum_{h=0}^{\alpha} t_h(n)\\ &=\sum_{h=0}^{\alpha}\sum_{\substack{n_1+\cdots+n_h=n\\ n_j\geqslant 1}}\binom{n}{n_1,\ldots,n_h} n_1^{n_2}n_2^{n_3}\cdots n_{h-1}^{n_h}\\ &=\sum\binom{n}{n_1,\ldots,n_\alpha} n_1^{n_2}n_2^{n_3}\cdots n_{\alpha-1}^{n_\alpha},\end{aligned}$$

where the last sum extends over those α-tuples $(n_1,\ldots,n_\alpha)$ of nonnegative integers with the property that $\Sigma n_j=n$ and $n_j=0\Rightarrow n_{j+1}=0$. Formula (3.10) follows now since the extra summands correspond to tuples $(n_1,\ldots,n_\alpha)$ such that $n_j=0$, $n_{j+1}\neq 0$ for some j and therefore vanish. ∎

Define a sequence of functions $\{\Delta_\alpha(Z)\}_{\alpha=0}^{\infty}$ recursively by

$$\begin{aligned} \Delta_0(Z) &= Z \\ \Delta_\alpha(Z) &= Z\exp(\Delta_{\alpha-1}(Z)), \qquad \alpha \geqslant 1. \end{aligned} \tag{3.11}$$

An immediate induction on s shows that for fixed $\alpha \geqslant 1$ and every $s = 1, ..., \alpha$ we have that

$$Z^{-1}\Delta_\alpha(Z) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} \frac{n_1^{n_2} n_2^{n_3} \cdots n_{s-1}^{n_s}}{n_1!\, n_2! \cdots n_s!} Z^{n_1 + \cdots + n_{s-1}} (\Delta_{\alpha-s}(Z))^{n_s}.$$

Using this for fixed $\alpha \geqslant 1$ and $s = \alpha$ yields the identity

$$Z^{-1}\Delta_\alpha(Z) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_\alpha=0}^{\infty} \frac{n_1^{n_2} n_2^{n_3} \cdots n_{\alpha-1}^{n_\alpha}}{n_1!\, n_2! \cdots n_\alpha!} Z^{n_1 + \cdots + n_\alpha}, \qquad \alpha \geqslant 1, \tag{3.12}$$

which in conjunction with (3.10) shows that for fixed α the exponential generating function of $T_\alpha(n)$ is given by

$$\sum_{n \geqslant 0} T_\alpha(n)\, Z^n/n! = Z^{-1}\Delta_\alpha(Z), \qquad \alpha \geqslant 0. \tag{3.13}$$

Now, by (3.9) and (3.13)

$$\begin{aligned} \sum_{n\in S} c(n)\, Z^n/n! &= \sum_{n\in S} \sum_{\mu \mid \beta-\alpha} \binom{n}{\mu} (\mu-1)! \\ &\quad \times \sum_{\substack{n_1+\cdots+n_\mu = n-\mu \\ n_j \geqslant 0}} \binom{n-\mu}{n_1, ..., n_\mu} T_\alpha(n_1) \cdots T_\alpha(n_\mu)\, Z^n/n! \\ &= \sum_{n\in S} \sum_{\mu \mid \beta-\alpha} \frac{1}{\mu} Z^\mu \sum_{\substack{n_1+\cdots+n_\mu = n-\mu \\ n_j \geqslant 0}} \frac{T_\alpha(n_1)\cdots T_\alpha(n_\mu)}{n_1! \cdots n_\mu!} Z^{n-\mu} \\ &= \sum_{\mu \mid \beta-\alpha} \frac{1}{\mu} Z^\mu \sum_{n\in S} \langle Z^{n-\mu}, Z^{-\mu}(\Delta_\alpha(Z))^\mu \rangle Z^{n-\mu} \\ &= \sum_{\mu \mid \beta-\alpha} \frac{1}{\mu} \sum_{n\in S} \langle Z^n, (\Delta_\alpha(Z))^\mu \rangle Z^n \\ &= \sum_{\mu \mid \beta-\alpha} \frac{1}{\mu} ((\Delta_\alpha(Z))^\mu)_S. \end{aligned}$$

Summarizing we have established the following.

PROPOSITION 2. *Given integers* α, β *with* $0 \leqslant \alpha < \beta$ *and sets* $T \subseteq \mathbb{Z}_+$ *and* $S \subseteq \mathbb{N}$, *the number* $s(n, S, T) = s_{\alpha, \beta}(n, S, T)$ *of solutions of the equation* $X^\alpha = X^\beta$ *in* T_n *with decomposition length in* T *and decomposition numbers in* S *has the generating function*

$$\sum_{n \geqslant 0} s(n, S, T)\, Z^n/n! = e_T\left(\sum_{\mu \mid \beta - \alpha} \frac{1}{\mu} ((\Delta_\alpha(Z))^\mu)_S \right), \tag{3.14}$$

where $\Delta_\alpha(Z)$ *is defined by* (3.11).

For $T = \mathbb{Z}_+$ and $S = \mathbb{N}$ this result specializes to the formula of Goulden and Jackson, but the right-hand side of (3.14) is readily calculated for every choice of S and T. As an illustration let us consider the case $\alpha = 1$, $\beta = 2$ of idempotents in T_n somewhat closer. Putting $s_{1,2}(n, S, T) = U_n(S, T)$ we find from (3.14) that

$$\sum_{n \geqslant 0} U_n(S, T)\, Z^n/n! = e_T(Ze_{S-1}(Z)). \tag{3.15}$$

Taking $T = 2\mathbb{Z}_+$ and $S = 2\mathbb{N} - 1$ the right-hand side of (3.15) becomes

$$\cosh(Z \cosh(Z)) = 1 + \frac{1}{2!} Z^2 + \frac{13}{4!} Z^4 + \frac{181}{6!} Z^6 + \frac{3865}{8!} Z^8 + \cdots,$$

while choosing $T = 2\mathbb{Z}_+ + 1$ and $S = 2\mathbb{N}$ gives

$$\sinh(Z \sinh(Z)) = \frac{2}{2!} Z^2 + \frac{4}{4!} Z^4 + \frac{126}{6!} Z^6 + \frac{3368}{8!} Z^8 + \frac{95770}{10!} Z^{10} + \cdots.$$

As a final example let us take $T = 2\mathbb{Z}_+$ and

$$S = S_0 = \{ n \in \mathbb{N} : n \equiv 1 \bmod 3 \;\&\; n \geqslant 4 \}.$$

By (3.15) the exponential generating function of the number $U_n(S_0, 2\mathbb{Z}_+)$ of idempotent elements in T_n with an even number of components all of whose lengths are congruent to 1 mod 3 and $\geqslant 4$ is given by

$$\begin{aligned} &\sum_{n \geqslant 0} U_n(S_0, 2\mathbb{Z}_+)\, Z^n/n! \\ &\quad = \cosh\left(\sum_{\kappa=1}^{\infty} \frac{1}{(3\kappa)!} Z^{3\kappa+1} \right) \\ &\quad = 1 + \frac{560}{8!} Z^8 + \frac{9240}{11!} Z^{11} + \frac{124124}{14!} Z^{14} + \frac{672672000}{16!} Z^{16} + \cdots. \end{aligned}$$

3. *Cyclic Sets*

Let A be an *almost finite* cyclic set, i.e., a set equipped with an action of the infinite cyclic group Γ such that (i) each orbit is finite and (ii) each subgroup of finite index in Γ has at most finitely many fixed points on A. Denote by Γ_n the subgroup of index n in Γ and put

$$f_n(A) := \#\text{fixed points of } \Gamma_n \text{ on } A.$$

On the other hand, consider, for every $m \geqslant 0$, the mth symmetric power

$$S^m(A) = \left\{ s \in \mathbb{Z}_+^A : \sum_a s(a) = m \right\}$$

of A. With the induced Γ-action $S^m(A)$ is, again, an almost finite cyclic set, in particular

$$f_1(S^m(A)) = \#\{s \in S^m(A) : s \text{ constant on } \Gamma\text{-orbits}\}$$

is finite for every m.

Proposition 3. *For an almost finite cyclic set A the sequences $f_n(A)$ and $f_1(S^m(A))$ are related by*

$$m f_1(S^m(A)) = \sum_{k=1}^{m} f_k(A)\, f_1(S^{m-k}(A)), \qquad m \geqslant 1. \tag{3.16}$$

Proof. Fix a generator γ of the infinite cyclic group Γ. Given a permutation $\sigma \in S(\Omega)$ of the finite set Ω and a map $f \in A^\Omega$ we call f *σ-invariant*, if

$$f(\sigma(\omega)) = \gamma \cdot f(\omega) \qquad \text{for all} \quad \omega \in \Omega.$$

Put

$$F_A(\Omega) := \{(\sigma, f) \in S(\Omega) \times A^\Omega : f\ \sigma\text{-invariant}\}.$$

Clearly, $F_A(\cdot)$ is functorial and there is an obvious natural transformation $\eta_A : F_A \times F_A \to F_A \circ \cup$ such that condition (D1) holds (the pair (F_A, η_A) also satisfies (D2) but we will not use this here). To determine the cardinality of $F_A(\Omega)$ consider that map $\alpha : F_A(\Omega) \to S^{\#\Omega}(A)$, given by $\alpha(\sigma, f) = s_f$, where $s_f(a) := \# f^{-1}(a)$. If $s \in S^{\#\Omega}(A)$ is not shift-invariant then $\alpha^{-1}(s) = \varnothing$. Suppose, on the other hand, that s is constant on Γ-orbits. There exist $\#\Omega!/\prod_a s(a)!$ functions $f \in A^\Omega$ with $s_f = s$, and, given such a function f, a permutation $\sigma \in S(\Omega)$ identifying the preimage $f^{-1}(a)$

with $f^{-1}(\gamma a)$ for every $a \in A$ can be chosen in precisely $\prod_a s(a)!$ many ways. Therefore, we have in this case

$$\#\alpha^{-1}(s) = \sum_{\substack{f \in A^\Omega \\ s_f = s}} \#\{\sigma \in S(\Omega) : f(\sigma(\omega)) = \gamma f(\omega) \text{ for all } \omega \in \Omega\}$$

$$= \frac{\#\Omega!}{\prod_a s(a)!} \prod_a s(a)! = \#\Omega!.$$

It follows that $\#F_A(\Omega) = \#\Omega! \cdot f_1(S^{\#\Omega}(A))$, in particular, $F_A(\Omega)$ is finite for finite Ω; that is, F_A is a functor on Ens. Furthermore, we have for $\Omega \neq \varnothing$

$$(F_A)_\eta\,(\Omega) = \{(\sigma, f) \in F_A(\Omega) : \sigma \text{ a full cycle}\}$$

and for a given full cycle σ on Ω the σ-invariant functions $f \in A^\Omega$ correspond in a 1–1 fashion to the fixed points of the group $\Gamma_{\#\Omega}$ on A. Hence, $\#(F_A)_\eta\,(\Omega) = (\#\Omega - 1)! \cdot f_{\#\Omega}(A)$ and (3.16) follows from (1.6). ∎

In terms of the generating functions

$$F(Z) := \sum_{n=0}^{\infty} f_{n+1}(A)\, Z^n, \qquad S(Z) = S_A(Z) := \sum_{m=0}^{\infty} f_1(S^m(A))\, Z^m,$$

Eq. (3.16) can be expressed in the form

$$\frac{d}{dZ}(\log S(Z)) = F(Z),$$

or, equivalently,

$$S(Z) = \exp\left(\int_0^Z F(\zeta)\, d\zeta\right).$$

Now note that if A is the disjoint union of two cyclic sets A_1 and A_2, then $S_A(Z) = S_{A_1}(Z) \cdot S_{A_2}(Z)$, since $S^m(A) = \bigcup_{i=0}^m S^i(A_1) \times S^{m-i}(A_2)$. More generally, if we decompose A into its Γ-orbits, $A = \dot{\bigcup}_{m=1}^{\infty} M(A, m) \cdot \Gamma/\Gamma_m$, with $M(A, m)$ the number of Γ-orbits of type Γ/Γ_m, then we have

$$S_A(Z) = \prod_{m=1}^{\infty} S_{\Gamma/\Gamma_m}(Z)^{M(A, m)}.$$

Obviously, $S_{\Gamma/\Gamma_m}(Z) = 1/(1-Z^m)$ and $\sum_{d\mid m} dM(A, d) = f_m(A)$; i.e., $M(A, m) = (1/m) \sum_{d\mid m} \mu(d)\, f_{m/d}(A)$ and, hence, we obtain

$$\prod_{m=1}^{\infty} \left(\frac{1}{1-Z^m}\right)^{(1/m) \sum_{d \mid m} \mu(d)\, f_{m/d}(A)} = \exp\left(\int_0^Z F(\zeta)\, d\zeta\right), \tag{3.17}$$

an identity, which could also have been proved by first establishing it directly for a transitive Γ-set A of type, say, Γ/Γ_m, in which case both sides coincide with $1/(1-Z^m)$, and then using the multiplicativity of both sides to establish the general case of an arbitrary almost finite cyclic set.

In the special case where $A = P(F)$ is the set of all periodic functions from $\mathbb{Z}$ into a finite set F of cardinality q with the infinite cyclic group Γ acting on A by translations, we have $f_n(A) = q^n$ for all $n \in \mathbb{N}$, since a map $\mathbb{Z} \to F$ of period n is completely determined by its values on, say, $\{0, 1, \ldots, n-1\}$, which in turn can be chosen freely in F. Hence, $f_1(S^m(A)) = q^m$ is the unique solution of (3.16); i.e., we have in this case

$$S(Z) = \sum_{m=0}^{\infty} f_1(S^m(A))\, Z^m = \frac{1}{1-qZ},$$

and (3.17) yields immediately the so-called *cyclotomic identity*,

$$\prod_{m=1}^{\infty} \left(\frac{1}{1-Z^m}\right)^{(1/m) \sum_{d \mid m} \mu(d)\, q^{m/d}} = \frac{1}{1-qZ}. \tag{3.18}$$

Of course, as is well known, this identity can also be proved quite easily by taking logarithmic derivatives of both sides, and the "combinatorial" proof of this identity, suggested by our approach, coincides essentially with the proof given in [VW] (in that both proofs give proper combinatorial interpretations of the terms occurring in the logarithmic derivatives, only). This is to be distinguished from the (much more demanding) combinatorial proof by Metropolis and Rota (cf. [MR]; see also [DS 1-5]) which is based on a combinatorial interpretation of the terms occurring in the identity itself.

4. TOPOS-LIKE CATEGORIES

The example leading to Proposition 3 can be generalized in different ways. A convenient framework for many such generalizations is the following one.

Let $\mathscr{C}$ denote a category and assume that—as in the case where $\mathscr{C}$ is the category of cyclic sets or, more generally, the category of G-sets for a given group G—we have a forgetful functor $V: \mathscr{C} \to \widetilde{\widetilde{\mathrm{Ens}}}$ from $\mathscr{C}$ into the category of (all) sets (and arbitrary maps between sets), such that

(T1) *for any bijection* $\alpha: \Omega_1 \simeq \Omega_2$ *between two finite sets and any object* C_1 *in* $\mathscr{C}$ *with* $V(C_1) = \Omega_1$, *there exists a unique pair* (C_2, γ), *consisting of an object* $C_2 = C(C_1, \alpha)$ *in* $\mathscr{C}$ *with* $V(C_2) = \Omega_2$ *and an isomorphism* $\gamma = \gamma(C_1, \alpha) \in \mathrm{Iso}_{\mathscr{C}}(C_1, C_2)$ *with* $V(\gamma) = \alpha$,

that is, $\mathscr{C}$ allows "transport of structure" relative to V and bijections between finite sets in $\widetilde{\widetilde{\mathrm{Ens}}}$. By uniqueness, we have $C(C_1, \mathrm{id}_{\Omega_1}) = C_1$ and $C(C_2, \beta) = C(C_1, \beta \circ \alpha)$ as well as $\gamma(C_1, \beta \circ \alpha) = \gamma(C_2, \beta) \circ \gamma(C_1, \alpha)$ for every bijection $\beta: \Omega_2 \simeq \Omega_3$, where $C_2 = C(C_1, \alpha)$. In particular we find that $C(C_2, \alpha^{-1}) = C_1$, whence it follows that the map $C_1 \mapsto C(C_1, \alpha)$ defines a bijection between the class $V^{-1}(\Omega_1)$ of objects in $\mathscr{C}$ with $V(C_1) = \Omega_1$ and the correspondingly defined class $V^{-1}(\Omega_2)$, which we denote by $V^{-1}(\alpha)$ and which satisfies $V^{-1}(\mathrm{id}_{\Omega_1}) = \mathrm{id}_{V^{-1}(\Omega_1)}$ as well as $V^{-1}(\beta \circ \alpha) = V^{-1}(\beta) \circ V^{-1}(\alpha)$ for any bijection $\beta: \Omega_2 \simeq \Omega_3$ as above. Hence, assuming that the cardinal numbers $c_V(m) = c_{(\mathscr{C}, V)}(m) := \# V^{-1}(\{1, ..., m\})$ satisfy

(T2) $c_V(m) < \infty$ *for all* $m \geqslant 0$ *and* $c_V(m_1) > 0$ *for some* $m_1 \geqslant 0$,

we obtain a functor $\emptyset \neq V^{-1}$: Ens $\to$ Ens, $\Omega \mapsto V^{-1}(\Omega)$ and $\alpha \mapsto V^{-1}(\alpha)$. Axiom (T2) holds for the category of G-sets at least if G is finitely generated or of finite Prüfer rank. Furthermore, let us assume that

(T3) $\mathscr{C}$ *contains and* V *commutes with coproducts* $C_1 \sqcup C_2$ *of two objects* C_1 *and* C_2.

Again, this holds for the category of G-sets for every group G. Using (T3), together with (T1), we get a unique natural transformation $\eta = \eta^V = \eta^{(\mathscr{C}, V)}: V^{-1} \times V^{-1} \to V^{-1} \mathbin{\dot\cup}$ by associating, for any two finite sets Ω_1 and Ω_2, to each pair $(C_1, C_2) \in V^{-1}(\Omega_1) \times V^{-1}(\Omega_2)$ the coproduct $C_1 \sqcup C_2$ in $\mathscr{C}$, or, more precisely, the unique object $C(C_1 \sqcup C_2, \alpha)$ in $V^{-1}(\Omega_1 \mathbin{\dot\cup} \Omega_2)$ (which we will identity with $C_1 \sqcup C_2$), defined by applying "transport of structure" to the coproduct $C_1 \sqcup C_2$ relative to the canonical isomorphism $\alpha: V(C_1 \sqcup C_2) \simeq V(C_1) \mathbin{\dot\cup} V(C_2) = \Omega_1 \mathbin{\dot\cup} \Omega_2$ whose existence is guaranteed by our assumption that V transforms coproducts in $\mathscr{C}$ into coproducts in $\widetilde{\widetilde{\mathrm{Ens}}}$ (as η shows yet no sign of injectivity, $V^{-1} \times V^{-1}$ and $V^{-1} \circ \mathbin{\dot\cup}$ have for the moment to be interpreted as functors $\mathrm{Ens}^2 \to \widetilde{\widetilde{\mathrm{Ens}}}$). Obviously, for any three (pairwise disjoint) sets Ω_1, Ω_2, and Ω_3 we have

$$\eta_{(\Omega_1 \dot\cup \Omega_2, \Omega_3)} \circ (\eta_{(\Omega_1, \Omega_2)} \times \mathrm{id}_{V^{-1}(\Omega_3)}) = \eta_{(\Omega_1, \Omega_2 \dot\cup \Omega_3)} \circ (\mathrm{id}_{V^{-1}(\Omega_1)} \times \eta_{(\Omega_2, \Omega_3)}) \tag{4.1}$$

and

$$\eta_{(\Omega_2, \Omega_1)} \circ \mathit{switch}_{V^{-1}(\Omega_1) \times V^{-1}(\Omega_2)} = \eta(\Omega_1, \Omega_2). \tag{4.2}$$

Finally, let us make the following further assumption which, again, is satisfied for $\mathscr{C}$ the category of G-sets for an arbitrary group G.

(T4) *$\mathscr{C}$ contains and V commutes with pullbacks and for every three objects C_1, C_2, and C in $\mathscr{C}$ and every morphism $\gamma: C \to C_1 \sqcup C_2$ the object C is the coproduct of the pullbacks C_1' and C_2' of C and C_1 or C_2, respectively, over $C_1 \sqcup C_2$:*

$$\begin{array}{ccccc} C_1' & \longrightarrow & C & \longleftarrow & C_2' \\ \downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ C_1 & \underset{\iota_1}{\longrightarrow} & C_1 \sqcup C_2 & \underset{\iota_2}{\longleftarrow} & C_2 \end{array}$$

Note that this special property (T4) of a category is also discussed in the theory of toposes (or, rather, topoi); cf. [G]. A category $\mathscr{C}$ together with a forgetful functor $V: \mathscr{C} \to \underset{\approx}{\mathrm{Ens}}$ will be called a *topos-like category*, if the pair $(\mathscr{C}, V)$ satisfies the axioms (T1)–(T4).

It follows from these assumptions that for any finite set Ω, any two partitions $\Omega = \Omega_1 \mathbin{\dot\cup} \Omega_2 = \Omega_1' \mathbin{\dot\cup} \Omega_2'$ of Ω into a pair of disjoint subsets Ω_1, Ω_2 and Ω_1', Ω_2', and any four objects C_1, C_2, C_1', C_2' in $\mathscr{C}$ with $V(C_i) = \Omega_i$, $V(C_j') = \Omega_j'$ and $C_1 \sqcup C_2 = C_1' \sqcup C_2' =: C$ we have the diagram

$$\begin{array}{ccccc} C_{11} & \longrightarrow & C_1' & \longleftarrow & C_{21} \\ \downarrow & & \downarrow & & \downarrow \\ C_1 & \longrightarrow & C & \longleftarrow & C_2 \\ \uparrow & & \uparrow & & \uparrow \\ C_{12} & \longrightarrow & C_2' & \longleftarrow & C_{22} \end{array}$$

of pullbacks and coproducts which, under V, transforms into the corresponding diagram

$$\begin{array}{ccccc} \Omega_1 \cap \Omega_1' & \longrightarrow & \Omega_1' & \longleftarrow & \Omega_2 \cap \Omega_1' \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_1 & \longrightarrow & \Omega & \longleftarrow & \Omega_2 \\ \uparrow & & \uparrow & & \uparrow \\ \Omega_1 \cap \Omega_2' & \longrightarrow & \Omega_2' & \longleftarrow & \Omega_2 \cap \Omega_2' \end{array}$$

and therefore guarantees in particular (i) that $\eta_{(\Omega_1, \Omega_2)}$ is injective (choose $\Omega_1' = \Omega_1$ and $\Omega_2' = \Omega_2$) and (ii) that η is a weak decomposition of V^{-1} in the sense of Section 1 (one part of (1.2) follows from the remarks above, for the other inclusion use Eqs. (4.1) and (4.2)). For a nonempty finite set Ω the set $(V^{-1})_\eta(\Omega)$ consists precisely of those objects C in $\mathscr{C}$ with $V(C)=\Omega$ which are *indecomposable*; i.e., $C=C_1 \sqcup C_2$ implies $C_1 \cong C$ or $C_2 \cong C$. More generally, $(V^{-1})_\eta^{(k)}(\Omega)$ consists of those objects C with $V(C)=\Omega$ which are coproducts of k indecomposable objects and the above observation also shows that such a decomposition is uniquely determined up to order and isomorphism by C; in particular the pair (V^{-1}, η^V) also satisfies (D2); i.e., the functor V^{-1} is decomposable and η^V is a decomposition of V^{-1}. Hence, denoting by $c_V^\eta(n)=c_{(\mathscr{C},V)}^\eta(n) := \#(V^{-1})_\eta(\{1, ..., n\})$ the number of these "simple" objects over the n-set $\{1, ..., n\}$ and putting $c_V^\eta(n,k)=c_{(\mathscr{C},V)}^\eta(n,k) := \#(V^{-1})_\eta^{(k)}(\{1, ..., n\})$ our theorem yields the following.

PROPOSITION 4. *For a topos-like category $\mathscr{C}$ with forgetful functor V we have*

$$c_V^\eta(n,k)=\sum_{\mu=1}^{n}\binom{n-1}{\mu-1} c_V^\eta(\mu)\, c_V^\eta(n-\mu, k-1) \qquad (n, k \geqslant 1) \tag{4.3}$$

and

$$c_V(m)=\sum_k c_V^\eta(m,k) \qquad (m \geqslant 0). \tag{4.4}$$

Note that the examples leading to Propositions 1 and 2 correspond to special choices of the category $\mathscr{C}$.

In the first case $\mathscr{C}$ has to be chosen as the full subcategory of the category of G-sets, consisting of those G-sets Ω on which each element in Σ acts without fixed points and whose orbit lengths are in the set S, and V, of course, as the forgetful functor into the category of sets. Obviously, $V^{-1}(\Omega)$ is finite for each finite set Ω if and only if the triple (G, Σ, S) is admissible.

In the second case $\mathscr{C}$ has to be chosen as the category, whose objects are maps $f: \Omega \to \Omega$ of an arbitrary set Ω into itself with the properties that (i) $f^\alpha = f^\beta$ and (ii) the component lengths of f are in S and whose morphisms $\gamma \in \mathrm{Mor}_{\mathscr{C}}(f_1: \Omega_1 \to \Omega_1, f_2: \Omega_2 \to \Omega_2)$ are maps $\gamma: \Omega_1 \to \Omega_2$ such that $\gamma \circ f_1 = f_2 \circ \gamma$. The functor $V: \mathscr{C} \to \mathrm{Ens}$ is, of course, given by $V(f)=\Omega$ and $V(\gamma)=\gamma$.

Now consider objects A in $\mathscr{C}$ such that for every object C in $\mathscr{C}$ with $V(C)$ finite there exist only finitely many $\mathscr{C}$-morphisms from C to A. With each such distinguished object A in $\mathscr{C}$ we associate a functor $F_A: \mathrm{Ens} \to \mathrm{Ens}$, given by

$$F_A(\Omega) := \{(C, \gamma) \in |\mathscr{C}| \times \mathrm{Mor}(\mathscr{C}) \colon V(C)=\Omega, \gamma \in \mathrm{Mor}_{\mathscr{C}}(C, A)\}.$$

The bijection $F_A(\alpha)$ induced by a morphism $\alpha\colon \Omega_1 \simeq \Omega_2$ in Ens is, of course, defined via "transport of structure" as $F_A(\alpha)(C_1, \gamma_1) = (C_2, \gamma_2)$, where $C_2 := C(C_1, \alpha)$ and $\gamma_2 := \gamma_1 \circ \gamma(C_1, \alpha)^{-1}$. Then

$$F_A(\Omega_1) \times F_A(\Omega_2) \ni ((C_1, \gamma_1), (C_2, \gamma_2)) \mapsto (C_1 \sqcup C_2, \gamma_1 \sqcup \gamma_2) \in F_1(\Omega_1 \dot\cup \Omega_2)$$

defines a decomposition $\eta_A\colon F_A \times F_A \to F_A \circ \dot\cup$ of F_A; indeed the pair (F_A, η_A) is of the form (V^{-1}, η^V) for the category $\mathscr{C}/A$ of objects in $\mathscr{C}$ over A and the forgetful functor $V_A\colon \mathscr{C}/A \to \widetilde{\widetilde{\text{Ens}}}$ defined by combining $V\colon \mathscr{C} \to \widetilde{\widetilde{\text{Ens}}}$ with the canonical forgetful functor $\mathscr{C}/A \to \mathscr{C}$. The example leading to Proposition 3 fits into this setup: For $\mathscr{C}$ the category of cyclic sets, or, more generally, the category of G-sets, G an admissible group (i.e., $\mathrm{Hom}(G, S_n)$ finite for all n), the almost finite sets are distinguished objects in the sense above, and for an almost finite cyclic set A the pair (F_A, η_A) is naturally isomorphic to the corresponding data constructed in Section 3. Indeed, all instances of a relation between two generating functions of the type described in the theorem we have seen so far appear to be interpretable in this framework which, of course, invites many further applications.

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